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f-BIHARMONIC SUBMANIFOLDS AND *f*-BIHARMONIC INTEGRAL SUBMANIFOLDS IN LOCALLY CONFORMAL ALMOST COSYMPLECTIC SPACE FORMS

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Abstract. In this paper, we have studied f-biharmonic submanifolds in locally conformal almost cosymplectic space forms and have derived condition on second fundamental form for f-biharmonic submanifolds. Also, we have discussed its integral submanifolds in locally conformal almost cosymplectic space forms.

1. Introduction

Harmonic maps and biharmonic maps are important fields of research being the critical points of energy functional and bienergy functional. Because of both geometric and analytical aspects, harmonic maps are upward trend of researches. The idea behind the biharmonic maps is old and attractive subject of research. The biharmonic maps have been studied in 1862 by Maxwell and Airy to describe a mathematical model of elasticity. Biharmonic maps are a generalization of harmonic maps and first regular studied by Eells and Lemaire in 1978 [5]. In 1986, Jiang [10] discussed first and second variations formulas for bienergy functional. In 2015, Lu introduced f-biharmonic maps [15]. The first variation of the f-biharmonic maps and the equation for the f-biharmonic conformal maps between the same dimensional manifolds are calculated in [15]. In [19], Ou considered f-biharmonic maps and f-biharmonic submanifolds.

A map F between two Riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ is called harmonic and biharmonic, respectively if it is a critical point of $E(F)$ and $E_2(F)$

$$E(F) = \frac{1}{2}\int_{\mathcal{M}} \|dF\|^2 \, dv_g,$$

and

$$E_2(F) = \frac{1}{2}\int_{\mathcal{M}} \|\tau(F)\|^2 \, dv_g,$$

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where $(\mathcal{M}, g)$ is a compact Riemannian manifold and dv_g is the volume measure associated with the metric g on $\mathcal{M}$. A map $F : (\mathcal{M}, g) \to (\mathcal{N}, h)$ is harmonic and biharmonic, respectively if and only if $\tau(F) = 0$ and $\tau_2(F) = 0$, where $\tau(F)$ and $\tau_2(F)$ are called the tension field [6] and the bitension field [10], respectively which are given by

$$\tau(F) = trace_g(\nabla dF) = 0,$$

and

$$\tau_2(F) = trace_g(\nabla^F \nabla^F - \nabla^F_{\nabla})(\tau(F)) - trace_g(R^{\mathcal{N}}(dF, \tau(F))dF) = 0. \tag{1}$$

A map F is called f-biharmonic if it is the critical point of $E_{2,f}(F)$ [15],

$$E_{2,f}(F) = \frac{1}{2}\int_{\mathcal{M}} f \, \|\tau(F)\|^2 \, dv_g,$$

where $f : \mathcal{M} \to \mathbb{R}$ is a differentiable function. A map F is f-biharmonic if and only if $\tau_{2,f}(F) = 0$ where $\tau_{2,f}(F)$ is called the f-bitension field [15], which is given by

$$\tau_{2,f}(F) = f\tau_2(F) + \triangle f \tau(F) + 2\nabla^F_{grad\ f}\tau(F) = 0. \tag{2}$$

An f-biharmonic map is called proper f-biharmonic if it is neither harmonic nor biharmonic [19]. Moreover, if f is a constant, then an f-biharmonic map turns into a biharmonic map [15].

Recently, many geometers studied biharmonic and f-biharmonic submanifolds in different ambient spaces [4, 7, 13, 16, 20]. In [1], Baikoussis and Blair gave a classification of 3-dimensional flat integral C-parallel submanifolds in the unit sphere $\mathbb{S}^7(1)$. In [8], Fetcu and Oniciuc studied integral C-parallel submanifolds in 7-dimensional Sasakian space form. They also studied biharmonic integral C-parallel submanifolds in 7-dimensional Sasakian space forms and give its classification [9]. In [11], Karaca studied f-biharmonic integral submanifolds in generalized Sasakian space forms. In [21], Roth and Upadhyay studied f-biharmonic submanifolds in both generalized complex and Sasakian space forms. In [12], Karaca studied f-biminimal submanifolds of generalized space form. Motivated by these studies, in present paper, we consider f-biharmonic submanifolds and f-biharmonic integral submanifolds in locally conformal almost cosymplectic space forms. We obtain the necessary and sufficient conditions for submanifolds in locally conformal almost cosymplectic space forms to be f-biharmonic. Then, we also obtain the necessary and sufficient conditions for integral and integral C-parallel submanifolds in locally conformal almost cosymplectic space forms to be f-biharmonic.

2. Preliminaries

Let $\mathcal{N}^{2n+1} = (\mathcal{N}, \varphi, \xi, \eta)$ be an almost contact manifold [2] with an almost contact structure (φ, ξ, η) which satisfies

$$\varphi^2 = -I + \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1, \tag{3}$$

where φ, ξ and η are a $(1,1)$ tensor field, a vector field and 1-form, respectively. Clearly, (3) gives

$$\varphi(\xi) = 0 \quad \text{and} \quad \eta \circ \varphi = 0.$$

Define an almost complex structure J on the product manifold $\mathcal{N} \times \mathbb{R}$ defined by

$$J(X, \lambda \frac{d}{dt}) = \left(\varphi X - \lambda \xi, \eta(X) \frac{d}{dt}\right),$$

where X is tangent to $\mathcal{N}$, t the coordinate of $\mathbb{R}$ and λ a smooth function on $\mathcal{N} \times \mathbb{R}$. The manifold $\mathcal{N}$ is called normal if the almost complex structure J is integrable. The necessary and sufficient condition for $\mathcal{N}$ to be normal is

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ. There exists a compatible Riemannian metric g which satisfies

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \quad \text{and} \quad g(X, \xi) = \eta(X)$$

for all $X, Y \in T\mathcal{N}$. Then, $\mathcal{N}$ becomes an almost contact metric manifold with an almost contact metric structure (φ, ξ, η, g). If the fundamental 2-form Φ and 1-form η are closed, where

$$\Phi\,(X, Y) = g(X, \varphi Y)$$

then, $\mathcal{N}$ is said to be almost cosymplectic manifold. It is well known that a normal almost cosymplectic manifold is cosymplectic [2]. The manifold $\mathcal{N}$ is said to be a locally conformal almost cosymplectic manifold [22] if there exists a 1-form ω such that

$$d\Phi = 2\omega \wedge \Phi, \quad d\eta = \omega \wedge \eta \quad \text{and} \quad d\omega = 0.$$

A structure (φ, ξ, η, g) to be normal locally conformal almost cosymplectic [17] if and only if

$$(\nabla_X \varphi)Y = u\left(g(\varphi X, Y)\xi - \eta(Y)\varphi X\right), \tag{4}$$

where ∇ is the Levi-Civita connection of the Riemannian metric g and $\omega = u\eta$. From the equation (4), it follows that

$$\nabla_X \xi = u(X - \eta(X)\xi).$$

A locally conformal almost cosymplectic manifold $\mathcal{N}$ of dimension ≥ 5 is of pointwise constant φ-sectional curvature if and only if its curvature tensor $\tilde{R}^{\mathcal{N}}$ of the form

$$\begin{aligned} \tilde{R}^{\mathcal{N}}(X, Y)Z = {} & \frac{(c - 3u^2)}{4}\{g(Y, Z)X - g(X, Z)Y\} \\ & + \frac{(c + u^2)}{4}\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \\ & - (\frac{c + u^2}{4} + u')\{\eta(Y)\eta(Z)X - g(X, Z)\eta(Y)\xi \\ & \qquad + g(Y, Z)\eta(X)\xi - \eta(X)\eta(Z)Y\}, \end{aligned} \tag{5}$$

where u is the function such that $\omega = u\eta$, $u' = \xi(u)$ and c the pointwise φ-sectional curvature of $\mathcal{N}$ [18].

Let $\mathcal{M}$ be an m-dimensional submanifold immersed in $\mathcal{N}$. Let $X \in T\mathcal{M}$ and $V \in T\mathcal{M}^{\perp}$. The decompositions of φX and φV into tangent and normal components can be written as

$$\varphi X = TX + NX \quad \text{and} \quad \varphi V = tV + SV, \tag{6}$$

where TX and NX are tangent component and normal component of φX, respectively, whereas tV and SV are tangent component and normal component of φV, respectively. A submanifold $\mathcal{M}$ of a locally conformal almost cosymplectic manifold $\mathcal{N}$ is called anti-invariant (resp. invariant) if T (resp. N) vanishes identically. Moreover, it is known that $\varphi(T_X\mathcal{M}) \subset T_X^{\perp}\mathcal{M}$ for all $X \in \mathcal{M}$, then $\mathcal{M}$ is anti-invariant [14, 23].

3. *f*-biharmonic submanifolds in locally conformal almost cosymplectic space forms

Denote by B, A, H, ∇ and $\nabla^{\perp}$, the second fundamental form, the shape operator, the mean curvature vector field, the connection and the Laplacian in normal bundle, respectively.

We have the following theorem:

Theorem 3.1. *Let $\mathcal{M}^m$ be a submanifold of locally conformal almost cosymplectic space form $\mathcal{N}^{2n+1}$. Then $\mathcal{M}^m$ is an f-biharmonic submanifold of $\mathcal{N}^{2n+1}$ if and only if*

$$\begin{aligned}
&-\Delta^{\perp}H - trace(B\cdot, A_H(\cdot)) + \frac{\Delta f}{f}H + 2(\nabla^{\perp}_{grad(\ln f)}H)\\
=\ &-mH\frac{(c-3u^2)}{4} + (\frac{(c+u^2)}{4} + u')\{|\xi^t|^2 H + m\eta(H)\xi^{\perp}\}\\
&+\frac{3(c+u^2)}{4}NtH,
\end{aligned}\tag{7}$$

and

$$\begin{aligned}
&-\frac{m}{2}\ grad(|H|)^2 - 2trace(A_{\nabla^{\perp}_H}(\cdot)) - 2A_H\ grad(\ln f)\\
&= (\frac{c+u^2}{4} + u')(m-1)\eta(H)\xi^t + \frac{3(c+u^2)}{4}TtH.
\end{aligned}\tag{8}$$

Proof. Let $\{E_i\}$ $(1 \le i \le m)$ be a local geodesic orthonormal frame on $\mathcal{M}$. Using the equation (5) and $H \in span\{\varphi E_i : i = 1, \ldots, m\}$, we have

$$\begin{aligned}
\tilde{R}^{\mathcal{N}}(E_i, H)E_i = &\frac{(c-3u^2)}{4}\{g(H, E_i)E_i - g(E_i, E_i)H\}\\
&+\frac{(c+u^2)}{4}\{g(E_i, \varphi E_i)\varphi H - g(H, \varphi E_i)\varphi E_i + 2g(E_i, \phi H)\varphi E_i\}\\
&-(\frac{c+u^2}{4} + u')\{\eta(H)\eta(E_i)E_i - g(E_i, E_i)\eta(H)\xi
\end{aligned}$$

$$+g(H,E_i)\eta(E_i)\xi - \eta(E_i)\eta(E_i)H)\} . \tag{9}$$

After a straightforward computation, we obtain

$$\begin{aligned} trace_g\left(\tilde{R}^{\mathcal{N}}(E_i,H)E_i\right) = & -mH\frac{(c-3u^2)}{4} + \frac{3(c+u^2)}{4}(TtH+NtH) \\ & +(\frac{c+u^2}{4}+u')\{|\xi^t|^2H - \eta(H)\xi^t + m\eta(H)\xi\}. \end{aligned} \tag{10}$$

Using tension field of $F:(\mathcal{M},g)\to(\mathcal{N},h)$, we can write

$$\tau(F) = trace\,(\nabla dF) = mH. \tag{11}$$

Making use of (1) and (11), we obtain

$$\tau_2(F) = -mtrace_g(\nabla^F\nabla^F - \nabla^F_{\nabla})(\tau(F))H - mtrace_g\tilde{R}^{\mathcal{N}}(dF,H)dF. \tag{12}$$

From the well known computation, we have

$$\begin{aligned} trace_g(\nabla^F\nabla^F - \nabla^F_{\nabla})(\tau(F)) = & -\Delta^{\perp}H - \frac{m}{2}\,grad(|H|)^2 \\ & - trace(B\cdot, A_H\cdot) - 2trace(A_{\nabla^{\perp}H}(\cdot)). \end{aligned} \tag{13}$$

Putting (13) into (12), we get

$$\begin{aligned} \tau_2(F) = & -m\Delta^{\perp}H - m\frac{m}{2}\,grad(|H|)^2 - mtrace(B\cdot, A_H\cdot) \\ & -2mtrace(A_{\nabla^{\perp}}H(\cdot)) - mtrace_g\tilde{R}^{\mathcal{N}}(dF,H)dF. \end{aligned} \tag{14}$$

From (2), we have

$$\tau_2(F) + m\frac{\Delta f}{f}H - 2mA_H\,grad(\ln f) + 2m\nabla^{\perp}_{grad(\ln f)}H = 0. \tag{15}$$

Substituting (14) into (15), we obtain

$$\begin{aligned} & -\Delta^{\perp}H - \frac{m}{2}\,grad(|H|)^2 - trace(B\cdot, A_H\cdot) - 2trace(A_{\nabla^{\perp}H}(\cdot)) \\ & - trace_g\tilde{\mathcal{R}}^N(dF,H)dF + \frac{\Delta f}{f}H - 2A_H\,grad(\ln f) + 2\nabla^{\perp}_{grad(\ln f)}H = 0. \end{aligned} \tag{16}$$

When $\mathcal{M}^m$ is an f-biharmonic submanifold of $\mathcal{N}^{2n+1}$, substituting (10) in (16) and comparing normal and tangential components, we have desired result. □

Corollary 3.2. *Let $\mathcal{M}^m$ be a submanifold of locally almost cosymplectic space form $\mathcal{N}^{2n+1}$.*

1. *If $\mathcal{M}^m$ is invariant, then $\mathcal{M}^m$ is f-biharmonic if and only if*

$$\begin{aligned} & -\Delta^{\perp}H - trace(B\cdot, A_H(\cdot)) + \frac{\Delta f}{f}H + 2(\nabla^{\perp}_{grad(\ln f)}H) \\ = & -mH\frac{(c-3u^2)}{4} + (\frac{c+u^2}{4}+u')\{|\xi^t|^2H + m\eta(H)\xi^{\perp}\}, \end{aligned} \tag{17}$$

and

$$-\frac{m}{2}\,grad(|H|)^2 - 2trace(A_{\nabla^{\perp}_H}(\cdot)) - 2A_H\,grad(\ln f)$$

$$= (\frac{c+u^2}{4} + u')(m-1)\eta(H)\xi^t + \frac{3(c+u^2)}{4}TtH. \tag{18}$$

2. *If $\mathcal{M}^m$ is anti-invariant, then $\mathcal{M}^m$ is f-biharmonic if and only if*

$$\begin{aligned} & -\Delta^\perp H - trace(B\cdot, A_H(\cdot)) + \frac{\Delta f}{f}H + 2(\nabla^\perp_{grad(\ln f)}H) \\ = & -mH\frac{(c-3u^2)}{4} + (\frac{c+u^2}{4} + u')\{|\xi^t|^2 H + m\eta(H)\xi^\perp\} \\ & + \frac{3(c+u^2)}{4}NtH, \end{aligned} \tag{19}$$

and

$$\begin{aligned} & -\frac{m}{2}\ grad(|H|)^2 - 2trace(A_{\nabla^\perp_H}(\cdot)) - 2A_H\ grad(\ln f) \\ = & (\frac{c+u^2}{4} + u')(m-1)\eta(H)\xi^t. \end{aligned} \tag{20}$$

Proof. 1. For $\mathcal{M}^m$ is invariant, we take $N = 0$ in (7) and (8). Then, we obtain result.

2. For $\mathcal{M}^m$ is anti-invariant, we take $T = 0$ in (7) and (8). Then, we obtain the desired result. □

Corollary 3.3. *Let $\mathcal{M}^m$ be a submanifold of locally almost cosymplectic space form $\mathcal{N}^{2n+1}$.*

1. *Let ξ be normal to $\mathcal{M}^m$, then $\mathcal{M}^m$ is f-biharmonic if and only if*

$$\begin{aligned} & -\Delta^\perp H - trace(B\cdot, A_H(\cdot)) + \frac{\Delta f}{f}H + 2(\nabla^\perp_{grad(\ln f)}H) \\ = & -mH\frac{(c-3u^2)}{4} + (\frac{c+u^2}{4} + u')(m\eta(H)\xi) + \frac{3(c+u^2)}{4}NtH, \end{aligned}$$

and

$$-\frac{m}{2}\ grad(|H|)^2 - 2trace(A_{\nabla^\perp_H}(\cdot)) - 2A_H\ grad(\ln f) = 0.$$

2. *Let ξ be tangent to $\mathcal{M}^m$, then $\mathcal{M}^m$ is f-biharmonic if and only if*

$$\begin{aligned} & -\Delta^\perp H - trace(B\cdot, A_H(\cdot)) + \frac{\Delta f}{f}H + 2(\nabla^\perp_{grad(\ln f)}H) \\ = & -mH\frac{(c-3u^2)}{4} + (\frac{c+u^2}{4} + u') + \frac{3(c+u^2)}{4}NtH, \end{aligned}$$

and

$$-\frac{m}{2}\ grad(|H|)^2 - 2trace(A_{\nabla^\perp_H}(\cdot)) - 2A_H\ grad(\ln f) = \frac{3(c+u^2)}{4}TtH.$$

Proof. 1. Since ξ is normal to $\mathcal{M}^m$, then the tangential component of ξ vanishes and take $\mathcal{M}^m$ as anti-invariant, $T = 0$. Taking $\xi^t = 0$, $\xi^\perp = \xi$ and $T = 0$ in (7) and (8), we get the result.

2. Since ξ is tangent to $\mathcal{M}^m$, then $\xi^\perp$ vanishes and taking $\eta(H) = 0$ in (7) and (8), we obtain the result. □

Corollary 3.4. *Let $\mathcal{M}^{2n}$ be a hypersurface of locally almost cosymplectic space form $\mathcal{N}^{2n+1}$. Then $\mathcal{M}^{2n}$ is f-biharmonic if and only if*

$$\begin{aligned}
&-\Delta^{\perp}H - trace(B\cdot, A_H(\cdot)) + \frac{\Delta f}{f}H + 2(\nabla^{\perp}_{grad(\ln f)}H) \\
&= \left(-(2n)\frac{(c-3u^2)}{4} + (\frac{c+u^2}{4} + u')|\xi^t|^2 - \frac{3(c+u^2)}{4}\right)H \\
&+ \left(2n(\frac{(c+u^2)}{4} + u') + \frac{3(c+u^2)}{4}\right)\eta(H)\xi^{\perp},
\end{aligned}$$

and

$$\begin{aligned}
&-n\ grad(|H|)^2 - 2trace(A_{\nabla^{\perp}_H}(\cdot)) - 2A_H\ grad(\ln f) \\
&= \left((\frac{c+u^2}{4} + u')(2n-1) + \frac{3(c+u^2)}{4}\right)\eta(H)\xi^t.
\end{aligned}$$

Proof. Let $\mathcal{M}^{2n}$ be a hypersurface. Thus, we have φH is tangent. Using the equation (6), we get $SH = 0$. Then, we obtain $-H + \eta(H)\xi = TtH + NtH$. By comparing the tangential and normal parts, $TtH = \eta(H)\xi^t$ and $NtH = -H + \eta(H)\xi^{\perp}$ which gives the result. □

Proposition 3.5. *Let $\mathcal{M}^{2n}$ be a hypersurface of locally almost cosymplectic space form $\mathcal{N}^{2n+1}$ with non zero constant mean curvature H and ξ tangent to $\mathcal{M}^{2n}$. Then $\mathcal{M}^{2n}$ is proper f-biharmonic if and only if*

$$\|B\|^2 = \frac{\Delta f}{f} + \frac{n(c-3u^2)}{2} + \frac{3(c+u^2)}{4} - \left(\frac{c+u^2}{4} + u'\right),$$

and

$$A_H\ grad\ f = 0,$$

or equivalently, if and only if

$$\begin{aligned}
Scal_{\mathcal{M}} &= (c-3u^2)n(n-1) + \frac{3(c+u^2)}{2}(n-1) \\
&+ \left(\frac{c+u^2}{4} + u'\right)(3-4n) - \frac{\Delta f}{f} + 4n^2H^2,
\end{aligned} \tag{21}$$

and

$$A_H\ grad\ f = 0.$$

Proof. Let $\mathcal{M}^{2n}$ be an f-biharmonic hypersurface of $\mathcal{N}^{2n+1}$ and ξ is tangent to $\mathcal{M}^{2n}$, then $\eta(H) = 0$. Therefore, we can write

$$\varphi^2 H = -H + \eta(H)\xi = -H.$$

This implies that

$$TtH = 0 \quad \text{and} \quad NtH = -H. \tag{22}$$

Taking $\eta(H) = 0$ and the equation (22) in Corollary 3.4, we get

$$\begin{aligned} & -\Delta^{\perp} H - trace(B\cdot, A_H(\cdot)) + \frac{\Delta f}{f} H + 2(\nabla^{\perp}_{grad(\ln f)} H) \\ = & \left(-(2n)\,\frac{(c-3u^2)}{4} + (\frac{c+u^2}{4} + u') - \frac{3(c+u^2)}{4} \right) H, \end{aligned}$$

and

$$-n\ grad(|H|)^2 - 2trace(A_{\nabla^{\perp}_{H}}(\cdot)) - 2A_H\ grad(\ln f) = 0.$$

For constant mean curvature, we have

$$trace(B\cdot, A_H(\cdot)) = \left(\frac{\Delta f}{f} + \frac{n(c-3u^2)}{2} - (\frac{c+u^2}{4} + u') + \frac{3(c+u^2)}{4} \right) H, \tag{23}$$

and

$$A_H\ grad\ f = 0.$$

From the equation (23), we obtain

$$\|B\|^2 = \frac{\Delta f}{f} + \frac{n(c-3u^2)}{2} + \frac{3(c+u^2)}{4} - \left(\frac{c+u^2}{4} + u' \right). \tag{24}$$

Using Gauss equation, we have

$$Scal_{\mathcal{M}} = \sum_{i,j=1}^{2n} \tilde{R}^{\mathcal{N}}(E_i, E_j, E_j, E_i) - \|B\|^2 + 4n^2H^2. \tag{25}$$

Then, we calculate the following equation

$$\begin{aligned} \sum_{i,j=1}^{2n} \tilde{R}^{\mathcal{N}}(E_i, E_j, E_j, E_i) = {} & (2n-1)\frac{n(c-3u^2)}{2} + \frac{3(c+u^2)}{4}(2n-1) \\ & - 2(2n-1)(\frac{c+u^2}{4} + u'). \end{aligned} \tag{26}$$

Making use of (24), (25) and (26), we have

$$\begin{aligned} Scal_{\mathcal{M}} = {} & (c-3u^2)n(n-1) + \frac{3(c+u^2)}{2}(n-1) \\ & + \left(\frac{c+u^2}{4} + u' \right)(3-4n) - \frac{\Delta f}{f} + 4n^2H^2. \end{aligned}$$

This concludes the proof. □

Remark 3.6. Let $\mathcal{M}^{2n}$ be a constant mean curvature hypersurface with ξ tangent to $\mathcal{M}^{2n}$ on locally almost cosymplectic space form $\mathcal{N}^{2n+1}$. If the functions u and f satisfy the inequality

$$\frac{\Delta f}{f} \leq \left(\frac{c+u^2}{4} + u' \right) - \frac{n(c-3u^2)}{2} - \frac{3(c+u^2)}{4}$$

on $\mathcal{M}^{2n}$, then $\mathcal{M}^{2n}$ is not f-biharmonic.

4. f-biharmonic integral submanifolds in locally conformal almost cosymplectic space forms

A submanifold $\mathcal{M}^m$ of a contact manifold $\mathcal{N}^{2n+1}$ is called an integral submanifold if $\eta(X) = 0$ for every tangent vector field X [3]. An integral submanifold $\mathcal{M}^m$ of a contact manifold $\mathcal{N}^{2n+1}$ is said to be integral C-parallel [3] if $\nabla^\perp B$ is parallel to the characteristic vector field and $\nabla^\perp B$ is given by

$$\nabla^\perp B(X, Y, Z) = \nabla^\perp_X B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z),$$

for every tangent vector fields X, Y, Z, $\nabla^\perp$ and ∇ being the normal connection and the Levi-Civita connection on $\mathcal{M}$, respectively.

Let $\mathcal{N}^{2n+1}$ be a locally conformal almost cosymplectic space form with constant φ-sectional curvature c and $\mathcal{M}^m$ a submanifold of $\mathcal{N}^{2n+1}$.

We have the following theorem:

Theorem 4.1. *Let $\mathcal{M}^m$ be an integral submanifold of $\mathcal{N}^{2n+1}$. Then, $\mathcal{M}^m$ is f-biharmonic if and only if*

$$\begin{aligned}&\Delta^\perp H + trace(B\cdot, A_H(\cdot)) - 2\nabla^\perp_{grad(\ln f)} H\\ &= \left(\frac{(c-3u^2)}{4}m + 3\frac{(c+u^2)}{4} + \frac{\Delta f}{f}\right) H,\end{aligned} \tag{27}$$

and

$$\frac{m}{2}\ grad(|H|)^2 + 2trace(A_{\nabla^\perp_H}(\cdot)) + 2A_H\ grad(\ln f) = 0.$$

Proof. Let $\mathcal{M}^m$ be an integral submanifold of $\mathcal{N}^{2n+1}$. Let $\{E_i\}_{i=1}^m$ be a local orthonormal frame on $\mathcal{M}$, $\{E_i, \varphi E_j, \xi\}_{i,j=1}^n$ is a local orthonormal frame on $\mathcal{N}$. Using the equation (9) and $H \in span\{\varphi E_i : i = 1, \dots, n\}$, we can write

$$\tilde{R}^{\mathcal{N}}(E_i, H)E_i = -\frac{(c-3u^2)}{4} g(E_i, E_i)H + \frac{(c+u^2)}{4} 3g(E_i, \varphi H)\varphi E_i. \tag{28}$$

Hence, we have

$$\sum_{i=1}^m \tilde{\mathcal{R}}^N(E_i, H)E_i = -\left(\frac{(c-3u^2)}{4}m + \frac{3(c+u^2)}{4}\right) H. \tag{29}$$

Using equation (29) into (16), we obtain

$$\begin{aligned}&-\Delta^\perp H - \frac{m}{2}\ grad(|H|)^2 - trace(B\cdot, A_H\cdot) - 2trace(A_{\nabla^\perp H}(\cdot))\\ &+\left(\frac{(c-3u^2)}{4}m + \frac{3(c+u^2)}{4} + \frac{\Delta f}{f}\right) H - 2A_H\, grad(\ln f) + 2\nabla^\perp_{grad(\ln f)} H = 0.\end{aligned}$$

Finally, separating the tangential and normal components, we have the desired result. □

Corollary 4.2. *There does not exist a proper f-biharmonic integral submanifold $\mathcal{M}^m$ such that $\frac{\Delta f}{f} + \frac{(c-3u^2)}{4}m + \frac{3(c+u^2)}{4} < 0$ with constant mean curvature $||H||$ in $\mathcal{N}^{2n+1}$.*

Proof. Let $\mathcal{M}^m$ be an f-biharmonic integral submanifold with constant mean curvature $||H||$ in $\mathcal{N}^{2n+1}$. Then, taking the scalar product of (27) with H, we obtain

$$\begin{aligned} g(\Delta^{\perp}H,H) = & -g(trace(B\cdot,A_H(\cdot),H)+2g(\nabla^{\perp}_{grad(\ln f)}H,H) \\ & +\left(\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}\right)g(H,H) \\ = & -||A_H||^2+2g(\nabla^{\perp}_{grad(\ln f)}H,H) \\ & +\left(\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}\right)||H||^2. \end{aligned} \tag{30}$$

For constant mean curvature, we get

$$2g(\nabla^{\perp}_{grad(\ln f)}H,H)=0. \tag{31}$$

Making use of (30) and (31), we obtain

$$g(\Delta^{\perp}H,H)=-||A_H||^2+\left(\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}\right)||H||^2. \tag{32}$$

Using Weitzenböck formula for an f-biharmonic integral submanifold with constant mean curvature, we have

$$g(\Delta^{\perp}H,H)=||\nabla^{\perp}H||^2. \tag{33}$$

From (32) and (33), we obtain

$$||\nabla^{\perp}H||^2+||A_H||^2=\left(\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}\right)||H||^2. \tag{34}$$

Since, we assume that $\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}<0$, from (34), we get $||H||^2=0$, so $\mathcal{M}^m$ is minimal. This completes the proof. □

Corollary 4.3. *There does not exist a proper f-biharmonic compact integral submanifold $\mathcal{M}^m$ such that $\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}\leq 0$ in $\mathcal{N}^{2n+1}$.*

Proof. Let $\mathcal{M}^m$ be an f-biharmonic compact integral submanifold. By the use of the same method in the proof of Corollary 4.2, from the equation (32), $\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}\leq 0$ and Weitzenböck formula, we obtain the result. □

Proposition 4.4. *Let $\mathcal{M}^m$ be a integral C-parallel submanifold in $\mathcal{N}^{2n+1}$. Then, we have*

$$\left[\tau_{2,f}(F)\right]^T=A_H\ grad(\ln f)=0.$$

Proof. From [9], we have $||H||$ is constant and $\nabla^{\perp}H$ is parallel to ξ. Thus, we have $A_H\ grad(\ln f)=0$. This completes the proof. □

Proposition 4.5. *A non-minimal integral C-parallel submanifold $\mathcal{M}^m$ with constant mean curvature $||H||$ in $\mathcal{N}^{2n+1}$ is proper f-biharmonic if and only if*

$$\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}-1>0,$$

and

$$trace(B\cdot, A_H(\cdot))-2\nabla^{\perp}_{grad(\ln f)}H=\left[\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}-1\right]H.$$

Proof. From normal component of Theorem 4.1 and $\Delta^{\perp}H=H$ [9], we have

$$trace(B\cdot, A_H(\cdot))-2\nabla^{\perp}_{grad(\ln f)}H=\left[\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}-1\right]H. \tag{35}$$

Then taking the scalar product of the equation (35) with H, we obtain

$$||A_H||^2=\left[\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}-1\right]||H||^2.$$

Thus, it shows that

$$\frac{(c-3u^2)}{4}m+\frac{3(c+u^2)}{4}+\frac{\Delta f}{f}-1>0.$$

□

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References

[1] C. Baikoussis, D. E. Blair, and T. Koufogiorgos, *Integral submanifolds of Sasakian space forms $\overline{M}^7(k)$*, Results Math. **27** (1995), no. 3-4, 207–226. https://doi.org/10.1007/BF03322826

[2] D. E. Blair, *Contact manifolds. In Contact Manifolds in Riemannian Geometry*, Springer, Berlin, Heidelberg, (1976), 1–16.

[3] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, second edition, Progress in Mathematics, 203, Birkhäuser Boston, Ltd., Boston, MA, 2010. https://doi.org/10.1007/978-0-8176-4959-3

[4] R. Caddeo, S. Montaldo, and P. Piu, *On biharmonic maps*, in Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), 286–290, Contemp. Math., 288, Amer. Math. Soc., Providence, RI, 2001. https://doi.org/10.1090/conm/288/04836

[5] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), no. 1, 1–68. https://doi.org/10.1112/blms/10.1.1

[6] J. Eells, Jr., and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160. https://doi.org/10.2307/2373037

[7] D. Fetcu and C. Oniciuc, *Explicit formulas for biharmonic submanifolds in Sasakian space forms*, Pacific J. Math. **240** (2009), no. 1, 85–107. https://doi.org/10.2140/pjm.2009.240.85

[8] D. Fetcu and C. Oniciuc, *A note on integral C-parallel submanifolds in $\mathbb{S}^7(c)$*, Rev. Un. Mat. Argentina **52** (2011), no. 1, 33–45.

[9] D. Fetcu and C. Oniciuc, *Biharmonic integral C-parallel submanifolds in 7-dimensional Sasakian space forms*, Tohoku Math. J. (2) **64** (2012), no. 2, 195–222. https://doi.org/10.2748/tmj/1341249371

[10] G. Y. Jiang, *2-harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A **7** (1986), no. 4, 389–402.

[11] F. Karaca, *f-biharmonic integral submanifolds in generalized Sasakian space forms*, Filomat **33** (2019), no. 9, 2561–2570.

[12] F. Karaca, *f-biminimal submanifolds of generalized space forms*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat. **68** (2019), no. 2, 1301–1315. https://doi.org/10.31801/cfsuasmas.524498

[13] F. Karaca and C. Özgür, *f-biharmonic and bi-f-harmonic submanifolds of product spaces*, Sarajevo J. Math. **13(25)** (2017), no. 1, 115–129. https://doi.org/10.5644/sjm

[14] A. Lotta, *Slant submanifolds in contact geometry*, Bulletin mathématique de la Société des Sciences Mathé matiques de Roumanie (1996), 183–198.

[15] W. Lu, *On f-bi-harmonic maps and bi-f-harmonic maps between Riemannian manifolds*, Sci. China Math. **58** (2015), no. 7, 1483–1498. https://doi.org/10.1007/s11425-015-4997-1

[16] Y. Luo and Y.-L. Ou, *Some remarks on bi-f-harmonic maps and f-biharmonic maps*, Results Math. **74** (2019), no. 3, Paper No. 97, 19 pp. https://doi.org/10.1007/s00025-019-1023-x

[17] K. Matsumoto, I. Mihai, and R. Roşca, *A certain locally conformal almost cosymplectic manifold and its submanifolds*, Tensor (N.S.) **51** (1992), no. 1, 91–102.

[18] Z. Olszak, *Locally conformal almost cosymplectic manifolds*, Colloq. Math. **57** (1989), no. 1, 73–87. https://doi.org/10.4064/cm-57-1-73-87

[19] Y.-L. Ou, *On f-biharmonic maps and f-biharmonic submanifolds*, Pacific J. Math. **271** (2014), no. 2, 461–477. https://doi.org/10.2140/pjm.2014.271.461

[20] Y.-L. Ou, *f-biharmonic maps and f-biharmonic submanifolds II*, J. Math. Anal. Appl. **455** (2017), no. 2, 1285–1296. https://doi.org/10.1016/j.jmaa.2017.06.033

[21] J. Roth and A. Upadhyay, *f-biharmonic submanifolds of generalized space forms*, Results Math. **75** (2020), no. 1, Paper No. 20, 25 pp. https://doi.org/10.1007/s00025-019-1142-4

[22] M. M. Tripathi, *Almost semi-invariant submanifolds of trans-Sasakian manifolds*, J. Indian Math. Soc. (N.S.) **62** (1996), no. 1-4, 225–245.

[23] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, 3, World Scientific Publishing Co., Singapore, 1984.

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