A NOTE ON STATIC MANIFOLDS AND ALMOST RICCI SOLITONS

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Abstract. In this short paper, we investigate the existence of non-trivial almost Ricci solitones on static manifolds. As a result we show any compact nontrivial static manifold is isometric to a Euclidean sphere.

1. Introduction

In [2], Corvino studied localized scalar curvature deformation of a Riemannian metric and introduced the following definition:

Definition 1.1. A Riemannian metric $g$ is called static on a manifold $M$ if the linearized scalar curvature map at $g$ has a nontrivial cokernel, i.e., if there exists a nontrivial function $f$ on $M$ such that

\begin{align*}
-\Delta(f)g + \nabla^2 f - f\text{Ric} &= 0.
\end{align*}

Here $\nabla^2$, $\Delta$ and $\text{Ric}$ denote the Hessian, the Laplacian and the Ricci curvature of $g$, respectively.

A nontrivial solution $f$ to (1.1) has been called a static potential if it exists. It is proved that a static metric (as defined above) must have constant scalar curvature [2]. When this constant is zero (which is always the case for an asymptotically flat, static metric), (1.1) becomes

\begin{align*}
\nabla^2 f &= f\text{Ric} \quad \text{and} \quad \Delta f = 0.
\end{align*}

Some investigations around static manifolds can be found in [3, 5, 6].

On the other hand, the concept of almost Ricci soliton was introduced in a recent paper due to Pigola et al. [4], where essentially they modified the definition of Ricci solitons by adding the condition on the parameter $\lambda$ to be a variable function. More precisely:

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Definition 1.2. A Riemannian manifold \((M, g)\) is an almost Ricci soliton if there exist a complete vector filed \(X\) and a smooth soliton function \(\lambda : M \to \mathbb{R}\) satisfying:
\[
\text{Ric} + \frac{1}{2}L_X g = \lambda g,
\]
where \(L_X\) denotes the Lie derivative in the direction of \(X\).

Almost Ricci soliton will be called expanding, steady or shrinking, respectively if \(\lambda < 0\), \(\lambda = 0\) or \(\lambda > 0\). When the vector filed \(X\) is a gradient of a smooth function \(f : M \to \mathbb{R}\), the manifold will be called a gradient almost Ricci soliton. In this case the soliton equation (1.3) turns out to be:
\[
\text{Ric} + \nabla^2 f = \lambda g.
\]
Moreover, when \(X\) is a Killing vector filed, the almost Ricci soliton will be called trivial, otherwise it will be a nontrivial almost Ricci soliton [4].

Barros et al. proved that:

**Theorem 1.3** ([1]). Every compact nontrivial almost Ricci soliton with constant scalar curvature is gradient.

2. Main results

In this section, we investigate the existence of gradient almost Ricci solitones on static manifolds. As a result we prove any compact nontrivial static manifold is isometric to a Euclidean sphere. At first, we show:

**Theorem 2.1.** Almost Ricci solitons on static manifold \((M, g)\) are gradient.

*Proof.* Since static manifold \((M, g)\) has constant scalar curvature, the result is obtained from Theorem 1.3. □

Finally, we obtain the following theorem:

**Theorem 2.2.** Every compact static manifold \((M, g)\) with static potential \(f\) has a gradient almost Ricci solitons.

*Proof.* Let \((M, g)\) be a compact static manifold with static potential \(f\). Taking the trace of (1.1), we have
\[
\Delta f - n\Delta f - fR = 0,
\]
where \(R\) is the scalar curvature of \(g\). Thus we have
\[
(1 - n)\Delta f - fR = 0.
\]
Consequently
\[
\Delta f = \frac{R}{1 - n} f.
\]
On the other hand, computing the trace of equation (1.4) yields that
\[
R + \Delta f = n\lambda.
\]
Hence,
\begin{equation}
\Delta f = n\lambda - R.
\end{equation}

Then, from equations (2.5) and (2.3) we arrive at
\begin{equation}
n\lambda - R + \frac{R}{n-1}f = 0.
\end{equation}

Therefore we acquire the smooth function $\lambda$ as follows:
\begin{equation}
\lambda = \left( \frac{1}{n} - \frac{f}{n(n-1)} \right) R.
\end{equation}

So we proved on the compact static manifold $(M, g)$ there exist a gradient almost Ricci soliton with potential function $f$ and a soliton function $\lambda$ as given by equation (2.7).

\begin{proof}
Corollary 2.3. Any compact nontrivial static manifold is isometric to a Euclidean sphere.

Since we have shown every compact static manifold $(M, g)$ with static potential $f$ has a gradient almost Ricci solitons, applying Corollary 1 in [1] we conclude that the every compact nontrivial static manifold isometric to a Euclidean sphere.
\end{proof}

References


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