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GLOBAL ATTRACTOR FOR A SEMILINEAR STRONGLY DEGENERATE PARABOLIC EQUATION WITH EXPONENTIAL NONLINEARITY IN UNBOUNDED DOMAINS

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Abstract. We study the existence and long-time behavior of weak solutions to a class of strongly degenerate semilinear parabolic equations with exponential nonlinearities on $\mathbb{R}^N$. To overcome some significant difficulty caused by the lack of compactness of the embeddings, the existence of a global attractor is proved by combining the tail estimates method and the asymptotic *a priori* estimate method.

1. Introduction

In this paper we consider the following semilinear strongly degenerate parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - P_{\alpha,\beta}u + f(X,u) + \lambda u = g(X), & X=(x,y,z)\in\mathbb{R}^N,\ t>0, \\ u(X,0)=u_0(X), & X\in\mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\lambda > 0, \mathbb{R}^N = \mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\mathbb{R}^{N_3} (N_1, N_2, N_3 \geq 1)$, and $P_{\alpha,\beta}$ is a strongly degenerate operator of the form

$$P_{\alpha,\beta}u = \Delta_x u + \Delta_y u + |x|^{2\alpha}|y|^{2\beta}\Delta_z u, \quad \alpha,\beta\geq 0.$$

This operator is degenerate on two intersecting surfaces $x = 0$ and $y = 0$, and considered by Thuy and Tri [19]. It turns to be that this operator falls into the class of Δ_λ-Laplace operators [10].

The existence and asymptotic behavior of solutions to semilinear parabolic equations involving this strongly degenerate operator have been addressed by a number of authors in the last few years. One way to study the long-time behavior of solutions is to analyze the existence and properties of a global attractor for the continuous semigroup generated by solutions. Up to now, there

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are two main kinds of nonlinearities that have been considered in literatures. The first one is the class of nonlinearities that is locally Lipschitzian continuous and satisfies a Sobolev growth condition

$$|f(u) - f(v)| \leq C(1 + |u|^p + |v|^p)|u - v|, \ \ 0 < p \leq 4/(N_{\alpha,\beta} - 2),$$

where $N_{\alpha,\beta} = N_1 + N_2 + (\alpha+\beta+1)N_3$, and some suitable dissipative conditions. The second one is the class of nonlinearities that satisfies a polynomial growth

$$C_1|u|^p - C_0 \leq f(u)u \leq C_2|u|^p + C_0, \text{ for some } p \geq 2,$$

$$f'(u) \geq -\ell.$$

Under above types of nonlinearities, following closely the approach used in [4], Thuy and Tri [20] proved the existence of solutions and of a global attractor for the semigroup generated by problem (1.1) in bounded domains with homogeneous Dirichlet boundary conditions. The regularity of the global attractor obtained in [20] was investigated in [18]. The results in [18, 20] were extended to the case of unbounded domains in [1, 5], the more delicate case due to the lack of compactness of Sobolev embeddings. We also refer the interested reader to [2, 3, 6, 8, 11–15, 21] for some other related results. Note that in these papers, some restriction on the growth of the nonlinearity is imposed and an exponential nonlinearity, for example $f(u) = e^u$, does not hold.

In this paper we try to remove this restriction and we were able to prove the existence of weak solutions and existence of a global attractor for a very large class of nonlinearities and in the case of unbounded domains. This is the main novelty of our paper.

To study the problem (1.1) we assume that the initial datum $u_0 \in L^2(\mathbb{R}^N)$ is given, the nonlinearity f and the external force g satisfy the following conditions:

(F) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$f'_u(X, u) \geq -\ell, \tag{1.2}$$

$$f(X, u)u \geq -\mu u^2 - C_1(X), \tag{1.3}$$

where $\ell > 0$, $0 < \mu < \lambda$, $C_1(\cdot) \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is a nonnegative function;

(G) $g \in L^2(\mathbb{R}^N)$.

It follows from (1.2) that $0 \leq \int_0^u (f'_u(X, s)s + \ell s)ds$, and therefore by integrating by parts, we obtain

$$F(X, u) \leq f(X, u)u + \ell\frac{u^2}{2} \quad \text{for all } u \in \mathbb{R}, \tag{1.4}$$

where $F(X, u) = \int_0^u f(X, s)ds$ is a primitive of f.

To study the problem (1.1) we use the weighted Sobolev space $S^1(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ in the norm

$$\begin{aligned}\|u\|^2_{S^1(\mathbb{R}^N)} &:= \int_{\mathbb{R}^N}(|u|^2+|\nabla_x u|^2+|\nabla_y u|^2+|x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2)dX\\ &= \int_{\mathbb{R}^N}(|u|^2+|\nabla_{\alpha,\beta} u|^2)dX,\end{aligned}$$

with $\nabla_{\alpha,\beta}u := (\nabla_x u, \nabla_y u, |x|^\alpha|y|^\beta\nabla_z u)$.

This is a Hilbert space with respect to the following scalar product

$$((u,v))_{S^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N}(uv+\nabla_x u\cdot\nabla_x v+\nabla_y u\cdot\nabla_y v+|x|^{2\alpha}|y|^{2\beta}\nabla_z u\cdot\nabla_z v)dX.$$

We also use the space $S^2(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|^2_{S^2(\mathbb{R}^N)} := \int_{\mathbb{R}^N}(|u|^2+|P_{\alpha,\beta}u|^2)dX.$$

In a similar way, we also define the spaces $S_0^1(\Omega)$ and $S^2(\Omega)$ for a bounded domain Ω in $\mathbb{R}^N$.

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of global weak solutions to the problem (1.1). In Section 3, we show the existence of global attractors in various function spaces for the associated continuous semigroup by exploiting and combining the tail estimates method and the asymptotic *a priori* estimate method.

2. Existence and uniqueness of weak solutions

Definition. A function u is called a weak solution of the problem (1.1) on the interval $(0,T)$ if $u \in C([0,T]; L^2(\mathbb{R}^N)) \cap L^2(0,T; S^1(\mathbb{R}^N)), u(0) = u_0$, and

$$\langle u_t, w\rangle - \langle P_{\alpha,\beta}u, w\rangle + \langle f(X,u), w\rangle + \lambda\langle u, w\rangle = \langle g, w\rangle$$

for all test functions $w \in S^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and for a.e. $t\in(0,T)$.

Theorem 2.1. *Assume* (F) and (G) *hold. Then for any* $u_0 \in L^2(\mathbb{R}^N)$ *and* $T > 0$ *given, the problem* (1.1) *has a unique weak solution* u *on the interval* $(0,T)$. *Moreover, the mapping* $u_0 \mapsto u(t)$ *is continuous on* $L^2(\mathbb{R}^N)$.

Proof. (i) Existence. We consider a sequence of problems in bounded domains

$$\begin{cases}\dfrac{\partial u}{\partial t} - P_{\alpha,\beta}u + f(X,u) + \lambda u = g(X), & X=(x,y,z)\in B_R,\ t>0,\\ u(X,t)=0, & X\in\partial B_R,\ t>0,\\ u(X,0)=u_{0,R}(X), & X\in B_R,\end{cases} \tag{2.1}$$

where B_R is the open ball of radius $R \geq 1$ centered at 0, $u_{0,R} = u_0\psi_R(|X|)$ and ψ_R is a smooth function such that

$$\psi_R(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq R-1, \\ 0 \leq \psi_R(r) \leq 1 & \text{if } R-1 \leq r \leq R, \\ 0 & \text{if } r > R. \end{cases}$$

It was proved in [16] that for each $R \geq 1$, the problem (2.1) has a unique weak solution u_R. We will show that $\{u_R\}$ is uniformly bounded by a constant independent of R. We have

$$\frac{1}{2}\frac{d}{dt}\|u_R\|^2_{L^2(B_R)} + \int_{B_R} |\nabla_{\alpha,\beta} u_R|^2 dX + \int_{B_R} f(X, u_R)u_R dX + \lambda\|u_R\|^2_{L^2(B_R)}$$
$$= \int_{B_R} g u_R dX.$$

By (1.3), we have

$$\frac{1}{2}\frac{d}{dt}\|u_R\|^2_{L^2(B_R)} + \int_{B_R} |\nabla_{\alpha,\beta} u_R|^2 dX + (\lambda-\mu)\|u_R\|^2_{L^2(B_R)}$$
$$\leq \int_{B_R} g u_R dX + \int_{B_R} C_1(X) dX.$$

Using the Cauchy inequality and the assumption $C_1(\cdot) \in L^1(\mathbb{R}^N)$, we get

$$\frac{1}{2}\frac{d}{dt}\|u_R\|^2_{L^2(B_R)} + \int_{B_R} |\nabla_{\alpha,\beta} u_R|^2 dX + (\lambda-\mu)\|u_R\|^2_{L^2(B_R)}$$
$$\leq \frac{1}{2(\lambda-\mu)}\|g\|_{L^2(B_R)} + \frac{\lambda-\mu}{2}\|u_R\|^2_{L^2(B_R)} + C.$$

Therefore,

$$\frac{d}{dt}\|u_R\|^2_{L^2(B_R)} + 2\int_{B_R} |\nabla_{\alpha,\beta} u_R|^2 dX + (\lambda-\mu)\|u_R\|^2_{L^2(B_R)}$$
$$\leq \frac{1}{\lambda-\mu}\|g\|_{L^2(\mathbb{R}^N)} + C.$$

Integrating from 0 to t, $0 \leq t \leq T$, we get

$$\|u_R(t)\|^2_{L^2(B_R)} + 2\int_0^t \int_{B_R} |\nabla_{\alpha,\beta} u_R|^2 dX ds$$
$$+ (\lambda-\mu)\int_0^t \|u_R(s)\|^2_{L^2(B_R)} ds$$
$$\leq \frac{1}{\lambda-\mu}\|g(X)\|_{L^2(\mathbb{R}^N)} T + CT + \|u_0\psi_R(|X|)\|^2_{L^2(B_R)}. \tag{2.2}$$

Let $u_{r_j}, r_j \to +\infty$, be a sequence of solutions to the problem (2.1) in B_{r_j}. Then, by (2.2) it follows that

(2.3) $\{u_{r_j}\}$ is uniformly bounded in $L^\infty(0, T; L^2(B_{r_j})) \cap L^2(0, T; S^1(B_{r_j}))$.

We extend these solutions to be defined on $\mathbb{R}^N$ in the following way

$$\hat{u}_{r_j}(X) = \begin{cases} u_{r_j}(X)\psi_{r_j}(|X|) & \text{in } B_{r_j}, \\ 0 & \text{otherwise.} \end{cases}$$

By (2.3), $\{\hat{u}_{r_j}\}$ is a bounded sequence in

$$L^\infty(0,T;L^2(\mathbb{R}^N)) \cap L^2(0,T;S^1(\mathbb{R}^N)).$$

Hence, there exists a subsequence of $\{\hat{u}_{r_j}\}$ (denoted again by $\hat{u}_{r_j}$) such that

$$\begin{aligned} \hat{u}_{r_j} &\rightharpoonup u_\infty \text{ in } L^2(0,T;S^1(\mathbb{R}^N)), \\ \hat{u}_{r_j} &\overset{*}{\rightharpoonup} u_\infty \text{ in } L^\infty(0,T;L^2(\mathbb{R}^N)), \\ P_{\alpha,\beta}\hat{u}_{r_j} &\rightharpoonup P_{\alpha,\beta}u_\infty \text{ in } L^2(0,T;S^{-1}(\mathbb{R}^N)). \end{aligned} \tag{2.4}$$

We will prove that u_∞ is a weak solution of the problem (1.1).

Let r_k be fixed. Since $r_j \to +\infty$, we can assume $r_k \le r_j - 1$. We define the projections in B_{r_k} of $\hat{u}_{r_j}$ and denote them by $u_{kj} = L_k\hat{u}_{r_j}$. It is clear from (2.3) that $\{u_{kj}\}$ is bounded in $L^\infty(0,T;L^2(B_{r_k})) \cap L^2(0,T;S^1(B_{r_k}))$. It follows that there exists a subsequence (denoted again by u_{kj}) such that $u_{kj} = L_k\hat{u}_{r_j} \rightharpoonup u_{k\infty}$ in $L^2(0,T;S^1(B_{r_k}))$ and weakly-$*$ in $L^\infty(0,T;L^2(B_{r_k}))$. We now check that $L_k u_\infty = u_{k\infty}$. Indeed, letting $v \in C_0^\infty([0,T] \times B_{r_k})$, the weak-$*$ convergence in $L^\infty(0,T;L^2(B_{r_k}))$ gives

$$\int_0^T \int_{B_{r_k}} L_k\hat{u}_{r_j} v dX dt \to \int_0^T \int_{B_{r_k}} u_{k\infty} v dX dt.$$

On the other hand, noting that $v(t,X) = 0$ if $X \notin B_{r_k}$ and using (2.4) we have

$$\int_0^T \int_{B_{r_k}} L_k\hat{u}_{r_j} v dX dt \to \int_0^T \int_{B_{r_k}} \hat{u}_{r_j} v dX dt \to \int_0^T \int_{B_{r_k}} u_\infty v dX dt,$$

and

$$\int_0^T \int_{B_{r_k}} u_\infty v dX dt = \int_0^T \int_{B_{r_k}} L_k u_\infty v dX dt,$$

so that $L_k u_\infty = u_{k\infty}$. We claim that $L_k u_\infty$ is a weak solution in $Q_{r_k,T} = [0,T] \times B_{r_k}$. We have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_{kj}\|^2_{L^2(B_{r_k})} + \int_{B_{r_k}} |\nabla_{\alpha,\beta}u_{kj}|^2 dX + \int_{B_{r_k}} f(X,u_{kj})u_{kj} dX \\ &+ \lambda\|u_{kj}\|^2_{L^2(B_{r_k})} = \int_{B_{r_k}} g u_{kj} dX. \end{aligned} \tag{2.5}$$

Integrating (2.5) from 0 to T, we have

$$2\int_0^T \int_{B_{r_k}} |\nabla_{\alpha,\beta}u_{kj}|^2 dX dt + 2\int_{Q_{r_k,T}} f(X,u_{kj})u_{kj} dX dt + \lambda\int_0^T \|u_{kj}\|^2_{L^2(B_{r_k})} dt$$
$$\le \|u_{0,r_k}\|^2_{L^2(B_{r_k})} + \frac{1}{\lambda}\|g\|^2_{L^2(B_{r_k})}T.$$

Hence

$$\int_{Q_{r_k,T}} f(X,u_{kj})u_{kj}dXdt \leq C.$$

We now prove that $\{f(X,u_{kj})\}$ is bounded in $L^1(Q_{r_k,T})$. Putting $h(u_{kj}) = f(X,u_{kj}) + \bar{\mu}u_{kj}$, where $\bar{\mu} > \ell$. Note that $h(s)s \geq 0$ for all $s \in \mathbb{R}$, we have

$$\begin{aligned}\int_{Q_{r_k,T}} |h(u_{kj})|dXdt &\leq \int_{Q_{r_k,T}\cap\{|u_{kj}|>1\}} |h(u_{kj})u_{kj}|dXdt \\ &\quad + \int_{Q_{r_k,T}\cap\{|u_{kj}|\leq 1\}} |h(u_{kj})|dXdt \\ &\leq \int_{Q_{r_k,T}} h(u_{kj})u_{kj}dXdt + \sup_{|s|\leq 1}|h(s)||Q_{r_k,T}| \leq C.\end{aligned}$$

Hence it implies that $\{h(u_{kj})\}$, and therefore $\{f(X,u_{kj})\}$ is bounded in $L^1(Q_{r_k,T})$. Since

$$\frac{\partial u_{kj}}{\partial t} = P_{\alpha,\beta}u_{kj} - f(X,u_{kj}) - \lambda u_{kj} + g,$$

we deduce that $\{\frac{\partial u_{kj}}{\partial t}\}$ is bounded in $L^2(0,T;S^{-1}(B_{kj})) + L^1(Q_{r_k,T})$, and therefore in $L^1(0,T;S^{-1}(B_{kj}) + L^1(B_{kj}))$. Because $S^1_0(B_{kj}) \subset\subset L^2(B_{kj}) \subset S^{-1}(B_{kj}) + L^1(B_{kj})$, by the Aubin-Lions-Simon compactness lemma (see e.g. [7], Theorem II.5.16, p. 102), we have that $\{u_{kj}\}$ is compact in $L^2(0,T;L^2(B_{kj}))$. Hence we may assume, up to a subsequence, that we have $u_{kj} \to u_{k\infty}$ a.e. in $Q_{r_k,T}$ and then,

$$\int_{Q_{r_k,T}} f(X,u_{kj})\xi dXdt \to \int_{Q_{r_k,T}} f(X,u_{k\infty})\xi dXdt$$

for all $\xi \in C_0^\infty([0,T];S^{-1}(B_{kj}) \cap L^\infty(B_{kj}))$. Hence, we obtain that $u_{k\infty}$ is a weak solution in $[0,T]\times B_{r_k}$. Hence we get that u_∞ is a weak solution of the problem (1.1). Indeed, for any test function $v \in C_0^\infty(\mathbb{R}^{\mathbb{N}})$, there exists r_k such that $v \in C_0^\infty(B_{r_k})$. Using $u_{k\infty}$ solving (1.1) in $Q_{r_k,T}$, we can conclude that u_∞ is a weak solution of (1.1) in $[0,T]\times\mathbb{R}^N$.

(ii) Uniqueness and continuous dependence on the initial data. Let u and v be two weak solutions of (1.1) with initial data $u_0, v_0 \in L^2(\mathbb{R}^N)$, respectively. Putting $w = u - v$, we have

$$\begin{cases} \dfrac{\partial w}{\partial t} - P_{\alpha,\beta}w + \widetilde{f}(X,u) - \widetilde{f}(X,v) - \ell w + \lambda w &= 0, \\ \qquad w(0) &= u_0 - v_0, \end{cases} \tag{2.6}$$

where $\widetilde{f}(X,s) = f(X,s) + \ell s$. Here, because $w(t)$ does not belong to $W := S^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we cannot choose $w(t)$ as a test function as in [4]. Consequently, the proof will be more involved.

We use some ideas in [9]. Let $B_k : \mathbb{R} \to \mathbb{R}$ be the truncated function

$$B_k(s) = \begin{cases} k & \text{if } s > k, \\ s & \text{if } |s| \le k, \\ -k & \text{if } s < -k. \end{cases}$$

Consider the corresponding Nemytskii mapping $\widehat{B}_k : W \to W$ defined as follows

$$\widehat{B}_k(w)(x) = B_k(w(x)) \quad \text{for all } x \in \mathbb{R}^N.$$

By Lemma 2.3 in [9], we have that $\|\widehat{B}_k(w) - w\|_W \to 0$ as $k \to \infty$. Now multiplying the first equation in (2.6) by $\widehat{B}_k(w)$, then integrating over $\mathbb{R}^N \times (\varepsilon, t)$, where $t \in (0, T)$, we get

$$\begin{aligned} &\int_\varepsilon^t \int_{\mathbb{R}^N} \frac{d}{ds}\Big(w(s)\widehat{B}_k(w)(s)\Big) dxds - \int_\varepsilon^t \int_{\mathbb{R}^N} w \frac{d}{ds}\Big(\widehat{B}_k(w)(s)\Big) dxds \\ &+ \frac{1}{2}\int_\varepsilon^t \int_{\{x\in\mathbb{R}^N : |w(x,s)| \le k\}} |\nabla_{\alpha,\beta} w|^2 dxds \\ &+ \int_\varepsilon^t \int_{\mathbb{R}^N} \Big(\widetilde{f}(X,u) - \widetilde{f}(X,v)\Big)\widehat{B}_k(w) dxds \\ &- \ell \int_\varepsilon^t \int_{\mathbb{R}^N} w\widehat{B}_k(w) dxds + \lambda \int_\varepsilon^t \int_{\mathbb{R}^N} w\widehat{B}_k(w) dxds = 0. \end{aligned}$$

Nothing that $w\frac{d}{dt}(\widehat{B}_k(w)) = \frac{1}{2}\frac{d}{dt}(\widehat{B}_k(w))^2$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} w(t)\widehat{B}_k(w)(t) dx - \frac{1}{2}\|\widehat{B}_k(w)(t)\|^2_{L^2(\mathbb{R}^N)} \\ &+ \frac{1}{2}\int_\varepsilon^t \int_{\{x\in\mathbb{R}^N : |w(x,s)| \le k\}} |\nabla_{\alpha,\beta} w|^2 dxds + \int_\varepsilon^t \int_{\mathbb{R}^N} \widetilde{f}'(X,\xi) w\widehat{B}_k(w) dxds \\ = &\int_{\mathbb{R}^N} w(\varepsilon)\widehat{B}_k(w)(\varepsilon) dx - \frac{1}{2}\|\widehat{B}_k(w)(\varepsilon)\|^2_{L^2(\mathbb{R}^N)} + (\ell - \lambda)\int_\varepsilon^t \int_{\mathbb{R}^N} w\widehat{B}_k(w) dxds. \end{aligned}$$

Note that $\widetilde{f}'(X, s) \ge 0$ and $sB_k(s) \ge 0$ for all $s \in \mathbb{R}$, by letting $\varepsilon \to 0$ and $k \to \infty$ in the above equality, we obtain

$$\|w(t)\|^2_{L^2(\mathbb{R}^N)} \le \|w(0)\|^2_{L^2(\mathbb{R}^N)} + (2\ell - \lambda)\int_0^t \|w(s)\|^2_{L^2(\mathbb{R}^N)} ds.$$

Hence, by the Gronwall inequality of integral form, we get

$$\begin{aligned} \|w(t)\|^2_{L^2(\mathbb{R}^N)} &\le \|w(0)\|^2_{L^2(\mathbb{R}^N)} e^{(2\ell-\lambda)t} \\ &\le \|w(0)\|^2_{L^2(\mathbb{R}^N)} e^{(2\ell-\lambda)T} \text{ for all } t \in [0, T]. \end{aligned}$$

Hence, we get the continuous dependence on the initial data of the solutions, and in particular, the uniqueness when $u_0 = v_0$. □

3. Existence of global attractors

By Theorem 2.1, we can define a continuous semigroup $S(t) : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ associated to the problem (1.1) as follows

$$S(t)u_0 := u(t),$$

where $u(\cdot)$ is the unique global weak solution of (1.1) with the initial datum u_0. We will prove that the semigroup $S(t)$ has a global attractor $\mathcal{A}$ in the spaces $L^2(\mathbb{R}^N)$ and $S^1(\mathbb{R}^N)$.

3.1. Existence of bounded absorbing sets

For brevity, in the following lemmas, we give some formal calculation. The rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [17].

Lemma 3.1. *The semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $L^2(\mathbb{R}^N)$.*

Proof. Multiplying the first equation in (1.1) by u, we have

$$\begin{aligned}(3.1)\qquad &\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |\nabla_{\alpha,\beta}u|^2 dX + \int_{\mathbb{R}^N} f(X,u)udX + \lambda\|u\|^2_{L^2(\mathbb{R}^N)} \\ &= \int_{\mathbb{R}^N} gudX.\end{aligned}$$

By (1.3), we have

$$\begin{aligned}(3.2)\qquad &\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} |\nabla_{\alpha,\beta}u|^2 dX + 2(\lambda-\mu)\|u\|^2_{L^2(\mathbb{R}^N)} \\ &\leq 2\int_{\mathbb{R}^N} gudX + C \leq (\lambda-\mu)\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{\lambda-\mu}\|g\|^2_{L^2(\mathbb{R}^N)} + C.\end{aligned}$$

Thus, we have

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + (\lambda-\mu)\|u\|^2_{L^2(\mathbb{R}^N)} \leq \frac{1}{\lambda-\mu}\|g\|^2_{L^2(\mathbb{R}^N)} + C.$$

Hence, thanks to the Gronwall inequality, we obtain

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)} \leq \|u(0)\|^2_{L^2(\mathbb{R}^N)} e^{-(\lambda-\mu)t} + R_1,$$

where $R_1 = R_1(\lambda, \mu, C, \|g\|^2_{L^2(\mathbb{R}^N)})$. Hence, if choosing $\rho_1 = 2R_1$, we have

$$(3.3)\qquad \|u(t)\|^2_{L^2(\mathbb{R}^N)} \leq \rho_1 \quad \text{for all} \quad t \geq T_1 = T(B).$$

This completes the proof. □

Lemma 3.2. *The semigroup $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set in $S^1(\mathbb{R}^N)$.*

Proof. By (3.2), we have

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} |\nabla_{\alpha,\beta}u|^2 dX \leq \frac{1}{\lambda-\mu}\|g\|^2_{L^2(\mathbb{R}^N)} + C.$$

Integrating on $(t, t+1)$ and by Lemma 3.1, we have

$$\int_t^{t+1} \|\nabla_{\alpha,\beta} u(s)\|^2_{L^2(\mathbb{R}^N)} ds \le \rho_2 = \rho_2(C, \rho_1, \|g\|^2_{L^2(\mathbb{R}^N)}) \text{ for all } t \ge T_1. \tag{3.4}$$

Multiplying (1.1) by $-P_{\alpha,\beta}u$ and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)} + \|P_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)} \\
&= -\int_{\mathbb{R}^N} f'(X,u)|\nabla_{\alpha,\beta}u|^2 dX - \lambda \int_{\mathbb{R}^N} |\nabla_{\alpha,\beta}u|^2 dX - \int_{\mathbb{R}^N} g P_{\alpha,\beta} u dX \\
&\le \ell \int_{\mathbb{R}^N} |\nabla_{\alpha,\beta}u|^2 dX - \lambda \int_{\mathbb{R}^N} |\nabla_{\alpha,\beta}u|^2 dX + \frac{1}{2}\|P_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\|g\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

Hence, we have

$$\frac{d}{dt}\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)} \le 4\ell\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)} + \|g\|^2_{L^2(\mathbb{R}^N)}. \tag{3.5}$$

Combining (3.4)-(3.5) and using the uniform Gronwall inequality, we have

$$\|\nabla_{\alpha,\beta}u(t)\|^2_{L^2(\mathbb{R}^N)} \le \rho_2 \text{ for all } t \ge T_2 = T_1 + 1. \tag{3.6}$$

By (3.3) and (3.6), we finish the proof. □

Lemma 3.3. *Suppose* (F) and (G) *hold. Then for every bounded subset B in $L^2(\mathbb{R}^N)$, there exists a constant $T = T(B) > 0$ such that*

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} \le \rho_3 \quad \textit{for all} \quad u_0 \in B, \ \textit{and}\ s \ge T,$$

where $u_t(s) = \frac{d}{dt}(S(t)u_0)|_{t=s}$ and ρ_3 is a positive constant independent of B.

Proof. By differentiating (1.1) in time, we get

$$u_{tt} - P_{\alpha,\beta}u_t + f'_u(X,u)u_t + \lambda u_t = 0.$$

Taking the inner product of this equality with u_t in $L^2(\mathbb{R}^N)$ and using (1.2), in particular, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2(\mathbb{R}^N)} \le C\|u_t\|^2_{L^2(\mathbb{R}^N)}. \tag{3.7}$$

Multiplying the first equation in (1.1) by u_t, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} F(X,u)dX - \int_{\mathbb{R}^N} gudX\right) \\
&= -\|u_t\|^2_{L^2(\mathbb{R}^N)} \le 0.
\end{aligned} \tag{3.8}$$

On the other hand, integrating (3.1) from t to $t+1$ and using (3.3), we have

$$\begin{aligned}
&\int_t^{t+1} \left[\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)} + \lambda\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} f(X,u)udX - \int_{\mathbb{R}^N} gudX\right] ds \\
&\le \|u(t)\|^2_{L^2(\mathbb{R}^N)} \le \rho_1 \quad \text{for all } t \ge T_1.
\end{aligned}$$

Using the inequality (1.4), we deduce that

$$\begin{aligned}
&\int_t^{t+1}\Big[\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)}+\lambda\|u\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}f(X,u)udX-\int_{\mathbb{R}^N}gudX\Big]ds\\
\geq\ &\int_t^{t+1}\Big[\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)}+\lambda\|u\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}F(X,u)dX\\
&\qquad-\frac{\ell}{2}\|u\|^2_{L^2(\mathbb{R}^N)}-\int_{\mathbb{R}^N}gudX\Big]ds\\
\geq\ &\int_t^{t+1}\Big[\frac{1}{2}\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)}+\frac{\lambda}{2}\|u\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}F(X,u)dX\\
&\qquad-\int_{\mathbb{R}^N}gudX\Big]ds-\frac{\ell}{2}\rho_1\ \text{ for all } t\geq T_1,
\end{aligned}$$

where we have used the inequality (3.3). Hence,

$$\begin{aligned}
&\int_t^{t+1}\Big[\frac{1}{2}\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)}+\frac{\lambda}{2}\|u\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}F(X,u)dX\\
&\qquad-\int_{\mathbb{R}^N}gudX\Big]ds\leq(1+\frac{\ell}{2})\rho_1\ \text{ for all } t\geq T_1.
\end{aligned}\tag{3.9}$$

By the uniform Gronwall inequality, from (3.8) and (3.9), we deduce that

$$\frac{1}{2}\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)}+\frac{\lambda}{2}\|u\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}F(X,u)dX-\int_{\mathbb{R}^N}gudX\leq\rho_3\tag{3.10}$$

for all $t\geq T_2=T_1+1$. Integrating (3.8) from t to $t+1$ and using (3.10), we obtain

$$\int_t^{t+1}\|u_t(s)\|^2_{L^2(\mathbb{R}^N)}ds\leq\rho_3\ \text{ for all } t\geq T_2.\tag{3.11}$$

Combining (3.7) with (3.11) and using the uniform Gronwall inequality, we have

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)}\leq\rho_3\ \text{ for all } s\geq T_3=T_2+1.$$

This completes the proof. □

We now show the existence of a bounded absorbing set in $S^2(\mathbb{R}^N)$.

Lemma 3.4. *The semigroup $\{S(t)\}_{t\geq0}$ has a bounded absorbing set in $S^2(\mathbb{R}^N)$, i.e., there exists a constant $\rho_4>0$ such that for any bounded subset $B\subset L^2(\mathbb{R}^N)$, there is a $T_B>0$ such that*

$$\|P_{\alpha,\beta}u(t)\|^2_{L^2(\mathbb{R}^N)}+\|u(t)\|^2_{L^2(\mathbb{R}^N)}\leq\rho_4\ \text{ for any } t\geq T_B, u_0\in B.$$

Proof. Taking the L^2-inner product of (1.1) with $-P_{\alpha,\beta}u+\lambda u$, we have

$$\|P_{\alpha,\beta}u(t)\|^2_{L^2(\mathbb{R}^N)}+\lambda^2\|u(t)\|^2_{L^2(\mathbb{R}^N)}+\lambda\int_{\mathbb{R}^N}f(X,u)udX$$

$$\leq 2\lambda\int_{\mathbb{R}^N} uP_{\alpha,\beta}udX - \int_{\mathbb{R}^N} u_t(-P_{\alpha,\beta}u+\lambda u)dX + \int_{\mathbb{R}^N} f(X,u)P_{\alpha,\beta}udX + \int_{\mathbb{R}^N} g(-P_{\alpha,\beta}u+\lambda u)dX.$$

Using (1.3) and integrating by parts the third term on the right-hand side, we have

$$\begin{aligned}&\|P_{\alpha,\beta}u(t)\|^2_{L^2(\mathbb{R}^N)} + \lambda^2\|u(t)\|^2_{L^2(\mathbb{R}^N)} - \lambda\mu\|u(t)\|^2_{L^2(\mathbb{R}^N)}\\ &\leq 2\lambda\int_{\mathbb{R}^N} uP_{\alpha,\beta}udX - \int_{\mathbb{R}^N} u_t(-P_{\alpha,\beta}u+\lambda u)dX\\ &\quad - \int_{\mathbb{R}^N} f'_u(X,u)\big(|\nabla_x u|^2 + |\nabla_y u|^2 + |x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2\big)dX\\ &\quad + \int_{\mathbb{R}^N} g(-P_{\alpha,\beta}u+\lambda u)dX + \lambda\int_{\mathbb{R}^N} C_1(X)dX.\end{aligned}$$

By the Cauchy inequality and assumption (1.2), we have

$$\|P_{\alpha,\beta}u(t)\|^2_{L^2(\mathbb{R}^N)} + \|u(t)\|^2_{L^2(\mathbb{R}^N)} \leq C(1+\|u_t\|^2_{L^2(\mathbb{R}^N)} + \|u\|^2_{S^1(\mathbb{R}^N)} + \|g\|^2_{L^2(\mathbb{R}^N)}).$$

By Lemmas 3.1-3.3, there exists $\rho_4 > 0$ such that

$$\|P_{\alpha,\beta}u(t)\|^2_{L^2(\mathbb{R}^N)} + \|u(t)\|^2_{L^2(\mathbb{R}^N)} \leq \rho_4$$

for all t large enough. This completes the proof. □

Next, to prove the existence of a global attractor in $L^2(\mathbb{R}^N)$ and $S^1(\mathbb{R}^N)$. We will consider three functions $\varphi_R, \theta_R, \gamma_R$ such that

$$\varphi_R = \varphi(\frac{|x|^2}{R^2}),\ \theta_R = \theta(\frac{|y|^2}{R^2}),\ \gamma_R = \gamma(\frac{|z|^2}{R^{2(1+\alpha+\beta)}})$$

with $\varphi, \theta, \gamma \in C^\infty[0,+\infty)$,

$$0 \leq \varphi,\theta,\gamma \leq 1,\ \varphi,\theta,\gamma = 0 \text{ in } [0,\frac{1}{2}],\ \varphi,\theta,\gamma = 1 \text{ in } [1,+\infty).$$

Then there exists a constant $C > 0$ such that $|\varphi'(\cdot)|, |\theta'(\cdot)|, |\gamma'(\cdot)| \leq C$.

Moreover, letting

$$B^*_R = B_{\mathbb{R}^{N_1}}(0,R) \times B_{\mathbb{R}^{N_2}}(0,R) \times B_{\mathbb{R}^{N_3}}(0,R^{1+\alpha+\beta})$$

and

$$\Sigma_R = \mathbb{R}^N \backslash \big(B_{\mathbb{R}^{N_1}}(0,R/2) \times B_{\mathbb{R}^{N_2}}(0,R/2) \times B_{\mathbb{R}^{N_3}}(0,R^{1+\alpha+\beta}/2)\big).$$

3.2. Existence of a global attractor in $L^2(\mathbb{R}^N)$

Lemma 3.5. *Suppose* (F) *and* (G) *hold. Then for any $\epsilon > 0$ and any bounded subset $B \subset L^2(\mathbb{R}^N)$, there exist $T = T(\epsilon, B) > 0$ and $K = K(\epsilon, B) > 0$ such that for all $t \geq T$ and $R \geq K$,*

$$\int_{\mathbb{R}^N\backslash B^*_R} |u(X,t)|^2 dX \leq \epsilon.$$

Proof. Taking the inner product of (1.1) with $(\varphi_R\theta_R\gamma_R)u$ in $L^2(\mathbb{R}^N)$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|u|^2dX - \int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)uP_{\alpha,\beta}udX$$
$$+\lambda\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|u|^2dX + \int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)f(X,u)udX = \int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)ugdX.$$

Using (1.2) and (1.3), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|u|^2dX + (\lambda-\mu)\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|u|^2dX\\ &\le \int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)uP_{\alpha,\beta}udX + \int_{\sum_R}|C_1(X)|dX + \int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)ugdX.\end{aligned}\tag{3.12}$$

On the other hand, we have

$$\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)uP_{\alpha,\beta}udX \le \frac{1}{2}\int_{\mathbb{R}^N}(|u|^2+|P_{\alpha,\beta}u|^2)dX,\tag{3.13}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)ugdX &= \int_{\sum_R}(\varphi_R\theta_R\gamma_R)ugdX\\ &\le \frac{\lambda-\mu}{2}\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|u|^2dX + \frac{1}{2(\lambda-\mu)}\int_{\sum_R}|g|^2dX.\end{aligned}\tag{3.14}$$

It follows from (3.12)-(3.14) that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|u|^2dX + (\lambda-\mu)\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|u|^2dX\\ &\le 2\int_{\sum_R}|C_1(X)|dX + \frac{1}{\lambda-\mu}\int_{\sum_R}|g|^2dX + \int_{\mathbb{R}^N}(|u|^2+|P_{\alpha,\beta}u|^2)dX.\end{aligned}\tag{3.15}$$

Multiplying (3.15) by $e^{(\lambda-\mu)t}$ and then integrating over (T,t), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|u|^2dX\\ &\le e^{-(\lambda-\mu)t}\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|u(T)|^2dX\\ &\quad + 2e^{-(\lambda-\mu)t}\int_T^t e^{(\lambda-\mu)\xi}\int_{\sum_R}|C_1(X)|dXd\xi\\ &\quad + \frac{1}{\lambda-\mu}e^{-(\lambda-\mu)t}\int_T^t e^{(\lambda-\mu)\xi}\int_{\sum_R}|g|^2dXd\xi\\ &\quad + e^{-(\lambda-\mu)t}\int_T^t e^{(\lambda-\mu)\xi}\int_{\mathbb{R}^N}(|u|^2+|P_{\alpha,\beta}u|^2)dXd\xi\\ &\le e^{-(\lambda-\mu)t}\|u(T)\|^2_{L^2(\mathbb{R}^N)} + \frac{2}{\lambda-\mu}\int_{\sum_R}|C_1(X)|dX\end{aligned}$$

$$+\frac{1}{(\lambda-\mu)^2}\int_{\sum_R}|g|^2dX$$

$$+e^{-(\lambda-\mu)t}\int_T^t e^{(\lambda-\mu)\xi}\int_{\mathbb{R}^N}(|u|^2+|P_{\alpha,\beta}u|^2)dXd\xi. \tag{3.16}$$

Noting that for given $\epsilon>0$, there is $T_1=T_1(\epsilon)>0$ such that for all $t\geq T_1$,

$$e^{-(\lambda-\mu)t}\|u(T_1)\|^2_{L^2(\mathbb{R}^N)}\leq\frac{\epsilon}{4}. \tag{3.17}$$

Since $C_1(\cdot)\in L^1(\mathbb{R}^N)$, there exists $K_1=K_1(\epsilon)>0$ such that for all $R\geq K_1$,

$$\frac{2}{\lambda-\mu}\int_{\sum_R}|C_1(X)|dX\leq\frac{\epsilon}{4}. \tag{3.18}$$

On the other hand, since $g\in L^2(\mathbb{R}^N)$, there is $K_2=K_2(\epsilon)>K_1$ such that for all $R\geq K_2$,

$$\frac{1}{(\lambda-\mu)^2}\int_{\sum_R}|g|^2dX\leq\frac{\epsilon}{4}. \tag{3.19}$$

For the last term on the right-hand side of (3.16), it follows from Lemma 3.4 that there is $T_2>0$ such that for all $\xi\geq T_2$,

$$\int_{\mathbb{R}^N}(|u(X,\xi)|^2+|P_{\alpha,\beta}u(X,\xi)|^2)dX\leq\rho_4. \tag{3.20}$$

Therefore, there is $K_3=K_3(\epsilon)>K_2$ such that for all $R\geq K_3$ and $t\geq T_2$,

$$e^{-(\lambda-\mu)t}\int_{T_2}^t e^{(\lambda-\mu)\xi}(\int_{\mathbb{R}^N}(|u|^2+|P_{\alpha,\beta}u|^2)dX)d\xi\leq\frac{\epsilon}{4}. \tag{3.21}$$

Let $T=\max\{T_1,T_2\}$, then by (3.16)-(3.21) we find that for all $R\geq K\geq K_3$ and $t\geq T$,

$$\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|u(X,t)|^2dX\leq\epsilon,$$

and hence for all $R\geq K$ and $t\geq T$,

$$\int_{\mathbb{R}^N\backslash B_R^*}|u(X,t)|^2dX\leq\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|u(X,t)|^2dX\leq\epsilon,$$

which completes the proof. □

Now, we show the asymptotic compactness of $S(t)$ in $L^2(\mathbb{R}^N)$.

Lemma 3.6. *Suppose that* (F) *and* (G) *hold. Then $S(t)$ is asymptotically compact in $L^2(\mathbb{R}^N)$, that is, for any bounded sequence $\{x_n\}\subset L^2(\mathbb{R}^N)$ and any sequence $t_n\geq 0$, $t_n\to\infty$, $\{S(t_n)x_n\}$ has a convergent subsequence with respect to the topology of $L^2(\mathbb{R}^N)$.*

Proof. We use the uniform estimate on the tails of solutions to establish the precompactness of $\{u_n(t_n)\} := \{S(t_n)x_n\}$, that is, we prove that for every $\epsilon > 0$, the sequence $\{u_n(t_n)\}$ has a finite covering of balls of radii less than ϵ.

Given $K > 0$, denote

$$B_K^* = B_{\mathbb{R}^{N_1}}(0, K) \times B_{\mathbb{R}^{N_2}}(0, K) \times B_{\mathbb{R}^{N_3}}(0, K^{1+\alpha+\beta}) \text{ and } B_K^c = \mathbb{R}^N \backslash B_K^*.$$

Then, by Lemma 3.5, for the given $\epsilon > 0$, there exist $K = K(\epsilon) > 0$ and $T = T(\epsilon) > 0$ such that for $t \geq T$,

$$\|u_n(t)\|_{L^2(B_K^c)} \leq \epsilon.$$

Since $t_n \to \infty$, there is $N_1 = N_1(\epsilon) > 0$ such that $t_n \geq T$ for all $n \geq N_1$ and hence we obtain that, for all $n \geq N_1$,

$$\|u_n(t_n)\|_{L^2(B_K^c)} \leq \epsilon. \tag{3.22}$$

By Lemma 3.2, there exist $C > 0$ and $N_2 > 0$ such that for all $n \geq N_2$,

$$\|u_n(t_n)\|_{S^1(B_K^*)} \leq C.$$

Since compactness of the embedding $S^1(B_K^*) \hookrightarrow L^2(B_K^*)$ (see [19]), the sequence $\{u_n(t_n)\}$ is precompact in $L^2(B_K^*)$. Therefore, for the given $\epsilon > 0$, $\{u_n(t_n)\}$ has a finite covering in $L^2(B_K^*)$ of balls of radii less than ϵ, which along with (3.22) shows that $\{u_n(t_n)\}$ has a finite covering in $L^2(\mathbb{R}^N)$ of balls of radii less than ϵ, and thus $\{u_n(t_n)\}$ is precompact in $L^2(\mathbb{R}^N)$. □

We are now ready to prove the existence of a global attractor in $L^2(\mathbb{R}^N)$.

Theorem 3.7. *Suppose that* (F) *and* (G) *hold. Then the semigroup $S(t)$ generated by the problem* (1.1) *has a global attractor $\mathcal{A}_{L^2}$ in $L^2(\mathbb{R}^N)$.*

Proof. Denote

$$B = \left\{u : \|u\|_{L^2(\mathbb{R}^N)} \leq \rho_1\right\},$$

where ρ_1 is the positive constant in the proof of Lemma 3.1. Then B is a bounded absorbing set for $S(t)$ in $L^2(\mathbb{R}^N)$. In addition, $S(t)$ is asymptotically compact in $L^2(\mathbb{R}^N)$ since Lemma 3.6. Thus, we get the conclusion. □

3.3. Existence of a global attractor in $S^1(\mathbb{R}^N)$

Lemma 3.8. *Suppose that* (F) *and* (G) *hold. Then for any $\epsilon > 0$ and any bounded subset $B \subset L^2(\mathbb{R}^N)$, there exist $T = T(\epsilon, B) > 0$ and $K = K(\epsilon, B) > 0$ such that for all $t \geq T$ and $R \geq K$,*

$$\int_{\mathbb{R}^N \backslash B_R^*} |\nabla_{\alpha,\beta} u|^2 dX \leq \epsilon.$$

Proof. Taking the inner product of (1.1) with $-(\varphi_R\theta_R\gamma_R)P_{\alpha,\beta}u$ in $L^2(\mathbb{R}^N)$, we get

$$-\int_{\mathbb{R}^N} u_t(\varphi_R\theta_R\gamma_R)P_{\alpha,\beta}u dX + \int_{\mathbb{R}^N} (\varphi_R\theta_R\gamma_R)|P_{\alpha,\beta}u|^2 dX$$

$$\begin{aligned}&-\lambda\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)uP_{\alpha,\beta}udX-\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)f(X,u)P_{\alpha,\beta}udX\\ =&\ -\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)P_{\alpha,\beta}ugdX.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&-\int_{\mathbb{R}^N}u_t(\varphi_R\theta_R\gamma_R)P_{\alpha,\beta}udX\\ =&\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)\Big(\nabla_{\alpha,\beta}u_t,\nabla_{\alpha,\beta}u\Big)_{\mathbb{R}^N}dX\\ &+\int_{\mathbb{R}^N}u_t\Big(\nabla_{\alpha,\beta}(\varphi_R\theta_R\gamma_R),\nabla_{\alpha,\beta}u\Big)_{\mathbb{R}^N}dX\\ =&\ \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|\nabla_{\alpha,\beta}u|^2dX+\int_{\mathbb{R}^N}u_t\Big(\nabla_{\alpha,\beta}(\varphi_R\theta_R\gamma_R),\nabla_{\alpha,\beta}u\Big)_{\mathbb{R}^N}dX;\end{aligned}$$

$$\begin{aligned}&-\lambda\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)uP_{\alpha,\beta}udX\\ =&\ \lambda\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|\nabla_{\alpha,\beta}u|^2dX+\lambda\int_{\mathbb{R}^N}u\Big(\nabla_{\alpha,\beta}(\varphi_R\theta_R\gamma_R),\nabla_{\alpha,\beta}u\Big)_{\mathbb{R}^N}dX;\end{aligned}$$

$$\begin{aligned}&-\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)f(X,u)P_{\alpha,\beta}udX\\ =&\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)f'_u(X,u)|\nabla_{\alpha,\beta}u|^2dX\\ &+\int_{\mathbb{R}^N}f(X,u)\Big(\nabla_{\alpha,\beta}(\varphi_R\theta_R\gamma_R),\nabla_{\alpha,\beta}u\Big)_{\mathbb{R}^N}dX;\end{aligned}$$

$$\begin{aligned}-\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)P_{\alpha,\beta}ugdX&=-\int_{\sum_R}(\varphi_R\theta_R\gamma_R)P_{\alpha,\beta}ugdX\\ &\le\int_{\sum_R}(\varphi_R\theta_R\gamma_R)|P_{\alpha,\beta}u|^2dX+\frac{1}{4}\int_{\sum_R}|g|^2dX.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|\nabla_{\alpha,\beta}u|^2dX+\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|P_{\alpha,\beta}u|^2dX\\ &+\lambda\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|\nabla_{\alpha,\beta}u|^2dX\\ \le&\ -\int_{\mathbb{R}^N}u_t\Big(\nabla_{\alpha,\beta}(\varphi_R\theta_R\gamma_R),\nabla_{\alpha,\beta}u\Big)_{\mathbb{R}^N}dX\\ &-\lambda\int_{\mathbb{R}^N}u\Big(\nabla_{\alpha,\beta}(\varphi_R\theta_R\gamma_R),\nabla_{\alpha,\beta}u\Big)_{\mathbb{R}^N}dX\end{aligned}$$

$$
\begin{aligned}
&+ \ell \int_{\mathbb{R}^N} (\varphi_R \theta_R \gamma_R) |\nabla_{\alpha,\beta} u|^2 dX \\
&- \int_{\mathbb{R}^N} f(X,u) \Big(\nabla_{\alpha,\beta}(\varphi_R \theta_R \gamma_R), \nabla_{\alpha,\beta} u \Big)_{\mathbb{R}^N} dX \\
&+ \int_{\sum_R} (\varphi_R \theta_R \gamma_R) |P_{\alpha,\beta} u|^2 dX + \frac{1}{4} \int_{\sum_R} |g|^2 dX. \qquad (3.23)
\end{aligned}
$$

Because $\varphi'(s), \theta'(s), \gamma'(s) = 0$ for all $0 \le s < \frac{1}{2}$ and $s > 1$, we have

$$
\begin{aligned}
&\Big| - \int_{\mathbb{R}^N} u_t \Big(\nabla_{\alpha,\beta}(\varphi_R \theta_R \gamma_R), \nabla_{\alpha,\beta} u \Big)_{\mathbb{R}^N} dX \Big| \\
\le &\Big| \int_{\mathbb{R}^N} u_t \theta_R \gamma_R \Big(\nabla_x u, \nabla_x \varphi_R \Big)_{\mathbb{R}^N} dX \Big| + \Big| \int_{\mathbb{R}^N} u_t \varphi_R \gamma_R \Big(\nabla_y u, \nabla_y \theta_R \Big)_{\mathbb{R}^N} dX \Big| \\
&+ \Big| \int_{\mathbb{R}^N} u_t \varphi_R \theta_R |x|^{2\alpha|} |y|^{2\beta} \Big(\nabla_z u, \nabla_z \gamma_R \Big)_{\mathbb{R}^N} dX \Big| \\
\le &\frac{2}{R^2} \Big| \int_{\mathbb{R}^N} u_t \theta_R \gamma_R \varphi'(\frac{|x|^2}{R^2})(x \cdot \nabla_x u) dX \Big| \\
&+ \frac{2}{R^2} \Big| \int_{\mathbb{R}^N} u_t \varphi_R \gamma_R \theta'(\frac{|y|^2}{R^2})(y \cdot \nabla_y u) dX \Big| \\
&+ \frac{2}{R^{2(1+\alpha+\beta)}} \Big| \int_{\mathbb{R}^N} u_t \varphi_R \theta_R \gamma'(\frac{|z|^2}{R^{2(1+\alpha+\beta)}}) |x|^{2\alpha} |y|^{2\beta} (z \cdot \nabla_z u) dX \Big| \\
\le &\frac{2}{R^2} \Big| \int_{B_R^*} u_t \theta_R \gamma_R \varphi'(\frac{|x|^2}{R^2})(x \cdot \nabla_x u) dX \Big| \\
&+ \frac{2}{R^2} \Big| \int_{B_R^*} u_t \varphi_R \gamma_R \theta'(\frac{|y|^2}{R^2})(y \cdot \nabla_y u) dX \Big| \\
&+ \frac{2}{R^{2(1+\alpha+\beta)}} \Big| \int_{B_R^*} u_t \varphi_R \theta_R \gamma'(\frac{|z|^2}{R^{2(1+\alpha+\beta)}}) |x|^{2\alpha} |y|^{2\beta} (z \cdot \nabla_z u) dX \Big| \\
\le &\frac{2C}{R^2} \Big(\int_{B_R^*} |u_t|^2 dX \Big)^{1/2} \Big(\int_{B_R^*} |x|^2 |\nabla_x u|^2 dX \Big)^{1/2} \\
&+ \frac{2C}{R^2} \Big(\int_{B_R^*} |u_t|^2 dX \Big)^{1/2} \Big(\int_{B_R^*} |y|^2 |\nabla_y u|^2 dX \Big)^{1/2} \\
&+ \frac{2C}{R^{2(1+\alpha+\beta)}} \Big(\int_{B_R^*} |u_t|^2 dX \Big)^{1/2} \Big(\int_{B_R^*} |x|^{4\alpha} |y|^{4\beta} |z|^2 |\nabla_z u|^2 dX \Big)^{1/2} \\
\le &\frac{2C}{R} \Big(\int_{B_R^*} |u_t|^2 dX \Big)^{1/2} \Big(\int_{B_R^*} |\nabla_x u|^2 dX \Big)^{1/2} \\
&+ \frac{2C}{R} \Big(\int_{B_R^*} |u_t|^2 dX \Big)^{1/2} \Big(\int_{B_R^*} |\nabla_y u|^2 dX \Big)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2C}{R}\left(\int_{B_R^*}|u_t|^2dX\right)^{1/2}\left(\int_{B_R^*}|x|^{2\alpha}|y|^{2\beta}\nabla_z u|^2dX\right)^{1/2}\\
\le\ &\frac{2C}{R}\left[\frac{1}{2}\|u_t\|^2_{L^2(B_R^*)}+\frac{1}{2}\int_{B_R^*}|\nabla_x u|^2dX\right]\\
&+\frac{2C}{R}\left[\frac{1}{2}\|u_t\|^2_{L^2(B_R^*)}+\frac{1}{2}\int_{B_R^*}|\nabla_y u|^2dX\right]\\
&+\frac{2C}{R}\left[\frac{1}{2}\|u_t\|^2_{L^2(B_R^*)}+\frac{1}{2}\int_{B_R^*}|x|^{2\alpha}|y|^{2\beta}|\nabla_z u|^2dX\right]\\
\le\ &\frac{3C}{R}\|u_t\|^2_{L^2(B_R^*)}+\frac{C}{R}\int_{B_R^*}|\nabla_{\alpha,\beta}u|^2dX.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left|-\int_{\mathbb{R}^N}u_t\Big(\nabla_{\alpha,\beta}(\varphi_R\theta_R\gamma_R),\nabla_{\alpha,\beta}u\Big)_{\mathbb{R}^N}dX\right|\\
&\le\frac{3C}{R}\|u_t\|^2_{L^2(\mathbb{R}^N)}+\frac{C}{R}\int_{\mathbb{R}^N}|\nabla_{\alpha,\beta}u|^2dX.
\end{aligned}\tag{3.24}
$$

Analogously to (3.24), we have

$$
\begin{aligned}
&\left|-\lambda\int_{\mathbb{R}^N}u\Big(\nabla_{\alpha,\beta}(\varphi_R\theta_R\gamma_R),\nabla_{\alpha,\beta}u\Big)_{\mathbb{R}^N}dX\right|\\
&\le\frac{3\lambda C}{R}\|u\|^2_{L^2(\mathbb{R}^N)}+\frac{\lambda C}{R}\int_{\mathbb{R}^N}|\nabla_{\alpha,\beta}u|^2dX,
\end{aligned}\tag{3.25}
$$

and

$$
\begin{aligned}
&\left|-\int_{\mathbb{R}^N}f(X,u)\Big(\nabla_{\alpha,\beta}(\varphi_R\theta_R\gamma_R),\nabla_{\alpha,\beta}u\Big)_{\mathbb{R}^N}dX\right|\\
&\le\frac{3C}{R}\|f(X,u)\|^2_{L^2(\mathbb{R}^N)}+\frac{C}{R}\int_{\mathbb{R}^N}|\nabla_{\alpha,\beta}u|^2dX.
\end{aligned}\tag{3.26}
$$

From (3.23)-(3.26), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|\nabla_{\alpha,\beta}u|^2dX+2\lambda\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|\nabla_{\alpha,\beta}u|^2dX\\
\le\ &\frac{1}{2}\int_{\sum_R}|g|^2dX+2\ell\int_{\mathbb{R}^N}|\nabla_{\alpha,\beta}u|^2dX+\frac{6C}{R}\|u_t\|^2_{L^2(\mathbb{R}^N)}\\
&+\frac{6\lambda C}{R}\|u\|^2_{L^2(\mathbb{R}^N)}+\frac{6C}{R}\|f(X,u)\|^2_{L^2(\mathbb{R}^N)}\\
&+2\frac{(2+\lambda)C}{R}\int_{\mathbb{R}^N}|\nabla_{\alpha,\beta}u|^2dX\\
\le\ &\frac{1}{2}\int_{\sum_R}|g|^2dX+\frac{6C}{R}\|u_t\|^2_{L^2(\mathbb{R}^N)}+\frac{6C}{R}\|f(X,u)\|^2_{L^2(\mathbb{R}^N)}
\end{aligned}
$$

$$+ C_1 \int_{\mathbb{R}^N} (|u|^2 + |\nabla_{\alpha,\beta} u|^2) dX, \tag{3.27}$$

where $C_1 = \max\{\frac{6\lambda C}{R}; 2\ell + 2\frac{(2+\lambda)C}{R}\}$. Multiplying (3.27) by $e^{2\lambda t}$ and then integrating over (T, t), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} (\varphi_R \theta_R \gamma_R) |\nabla_{\alpha,\beta} u|^2 dX \\
&\leq e^{-2\lambda t} \int_{\mathbb{R}^N} (\varphi_R \theta_R \gamma_R) |\nabla_{\alpha,\beta} u(T)|^2 dX + \frac{1}{2} e^{-2\lambda t} \int_T^t e^{2\lambda \xi} \int_{\sum_R} |g|^2 dX d\xi \\
&\quad + \frac{6C}{R} e^{-2\lambda t} \int_T^t e^{2\lambda \xi} \|u_t\|^2_{L^2(\mathbb{R}^N)} d\xi \\
&\quad + \frac{6C}{R} e^{-2\lambda t} \int_T^t e^{2\lambda \xi} \|f(X,u)\|^2_{L^2(\mathbb{R}^N)} d\xi \\
&\quad + C_1 e^{-2\lambda t} \int_T^t e^{2\lambda \xi} \int_{\mathbb{R}^N} (|u|^2 + |\nabla_{\alpha,\beta} u|^2) dX d\xi \\
&\leq e^{-2\lambda t} \|u(T)\|^2_{S^1(\mathbb{R}^N)} + \frac{1}{4\lambda} \int_{\sum_R} |g|^2 dX \\
&\quad + \frac{6C}{R} e^{-2\lambda t} \int_T^t e^{2\lambda \xi} \|u_t\|^2_{L^2(\mathbb{R}^N)} d\xi \\
&\quad + \frac{6C}{R} e^{-2\lambda t} \int_T^t e^{2\lambda \xi} \|f(X,u)\|^2_{L^2(\mathbb{R}^N)} d\xi \\
&\quad + C_1 e^{-2\lambda t} \int_T^t e^{2\lambda \xi} \int_{\mathbb{R}^N} (|u|^2 + |\nabla_{\alpha,\beta} u|^2) dX d\xi.
\end{aligned} \tag{3.28}$$

Noting that for given $\epsilon > 0$, there is $T_1 = T_1(\epsilon) > 0$ such that for all $t \geq T_1$,

$$e^{-2\lambda t} \|u(T_1)\|^2_{S^1(\mathbb{R}^N)} \leq \frac{\epsilon}{5}. \tag{3.29}$$

On the other hand, since $g \in L^2(\mathbb{R}^N)$, there is $K_1 = K_1(\epsilon)$ such that for all $R \geq K_1$,

$$\frac{1}{4\lambda} \int_{\sum_R} |g|^2 dX \leq \frac{\epsilon}{5}. \tag{3.30}$$

For the third term on the right hand, it follows from Lemmas 3.1, 3.2 and 3.3 that there is $K_2 = K_2(\epsilon) \geq K_1$ such that for all $R \geq K_2$ and $T_2 > 0$ such that for all $t \geq T_2$,

$$C_1 e^{-2\lambda t} \int_{T_2}^t e^{2\lambda \xi} \int_{\mathbb{R}^N} (|u|^2 + |\nabla_{\alpha,\beta} u|^2) dX d\xi \leq \frac{\epsilon}{5}, \tag{3.31}$$

$$\frac{6C}{R} e^{-2\lambda t} \int_{T_2}^t e^{2\lambda \xi} \|u_t\|^2_{L^2(\mathbb{R}^N)} d\xi \leq \frac{\epsilon}{5}. \tag{3.32}$$

On the other hand, from (1.1) we have $f(X,u) = -u_t + P_{\alpha,\beta}u - \lambda u + g$. Using Lemmas 3.3 and 3.4, we obtain $f(X,u(t)) \in L^2(\mathbb{R}^N)$ for t large enough. Therefore, there is $K_3 = K_3(\epsilon) \geq K_2$ such that for all $R \geq K_3$ and $T_3 > 0$ such that for all $t \geq T_3$,

$$\frac{6C}{R}e^{-2\lambda t}\int_{T_3}^{t} e^{2\lambda\xi}\int_{\mathbb{R}^N}|f(X,u)|^2 dX d\xi \leq \frac{\epsilon}{5}. \tag{3.33}$$

Let $T = \max\{T_1, T_2, T_3\}$, then by (3.28)-(3.33) we find that for all $R \geq K \geq K_3$ and $t \geq T$,

$$\int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|\nabla_{\alpha,\beta}u|^2 dX \leq \epsilon,$$

and hence for all $R \geq K$ and $t \geq T$,

$$\int_{\mathbb{R}^N\setminus B_R^*}|\nabla_{\alpha,\beta}u|^2 dX \leq \int_{\mathbb{R}^N}(\varphi_R\theta_R\gamma_R)|\nabla_{\alpha,\beta}u|^2 dX \leq \epsilon.$$

Then, we complete the proof. □

Now, we show the asymptotic compactness of $S(t)$ in $S^1(\mathbb{R}^N)$.

Lemma 3.9. *Suppose that* (F) *and* (G) *hold. Then $S(t)$ is asymptotically compact in $S^1(\mathbb{R}^N)$, that is, for any bounded sequence $\{x_n\} \subset S^1(\mathbb{R}^N)$ and any sequence $t_n \geq 0$, $t_n \to \infty$, $\{S(t_n)x_n\}$ has a convergent subsequence with respect to the topology of $S^1(\mathbb{R}^N)$.*

Proof. Similarly to Lemma 3.6, given $K > 0$, denote

$$B_K^* = B_{\mathbb{R}^{N_1}}(0,K) \times B_{\mathbb{R}^{N_2}}(0,K) \times B_{\mathbb{R}^{N_3}}(0,K^{1+\alpha+\beta}) \text{ and } B_K^c = \mathbb{R}^N\setminus B_K^*.$$

Then, by Lemmas 3.5 and 3.8, for the given $\epsilon > 0$, there exist $K = K(\epsilon) > 0$ and $T = T(\epsilon) > 0$ such that for $t \geq T$,

$$\|u_n(t)\|_{S^1(B_K^c)} \leq \epsilon.$$

Since $t_n \to \infty$, there is $N_1 = N_1(\epsilon) > 0$ such that $t_n \geq T$ for all $n \geq N_1$ and hence we obtain that, for all $n \geq N_1$,

$$\|u_n(t_n)\|_{S^1(B_K^c)} \leq \epsilon. \tag{3.34}$$

By Lemma 3.4, there exist $C > 0$ and $N_2 > 0$ such that for all $n \geq N_2$,

$$||u_n(t_n)||_{S^2(B_K^*)} \leq C.$$

Since the compactness of the embedding $S^2(B_K^*) \hookrightarrow S^1(B_K^*)$ (see [5]), the sequence $\{u_n(t_n)\}$ is precompact in $S^1(B_K^*)$. Therefore, for the given $\epsilon > 0$, $\{u_n(t_n)\}$ has a finite covering in $S^1(B_K^*)$ of balls of radii less than ϵ, which along with (3.34) shows that $\{u_n(t_n)\}$ has a finite covering in $S^1(\mathbb{R}^N)$ of balls of radii less than ϵ, and thus $\{u_n(t_n)\}$ is precompact in $S^1(\mathbb{R}^N)$. □

We are now ready to prove the existence of a global attractor for $S(t)$ in $S^1(\mathbb{R}^N)$.

Theorem 3.10. *Suppose that* (F) *and* (G) *hold. Then the semigroup* $S(t)$ *generated by the problem* (1.1) *has a global attractor* $\mathcal{A}_{S^1}$ *in* $S^1(\mathbb{R}^N)$.

Proof. By Lemma 3.2, there is a bounded absorbing set for $S(t)$ in $S^1(\mathbb{R}^N)$. In addition, $S(t)$ is asymptotically compact in $S^1(\mathbb{R}^N)$ by Lemma 3.9. Thus, we get the conclusion. □

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