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HOMOGENEOUS CONDITIONS FOR STOCHASTIC TENSORS

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Abstract. Fix an integer $n \geq 1$. Then the simplex Π_n, Birkhoff polytope Ω_n, and Latin square polytope Λ_n each yield projective geometries obtained by identifying antipodal points on a sphere bounding a ball centered at the barycenter of the polytope. We investigate conditions for homogeneous coordinates of points in the projective geometries to locate exact vertices of the respective polytopes, namely crisp distributions, permutation matrices, and quasigroups or Latin squares respectively. In the latter case, the homogeneous conditions form a crucial part of a recent projective-geometrical approach to the study of orthogonality of Latin squares. Coordinates based on the barycenter of Ω_n are also suited to the analysis of generalized doubly stochastic matrices, observing that orthogonal matrices of this type form a subgroup of the orthogonal group.

1. Introduction

A number of hard open combinatorial problems involve quasigroups and Latin squares, for example the existence question for a projective plane of order 12, or determination of the maximum number (known from [6] to be less than 9) of mutually orthogonal Latin squares of order 10. Recently, some relaxation techniques have been proposed in order to open up such problems to geometrical methods using a notion of (weak) approximate quasigroup or Latin square [7]. The concept of orthogonality has been extended from exact to approximate quasigroups or Latin squares, based either on convex geometry or projective geometry [5]. The extended concept of orthogonality specifies a space of approximate orthogonal mates to a given set of exact quasigroups or Latin squares. Once such a space has been determined, a solution of the combinatorial problems involves the question of identifying any exact quasigroups or Latin

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squares that may lie in the space of approximate orthogonal mates. That question is the topic of this paper.

Successive sections of the paper consider the respective convex sets Π_n of probability distributions on an n-element set (§2), the *Birkhoff polytope* Ω_n of bistochastic matrices (§3), and the *Latin square polytope* Λ_n of tristochastic tensors, interpreted as (*weak*) *approximate quasigroups* or *Latin squares* (§5). Each section considers various conditions for recognizing exact structures within the convex sets, namely crisp distributions in Π_n, permutation matrices in Ω_n, and quasigroups or Latin squares in Λ_n. In other words, these exactness conditions apply to the convex geometry.

The barycenter O of each of the successive convex sets $\Pi_n, \Omega_n, \Lambda_n$ is the center of a small ball that lies entirely within the corresponding convex sets. The boundary of the ball is a sphere, and identification of diametrically opposed points on the sphere produces a projective geometry. This projective geometry is also defined by the set of lines through O in the affine hull of the corresponding convex set. Since the (extended) orthogonality of approximate quasigroups or Latin squares is well-defined on the intersection of these lines with Λ_n, orthogonality of these structures is a projective concept [5, §6]. Rather curiously, the orthogonality conditions [5, Th. 6.2] reduce to the usual linear-algebraic concept of orthogonality of two vectors, even though the combinatorial notion of Latin square orthogonality is not normally considered as being directly related to geometric orthogonality.

We are confronted with the issue of recognizing when a line through O, in each of the respective convex sets, contains an exact structure. The primary results of the paper, Theorems 2.6, 3.5, and 5.5, provide purely projective necessary and sufficient conditions for this to happen, based on the *Partition Condition* of Definition 2.5.

Now let $L_1, \dots, L_r$ be r mutually orthogonal $n \times n$ Latin squares. The subset of the Latin polytope Λ_n consisting of all approximate Latin squares orthogonal to each of $L_1, \dots, L_r$ yields a linear subspace $\mathcal{L}(L_1, \dots, L_r)$ or $\mathcal{L}$ in the projective geometry of Λ_n. Theorem 5.5 then recognizes any exact orthogonal mate of $L_1, \dots, L_r$ (which of course may or may not exist) by the satisfaction of the Partition Condition for each of the n^2 homogeneous n-dimensional coordinate vectors that serve to specify an approximate Latin square in the linear subspace $\mathcal{L}$. Search techniques for a witness of the partition conditions in $\mathcal{L}$ thus become a topic for active research that is opened up by the present paper. It is anticipated that hybrid methods, along the lines of the *augmented annealing* of [3], currently offer the best chance of success.

A secondary topic of the paper involves the affine hull $\mathbb{R}\Omega_n$ of the Birkhoff polytope Ω_n — the space of generalized doubly stochastic or biaffine matrices. Given the recent attention that has been paid to identifying orthogonal matrices of this type [4, 12], Section 4 examines some connections between that problem and the main themes of the current paper. In particular, Theorem 4.3 and its corollary parametrize the group structure of biaffine orthogonal matrices.

2. Stochastic vectors

Fix a dimension or *degree* $n > 1$. Then an n-dimensional vector $\mathbf{t}$ is said to be *stochastic* if its components are non-negative, and sum to 1. Such a vector is *exact* if there is a single non-zero component, namely 1. Consider the uniform stochastic vector $O = (\frac{1}{n}, \dots, \frac{1}{n})$. We will coordinatize the n-dimensional affine real space of n-dimensional vectors as a linear space with respect to O as an origin. Thus an arbitrary vector $\mathbf{t} = (t_1, \dots, t_n)$ of degree n is given by

$$t_i = X_i + \frac{1}{n} \tag{2.1}$$

for $1 \leq i \leq n$, with $X_i = 0$ corresponding to O.

Lemma 2.1. *Consider the vector* $\mathbf{t}$ *given by* (2.1). *Suppose that*

$$X_i \geq -\frac{1}{n}$$

for $1 \leq i \leq n$. *Then* $\mathbf{t}$ *is stochastic if and only if the equation*

$$\sum_{i=1}^{n} X_i = 0 \tag{2.2}$$

holds.

The linear condition (2.2) serves to specify an $(n-1)$-dimensional real subspace of the space of n-dimensional real vectors. The subspace is the linear hull of the simplex Π_n, the set of all the stochastic vectors with n coordinates. The space of lines in the linear hull of Π_n is the projective geometry $\mathsf{PG}(n-2, \mathbb{R})$. Here, we may take $[X_1, \dots, X_n]$ with (2.2), and not all X_i being zero, as homogeneous coordinates for the projective geometry, identifying $[X_1, \dots, X_n]$ and $[Y_1, \dots, Y_n]$ if there is a non-zero scalar λ such that $Y_i = \lambda X_i$ for $1 \leq i \leq n$.

Remark 2.2. Since the simplex that we are denoting here by Π_n is $(n-1)$-dimensional, it is usually described as Δ_{n-1}, with the suffix referring to the topological dimension. However, in the present context, where the simplex is taken as the space of probability distributions on an n-element set, considered along with the Birkhoff polytope Ω_n and Latin square polytope Λ_n, the suffix n is more appropriate, attached to the letter Π standing for the probability distributions.

Lemma 2.3. *Let* $\mathbf{t} = (t_1, \dots, t_n)$ *be a stochastic vector. Then the following are equivalent*:

(a) *The stochastic vector* $\mathbf{t}$ *is exact*;
(b) *The condition*

$$\sum_{i=1}^{n} t_i^2 = 1 \tag{2.3}$$

holds;

(c) *The condition*

$$\sum_{1\leq i\neq j\leq n} t_i t_j = 0 \tag{2.4}$$

holds;

(d) *The condition*

$$\forall\ 1 \leq i \neq j \leq n\,,\ t_i t_j = 0 \tag{2.5}$$

holds.

Proof. (a)⇒(b) is immediate.

(b)⇒(c): Suppose (2.3) holds. Then since **t** is stochastic, we have

$$1 = \Big(\sum_{i=1}^{n} t_i\Big)^2 = \sum_{i=1}^{n} t_i^2 + \sum_{1\leq i\neq j\leq n} t_i t_j = 1 + \sum_{1\leq i\neq j\leq n} t_i t_j\,,$$

so (2.4) holds.

(c)⇒(d): Because the components of **t** are non-negative, (2.4) implies that (2.5) holds.

(d)⇒(a): Suppose (2.5) holds. Then at most one component of **t** is non-zero. Since **t** is stochastic, this non-zero component must be 1, and then **t** is exact. □

Lemma 2.4. *Consider a stochastic vector* **t** *as given by* (2.1)*. Then the following are equivalent*:

(a) *The stochastic vector* **t** *is exact*;

(b) *The condition*

$$\sum_{i=1}^{n} X_i^2 = 1 - \frac{1}{n} \tag{2.6}$$

holds;

(c) *The condition*

$$\sum_{1\leq i\neq j\leq n} X_i X_j = \frac{1}{n} - 1 \tag{2.7}$$

holds;

(d) *The condition*

$$X_i X_j + \frac{1}{n}\big(X_i + X_j\big) + \frac{1}{n^2} = 0 \tag{2.8}$$

holds for all $1 \leq i \neq j \leq n$.

Proof. (a)⇔(b) By Lemma 2.3, the stochastic vector is exact if and only if

$$1 = \sum_{i=1}^{n}\Big(X_i + \frac{1}{n}\Big)\Big(X_i + \frac{1}{n}\Big) = \sum_{k=1}^{n} X_i^2 + \frac{2}{n}\sum_{i=1}^{n} X_i + \sum_{k=1}^{n}\frac{1}{n^2}\,.$$

By Lemma 2.1, the linear term sums to 0, and the result follows.

(a)⇔(c) Note that

$$0 = \left(\sum_{i=1}^{n} X_i\right)^2 = \sum_{i=1}^{n} X_i^2 + \sum_{1 \le i \ne j \le n} X_i X_j$$

by Lemma 2.1.

(a)⇔(d) By Lemma 2.3, the stochastic vector is exact if and only if

$$\begin{aligned} 0 &= \Big(X_i + \frac{1}{n}\Big)\Big(X_j + \frac{1}{n}\Big) \\ &= X_i X_j + \frac{1}{n}\big(X_i + X_j\big) + \frac{1}{n^2} \end{aligned}$$

for all $1 \le i \ne j \le n$. □

The theorem below uses the *product form* for integer partitions [11, p. 50]. In contrast with the non-homogenous exactness conditions (2.6)–(2.8), it gives a homogeneous condition for exactness.

Definition 2.5. Homogeneous coordinates $[X_1, \ldots, X_n]$ are described as satisfying the *Partition Condition* if there is an $(n-1) \cdot 1$-partition of values in the multiset $\{X_1, \ldots, X_n\}$. In other words, one of the coordinates takes an exceptional value, while all the remaining coordinates are equal.

Theorem 2.6. *Homogeneous coordinates $[X_1, \ldots, X_n]$ specify a line, in the linear hull of the simplex Π_n centered at O, that passes through an exact stochastic vector if and only if they satisfy the Partition Condition.*

Proof. Certainly, the necessity of the Partition Condition is immediate. Now suppose that the Partition Condition holds, so that the multiset $\{X_1, \ldots, X_n\}$ contains $n-1$ copies of a real number v, complemented by a single copy of $-(n-1)v$. It will be shown that there is a nonzero scalar λ such that $\lambda[X_1, \ldots, X_n]$ satisfies the conditions (2.8) of Lemma 2.4(d).

Consider distinct indices i and j for which $X_i = X_j = v$. Then the condition (2.8) on $\lambda[X_1, \ldots, X_n]$ reduces to the equation

$$\lambda^2 v^2 + 2\frac{\lambda v}{n} + \frac{1}{n^2} = 0\,,$$

satisfied by $\lambda = -1/(nv)$. Then for distinct indices i and j with $X_i = v$ and $X_j = -(n-1)v$, the condition (2.8) on $\lambda[X_1, \ldots, X_n]$ reduces to the equation

$$\lambda^2 v[-(n-1)v] + \frac{\lambda}{n}[-(n-1)v + v] + \frac{1}{n^2} = 0$$

which is also satisfied by $\lambda = -1/(nv)$. □

3. Doubly stochastic matrices

A square matrix T of degree n is considered as a stack $T = (\mathbf{t}_1, \dots, \mathbf{t}_n)$ of row vectors $\mathbf{t}_1, \dots, \mathbf{t}_n$. Thus T is specified by the vector components $(\mathbf{t}_i)_j$ for $1 \le i, j \le n$, the matrix entries $[T]_{ij} = (\mathbf{t}_i)_j$. The matrix T is *row stochastic* if each row vector $\mathbf{t}_i$, for $1 \le i \le n$, is a stochastic vector. It is *column stochastic* if its transpose is row stochastic. It is *doubly stochastic* or *bistochastic* if it is both row and column stochastic. It is said to be *exact* if it is a permutation matrix.

Lemma 3.1. *Let T be a doubly stochastic square matrix.*

(a) *T is exact if and only if $\big|\{(i,j) \mid [T]_{ij} = 1\}\big| = n$.*
(b) *T is exact if and only if its transpose is exact.*

Lemma 3.2. *Let T be a doubly stochastic matrix of degree $n > 1$. For $1 \le k \le n$, let $\mathbf{t}^{(j)}$ be the stochastic vector that is the transpose of the j-th column of T. Then the following are equivalent*:

(a) *T is exact.*
(b) *The condition*

$$\forall\ 1 \le i \ne j \le n\,,\ \sum_{k=1}^{n} [T]_{ik}[T]_{jk} = 0 \tag{3.1}$$

holds.
(c) *The condition*

$$\forall\ 1 \le i \ne j \le n\,,\ [T]_{ik}[T]_{jk} = 0 \tag{3.2}$$

holds for all $1 \le k \le n$.
(d) *The stochastic vector $\mathbf{t}^{(j)}$ is exact for each $1 \le j \le n$.*
(e) *The stochastic vector $\mathbf{t}_i$ is exact for each $1 \le i \le n$.*

Proof. (a)⇒(b) If T is a permutation matrix, then it is orthogonal, and the condition (3.1) holds.

(b)⇒(c) Suppose (3.1) holds. Since T is doubly stochastic, each entry of T is nonnegative, and so (3.2) holds.

(c)⇒(d) Condition (3.2) reduces to the condition (2.5) of Lemma 2.3 for $\mathbf{t}^{(k)}$. Thus $\mathbf{t}^{(k)}$ is exact.

(d)⇒(e): Consider a nonzero entry $[T]_{ij}$ of a row $\mathbf{t}_i$. Since $\mathbf{t}^{(j)}$ is exact by (d), $[T]_{ij} = 1$. Then since T is row stochastic, all the other components of $\mathbf{t}_i$ are zero, and so $\mathbf{t}_i$ is exact.

(e)⇒(a): Condition (e) implies that $\big|\{(i,j) \mid [T]_{ij} = 1\}\big| = n$, so T is exact by Lemma 3.1. □

Fix a degree $n > 1$. Consider the uniform doubly stochastic matrix $O = \frac{1}{n} J_n$, where J_n is the all-ones matrix of degree n. We coordinatize the n^2-dimensional affine real space of $(n \times n)$-matrices as a linear space with respect to O as an origin. Thus an arbitrary square matrix T of degree n is given by

$$[T]_{ij} = X_{ij} + \frac{1}{n} \tag{3.3}$$

for $1 \leq i, j \leq n$, with $X_{ij} = 0$ corresponding to O.

Lemma 3.3. *Consider the square matrix T given by* (3.3)*. Suppose that*

$$X_{ij} \geq -\frac{1}{n}$$

for $1 \leq i, j \leq n$.

(a) *The matrix T is row-stochastic if and only if*

$$\sum_{j=1}^{n} X_{ij} = 0 \tag{3.4}$$

for $1 \leq i \leq n$.

(b) *The matrix T is column-stochastic if and only if*

$$\sum_{i=1}^{n} X_{ij} = 0 \tag{3.5}$$

for $1 \leq j \leq n$.

(c) *The matrix T is doubly stochastic if and only if the conditions* (3.4) *and* (3.5) *hold.*

The $2n$ linear conditions listed in (3.4) and (3.5) serve to specify an $(n-1)^2$-dimensional real subspace of the n^2-dimensional real linear space of square matrices. (Note that $n^2 - 2n < (n-1)^2$; the conditions (3.4) and (3.5) are not independent.) The subspace is the linear hull of the polytope Ω_n centered at O. The space of lines in the linear hull of Ω_n is the projective geometry $\mathsf{PG}\big((n-1)^2 - 1, \mathbb{R}\big)$.

Lemma 3.4. *Consider a doubly stochastic matrix T, written in the notation* (3.3)*. The following conditions are equivalent*:

(a) *T is exact*;

(b) *The condition*

$$\forall\ 1 \leq i \neq j \leq n\,,\ \sum_{k=1}^{n} X_{ik} X_{jk} = -\frac{1}{n} \tag{3.6}$$

holds;

(c) *The condition*

$$\forall\ 1 \leq i \neq j \leq n\,,\ \forall\ 1 \leq k \leq n\,, \quad X_{ik} X_{jk} + \frac{1}{n}\big(X_{ik} + X_{jk}\big) + \frac{1}{n^2} = 0 \tag{3.7}$$

holds.

Proof. (a)⇔(b) By Lemma 3.2, the matrix T is a permutation matrix if and only if

$$\begin{aligned}0 &= \sum_{k=1}^{n}\Big(X_{ik}+\frac{1}{n}\Big)\Big(X_{jk}+\frac{1}{n}\Big)\\ &= \sum_{k=1}^{n} X_{ik}X_{jk}+\frac{1}{n}\sum_{k=1}^{n}\big(X_{ik}+X_{jk}\big)+\frac{1}{n}\end{aligned}$$

for all $1 \le i \ne j \le n$. By Lemma 3.3, the linear terms vanish, and the result follows.

(a)⇔(c) By Lemma 3.2, the matrix T is a permutation matrix if and only if

$$0 = \Big(X_{ik}+\frac{1}{n}\Big)\Big(X_{jk}+\frac{1}{n}\Big) = X_{ik}X_{jk}+\frac{1}{n}\big(X_{ik}+X_{jk}\big)+\frac{1}{n^2}$$

for all $1 \le i \ne j \le n$ and $1 \le k \le n$. □

The respective exactness conditions (3.6) and (3.7) of Lemma 3.4(b) and (c) are not homogeneous. A homogeneous condition for locating permutation matrices in Ω_n may be obtained as follows.

Theorem 3.5. *Let* $[X_{ij} \mid 1 \le i,j \le n]$ *be homogeneous coordinates that satisfy the conditions* (3.4) *and* (3.5). *Then in the linear hull of the polytope* Ω_n *centered at* O*, the coordinates specify a line that passes through a permutation matrix if and only if the homogeneous coordinates* $[X_{i1},\dots,X_{in}]$ *satisfy the Partition Condition for* $1 \le i \le n$*.*

Proof. By Lemma 3.2, a doubly stochastic matrix $T = (\mathbf{t}_1,\dots,\mathbf{t}_n)$ is exact if and only if each stochastic vector $\mathbf{t}_i$, for $1 \le i \le n$, is exact. By Theorem 2.6, this happens if and only if, for $1 \le i \le n$, the homogeneous coordinates $[X_{i1},\dots,X_{in}]$ satisfy the Partition Condition, writing T as in (3.3). □

4. Generalized doubly stochastic matrices

A matrix T is *generalized doubly stochastic* [4,12] or *biaffine* if each row and column sum is 1 (as is the case for doubly stochastic matrices), but the entries are not required to be non-negative. Like any matrix of degree n, a biaffine matrix T of degree n may be written in the form (3.3). The row and column sum conditions guaranteeing the generalized double stochasticity then take the form of (3.4) and (3.5). In other words, the linear hull $\mathbb{R}\Omega_n$ of the polytope Ω_n centered at O is precisely the set of biaffine matrices. The intention of this section is to demonstrate the utility of the representation (3.3), or its matrix version (4.1) below, in the study of biaffine matrices.

Lemma 4.1. *Let*

$$T = \frac{1}{n}J_n + X \tag{4.1}$$

be an $n \times n$ *matrix.*

(a) *The matrix T is biaffine if and only if the equations*

$$XJ_n = 0 = J_nX \tag{4.2}$$

are satisfied.

(b) *A biaffine matrix T is orthogonal if and only if the equation*

$$XX^T = I_n - \frac{1}{n}J_n \tag{4.3}$$

is satisfied.

Proof. (a) Note that

$$[XJ_n]_{ik} = \sum_{j=1}^{n} X_{ij}$$

for $1 \leq i, k \leq n$, so the left-hand equation in (4.2) is equivalent to

$$\sum_{j=1}^{n}[T]_{ij} = \sum_{j=1}^{n}\Big(\frac{1}{n} + X_{ij}\Big) = \sum_{j=1}^{n}\frac{1}{n} = 1$$

for $1 \leq i \leq n$. Similarly, the right-hand equation in (4.2) is equivalent to the column sums of T all being 1.

(b) Suppose that T is biaffine, and thus satisfies (4.2) by (a). Then

$$\begin{aligned} TT^\top &= \Big(\frac{1}{n}J_n + X\Big)\cdot\Big(\frac{1}{n}J_n + X\Big)^\top \\ &= \frac{1}{n}J_n + \frac{1}{n}(XJ_n)^\top + \frac{1}{n}XJ_n + XX^\top = \frac{1}{n}J_n + XX^\top . \end{aligned}$$

Thus the orthogonality condition $TT^\top = I_n$ is equivalent to (4.3), as required. □

Lemma 4.1 offers an alternative to other characterizations of biaffine orthogonal matrices, such as in [4, 12].

Lemma 4.2. *The set $\mathbb{R}\Omega_n$ of biaffine matrices is closed under matrix multiplication and transposition.*

Proof. Consider elements $T_1 = \frac{1}{n}J_n + X_1$ and $T_2 = \frac{1}{n}J_n + X_2$ of $\mathbb{R}\Omega_n$, so that X_1 and X_2 satisfy the conditions (4.2) of Lemma 4.1. Then

$$\begin{aligned} T_1T_2 &= \Big(\frac{1}{n}J_n + X_1\Big)\cdot\Big(\frac{1}{n}J_n + X_2\Big) \\ &= \frac{1}{n}J_n + \frac{1}{n}J_nX_2 + \frac{1}{n}X_1J_n + X_1X_2 = \frac{1}{n}J_n + X_1X_2 . \end{aligned} \tag{4.4}$$

Now $J_nX_1X_2 = 0$ and $X_1X_2J_n = 0$ by (4.2), so that $T_1T_2 \in \mathbb{R}\Omega_n$ by Lemma 4.1. It is clear from the definition that the set of biaffine matrices is closed under transposition. □

Theorem 4.3. *The set $\mathsf{AAO}_n = \mathbb{R}\Omega_n \cap \mathsf{O}_n(\mathbb{R})$ of orthogonal biaffine matrices forms a subgroup of the orthogonal group $\mathsf{O}_n(\mathbb{R})$.*

Proof. Apply the Subgroup Test [10, Prop. 4.43]. Consider elements T_1, T_2 of AAO_n. Since $\mathsf{O}_n(\mathbb{R})$ is a group (in which transposition acts as inversion), we have $T_1 T_2^\top \in \mathsf{O}_n(\mathbb{R})$. Again, we have $T_1 T_2^\top \in \mathbb{R}\Omega_n$ by Lemma 4.2. □

Remark 4.4. For $n > 1$, note

$$\begin{bmatrix} 2^{-1/2} & -2^{-1/2} \\ 2^{-1/2} & 2^{-1/2} \end{bmatrix} \oplus I_{n-2} \in \mathsf{O}_n(\mathbb{R}) \smallsetminus \mathsf{AAO}_n\,,$$

so AAO_n is a proper subgroup of the orthogonal group. The subgroup AAO_n corresponds geometrically to the elements of $\mathsf{O}_n(\mathbb{R})$ fixing the vector $[1, 1, \dots, 1]$, so abstractly it is isomorphic to $\mathsf{O}_{n-1}(\mathbb{R})$.

Corollary 4.5. *Let X_n be the set of all $n \times n$ matrices which satisfy the conditions* (4.2) *and* (4.3) *of Lemma 4.1. Then X_n forms a group under matrix multiplication, with identity element $I_n - \frac{1}{n} J_n$, and transposition as inversion.*

Proof. The mutually inverse maps

$$\xi\colon \mathsf{AAO}_n \to \mathsf{X}_n; T \mapsto T - \frac{1}{n} J_n \tag{4.5}$$

and

$$\tau\colon \mathsf{X}_n \to \mathsf{AAO}_n; X \mapsto \frac{1}{n} J_n + X \tag{4.6}$$

serve to transport the group structure from AAO_n to X_n. By (4.4), the group multiplication in X_n is just matrix multiplication. By (4.3), matrix transposition furnishes the group inversion. □

We conclude this section with the following observation which is often useful for the generation of orthogonal biaffine matrices.

Proposition 4.6. (a) *The set X_n is closed under negation.*
(b) *The set AAO_n is closed under the transformation $\sigma\colon T \mapsto \frac{2}{n} J_n - T$.*

Proof. (a) is immediate from the conditions of Lemma 4.1.
(b) Note $T^\sigma = (-T^\xi)^\tau$, using the respective maps (4.5) and (4.6) from the proof of Corollary 4.5. □

Example 4.7. The permutation matrix image

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^\sigma = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$

is the orthogonal biaffine matrix presented in [12].

5. Triply stochastic tensors

A cubic tensor T of degree n is considered as a stack $T = (T_1, \ldots, T_n)$ of matrices $T_1, \ldots, T_n$. Thus T is specified by the matrix entries $[T_i]_{jk}$ for $1 \leq i, j, k \leq n$. It is *triply stochastic* or *tristochastic* if and only if

$$\sum_{j=1}^{n} T_j = J_n$$

and each matrix T_i, for $1 \leq i \leq n$, is doubly stochastic. A triply stochastic cubic tensor $T = (T_1, \ldots, T_n)$ is *exact* if and only if each matrix T_i, for $1 \leq i \leq n$, is a permutation matrix. In this case, the corresponding set of matrices is *sharply transitive* [1], [2], [8], [9, §8.1].

Lemma 5.1. *Let $T = (T_1, \ldots, T_n)$ be a triply stochastic cubic tensor.*

(a) *T is exact if and only if $\big|\{(i,j,k) \mid [T_i]_{jk} = 1\}\big| = n^2$.*

(b) *T is exact if and only if any one of its parastrophes is exact.*

Lemma 5.2. *Let $S = (S_1, \ldots, S_n)$ be a triply stochastic cubic tensor. Then the following are equivalent*:

(a) *S is exact*;

(b) *The condition*

$$\forall\ 1 \leq i \neq j \leq n\,,\ \forall\ 1 \leq k, l \leq n\,,\ [S_i]_{kl}[S_j]_{kl} = 0 \tag{5.1}$$

holds.

(c) *The conditions*

$$\forall\ 1 \leq i \neq j \leq n\,,\ \operatorname{tr}(S_i S_j^*) = 0 \tag{5.2}$$

or

$$\forall\ 1 \leq i \neq j \leq n\,,\ \sum_{k=1}^{n}\sum_{l=1}^{n}[S_i]_{kl}[S_j]_{kl} = 0 \tag{5.3}$$

hold.

Proof. (a)⇔(b) The triply stochastic cubic tensor S is exact if and only if, for $1 \leq k, l \leq n$, there is a unique index $1 \leq h \leq n$ such that $[S_h]_{kl} = 1$.

(b)⇔(c) Certainly, (5.1) implies (5.3). The converse follows since each summand in (5.3) is nonnegative. □

Fix a degree $n > 1$. Consider the uniform triply stochastic cubic tensor $O = (\frac{1}{n}J_n, \ldots, \frac{1}{n}J_n)$. We coordinatize the n^3-dimensional affine real space of cubic tensors as a linear space with respect to O as an origin. Thus an arbitrary cubic tensor $T = (T_1, \ldots, T_n)$ of degree n is given by

$$[T_i]_{jk} = X_{ijk} + \frac{1}{n} \tag{5.4}$$

for $1 \leq i, j, k \leq n$, with $X_{ijk} = 0$ corresponding to O.

Lemma 5.3. *Consider the cubic tensor* $[T_i]_{jk}$ *given by* (5.4). *Suppose that*

$$X_{ijk} \geq -\frac{1}{n}$$

for $1 \leq i, j, k \leq n$.

(a) *Each matrix of the stack* $T = (T_1, \ldots, T_n)$ *is row-stochastic if and only if*

$$\sum_{k=1}^{n} X_{ijk} = 0 \tag{5.5}$$

for $1 \leq i, j \leq n$.

(b) *Each matrix of the stack* $T = (T_1, \ldots, T_n)$ *is column-stochastic if and only if*

$$\sum_{j=1}^{n} X_{ijk} = 0 \tag{5.6}$$

for $1 \leq i, k \leq n$.

(c) *The matrices* T_i *of the stack* $T = (T_1, \ldots, T_n)$ *sum to* J_n *if and only if*

$$\sum_{i=1}^{n} X_{ijk} = 0 \tag{5.7}$$

for $1 \leq j, k \leq n$.

(d) *The cubic tensor is triply stochastic if and only if the conditions* (5.5)–(5.7) *hold.*

Together, the $3n^2$ linear conditions (5.5)–(5.7) serve to specify an $(n-1)^3$-dimensional real subspace of the n^3-dimensional real linear space of cubic tensors. (Note that $n^3 - 3n^2 < (n-1)^3$; indeed the conditions (5.5)–(5.7) are not independent.) The subspace is the linear hull of the polytope Λ_n centered at O. The space of lines in this linear hull of Λ_n is the projective geometry $\mathsf{PG}\big((n-1)^3 - 1, \mathbb{R}\big)$.

The following result expresses the exactness of a triply stochastic cubic tensor (5.4) in terms of the homogeneous coordinates X_{ijk}.

Lemma 5.4. *A triply stochastic cubic tensor* $[T_i]_{jk}$ *as given by* (5.4) *is exact if and only if the condition*

$$\forall\ 1 \leq i \neq j \leq n\,,\ \sum_{k=1}^{n}\sum_{l=1}^{n} X_{ikl}X_{jkl} = -1 \tag{5.8}$$

holds.

Proof. By Lemma 5.2, the triply stochastic cubic tensor is exact if and only if

$$0 = \sum_{k=1}^{n}\sum_{l=1}^{n}\Big(X_{ikl} + \frac{1}{n}\Big)\Big(X_{jkl} + \frac{1}{n}\Big)$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} X_{ikl} X_{jkl} + \frac{1}{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \big(X_{ikl} + X_{jkl} \big) + 1$$

for all $1 \leq i \neq j \leq n$. By Lemma 5.3, the linear terms vanish, and the result follows. $\square$

Note that the exactness condition (5.8) is not homogeneous. On the other hand, the following result provides a homogeneous condition for locating exact triply stochastic cubic tensors.

Theorem 5.5. *Suppose that* $[X_{ijk} \mid 1 \leq i, j, k \leq n]$ *are homogeneous coordinates satisfying the conditions* (5.5)–(5.7). *Then they specify a line, in the linear hull of the polytope* Λ_n *centered at* O*, passing through an exact triply stochastic cubic tensor if and only if the homogeneous coordinates* $[X_{ij1}, \ldots, X_{ijn}]$ *satisfy the Partition Condition, for* $1 \leq i, j \leq n$.

Proof. A triply stochastic cubic tensor $[T_i]_{jk}$, written as in (5.4), is exact if and only if each doubly stochastic matrix T_i, for $1 \leq i \leq n$, is exact. By Theorem 3.5, this happens if and only if, for $1 \leq i, j \leq n$, the homogeneous coordinates $[X_{ij1}, \ldots, X_{ijn}]$ satisfy the Partition Condition. $\square$

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