THE EXCEPTIONAL SET OF ONE PRIME SQUARE
AND FIVE PRIME CUBES

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Abstract. For a natural number \( n \), let \( R(n) \) denote the number of representations of \( n \) as the sum of one square and five cubes of primes. In this paper, it is proved that the anticipated asymptotic formula for \( R(n) \) fails for at most \( O(N^{4/9} + \varepsilon) \) positive integers not exceeding \( N \).

1. Introduction

Let \( \tilde{R}(n) \) denote the number of representations of \( n \) in the form
\[
n = x^2 + x_1^3 + x_2^3 + \cdots + x_5^3,
\]
where \( x \in \mathbb{N} \) and \( x_i \in \mathbb{N} \) \((1 \leq i \leq 5)\). We write \( e(\alpha) \) for \( e^{2\pi i \alpha} \) and \( e_q(\alpha) \) for \( e(\alpha/q) \). Vaughan [5] proved that
\[
\tilde{R}(n) \sim \frac{\Gamma(\frac{3}{2})^5 \Gamma(\frac{4}{3})}{\Gamma(\frac{13}{6})} \tilde{S}(n)n^{\frac{7}{6}},
\]
where
\[
\tilde{S}(n) = \sum_{q=1}^{\infty} \sum_{(a,q)=1}^{r} S_2(q,a)S_3(q,a)^5 e_q(-an) q^6,
\]
with
\[
S_k(q,a) = \sum_{r=1}^{q} e_q(ar^k) \ (k = 2, 3).
\]

In view of Vaughan’s result, it is reasonable to propose the conjecture that for every sufficiently large even integer \( n \), the equation
\[
n = p^2 + p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3
\]

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is solvable, where and below the letter $p$, with or without subscripts, denotes a prime number, unless otherwise indicated. However, this conjecture is at present far from being solved.

Let $P_r$ denote an almost-prime with at most $r$ prime factors, counted according to multiplicity. By $R(n)$ we denote the number of representations of $n$ in the form (1.5). The previous best result of $R(n)$ is due to Cai [1] who obtained $R(n) \gg n^{19/18} \log n$ with $p$ an almost-prime $P_{36}$.

Another topic in the study of (1.1) and (1.5) concerns the exceptional sets $\tilde{E}(N)$ and $E(N)$. Let $\tilde{E}(N)$ denote the number of all the positive integers $n$ not exceeding $N$ which cannot be written as (1.1). A conventional application of Bessel’s inequality yields that $\tilde{E}(N) \ll \psi(N)^2 N^{1/3} + \varepsilon$, where $\psi(N)$ denotes a function of uniform growth with $\psi(N) = O(N^\delta)$ for some sufficiently small positive number $\delta$.

In 2001, Wooley [7] made an important breakthrough in the study of $\tilde{E}(N)$, and showed that $\tilde{E}(N) \ll \psi(N) \exp(c \log N / \log \log N)$, where $c$ is a positive number. Wooley’s methods avoid a conventional application of Bessel’s inequality in favour of explicit control of an exponential sum over the exceptional set itself.

It is of interest to investigate the exceptional set $E(N)$ of (1.5), which denotes the number of integers $n \leq N$ and cannot be represented as the sum of one square and five cubes of primes.

In this paper, we apply the Hardy-Littlewood method to establish the following result.

**Theorem 1.** For a natural number $n$, let $R(n)$ denote the number of representations of $n$ as the sum of one square and five cubes of primes, $\varphi(q)$ denote Euler’s function and $E(N)$ denote the number of integers $n \leq N$ such that the asymptotic formula

$$R(n) = \frac{\Gamma(\frac{3}{2}) \Gamma^5(\frac{1}{2})}{\Gamma(\frac{13}{6})} \mathcal{S}(n) \frac{n^{1/2}}{\log^6 n} + O \left( \frac{n^{1/2} \log \log n}{\log^7 n} \right)$$

fails to hold, where

$$\mathcal{S}(n) = \sum_{q=1}^{\infty} \sum_{(a,q)=1}^{\varphi(q)} \sum_{r=1}^{q} \frac{S_2^*(q,a) S_3^*(q,a) e_q(-an)}{\varphi^6(q)},$$

$$S_k^*(q,a) = \sum_{r=1 \atop (r,q)=1}^{q} e_q(a r^k).$$

Then for any $\varepsilon > 0$, we have $E(N) \ll N^{\frac{4}{5} + \varepsilon}$.

From Theorem 1 and Lemma 2.5, we get the following theorem.

**Theorem 2.** For any $\varepsilon > 0$, we have $E(N) \ll N^{\frac{4}{5} + \varepsilon}$. 

2. Notation and some preparatory lemmas

Throughout this paper, by \( n \) we denote a sufficiently large integer which satisfies \( 0.5N \leq n \leq N \). In addition, let \( A = 10^{10} \), \( \varepsilon \in (0, 10^{-10}) \) be an arbitrarily small positive constant and \( c \) be a positive constant not necessarily the same in different formulae. The letter \( p \), with or without subscripts, is reserved for a prime number. We use \( \ll \) and \( \gg \) to denote Vinogradov’s well-known notation, implicit constants depending at most on \( \varepsilon \) and \( A \). As usual, we use \( \varphi(q) \) and \( d(n) \) to denote Euler’s function and Dirichlet’s divisor function. \( e(\alpha) \) stands for \( e^{2\pi i \alpha} \) and \( e_q(\alpha) = e(\alpha/q) \). We denote by \( \sum_{r(q)} \), sums with \( r \) running over a reduced system of residues modulo \( q \). We set

\[
N^{\frac{1}{2}} \leq X \leq N^{\frac{1}{2}}/2, \quad Q_0 = \log^A N, \quad Q_1 = N^{\frac{1}{2}}, \quad Q_2 = N^{\frac{1}{2}}.
\]

Then we have the Farey dissection running over a reduced system of residues modulo \( q \). We set

\[
S_k(q,a) = \sum_{r(q)} e_q(an), \quad \mathcal{G}(n) = \sum_{q=1}^{\infty} \sum_{a=q}^{\infty} \frac{S_2^* (q,a) S_3^* (q,a)}{\varphi(q)^6}.
\]

For \((a,q) = 1, 1 \leq a \leq q, \) put

\[
\mathcal{M}_0 = \bigcup_{1 \leq q \leq Q_0} \bigcup_{a=1}^{q} \left( \frac{a}{q} - \frac{Q_0^A}{N}, \frac{a}{q} + \frac{Q_0^A}{N} \right],
\]

\[
\mathcal{M}_0 = \left( \frac{1}{Q_2}, 1 + \frac{1}{Q_2} \right], \quad m = \mathcal{M}_0 \setminus \mathcal{M}_0.
\]

Then we have the Farey dissection

\[
(2.1) \quad \mathcal{M}_0 = \mathcal{M}_0 \cup m.
\]

**Lemma 2.1.** Define \( \omega_3(q) = 3q^{-\frac{1}{2}} \) and let \( \mathcal{M}(X) \) be the union of the intervals \( \mathcal{M}(q,a,X) \) for \( 1 \leq a \leq q \leq X^{\frac{1}{2}} \) and \((a,q) = 1 \), where \( \mathcal{M}(q,a,X) = \{ \alpha : |qa-a| \leq X^{-\frac{1}{2}} \} \). Let

\[
\mathcal{J}_0(X) = \sup_{\gamma \in [0,1]} \int_{\mathcal{M}(X)} \frac{\omega_3^2(q)g_3(\alpha + \gamma)^2}{(1 + X^3 |\alpha - \frac{\gamma}{q}|)^2} \, d\alpha.
\]

Suppose that \( G(\alpha) \) is an integrable function of period one. Then we have

\[
\int_{m} h(\alpha) G(\alpha) g_3(\alpha) \, d\alpha \ll X^{\frac{1}{2}} \left( \int_{m} |G(\alpha)|^2 \, d\alpha \right)^{\frac{1}{2}} \mathcal{J}_i + X^{\frac{1}{2} + \epsilon} \mathcal{J},
\]

where

\[
\mathcal{J} = \int_{m} |G(\alpha)| g_3(\alpha) \, d\alpha,
\]

and

\[
(2.2) \quad \mathcal{J}_0(X) \ll N^{\frac{1}{2}} X^{-3} (\log N)^c.
\]
Proof. Lemma 2.1 is Lemma 3.1 in Zhao [10] and the estimate (2.2) is Lemma 2.2 in Zhao [10]. □

Lemma 2.2. For $\alpha \in m_1$, we have
$$g_3(\alpha) \ll N^{\frac{1}{12}+\varepsilon}.$$  

Proof. See Lemma 2.5 i) in Liu [4]. □

Lemma 2.3. For $\alpha = \frac{a}{q} + \lambda$, $(a, q) = 1$, $q \leq Q \leq X$ and $|\lambda| \leq \frac{Q}{X^2}$, we have
$$g_k(\alpha, X) \ll Q^{\frac{1}{2}} X^\frac{11}{20} + V_k(\alpha, X),$$

where
$$V_k(\alpha, X) = \frac{X(\log N)^c}{q^{\frac{1}{2}-\varepsilon}(1 + X^k|\lambda|)^{\frac{1}{2}}}.$$  

Proof. See Theorem 2 in Kumchev [3]. □

Lemma 2.4. Let
$$u_k(\lambda) = \sum_{2 < n \leq N} \frac{e(n\lambda)}{n^{1-\frac{1}{k}}} \log n.$$  

Then for $\alpha = \frac{a}{q} + \lambda \in M_0$, we have
$$g_k(\alpha) = S^*_k(q, a) \varphi(q) u_k(\lambda) + O \left(N^{\frac{1}{2}} \exp(-\log^2 N)\right).$$  

Proof. See Hua [2, Lemma 7.15]. □

Lemma 2.5. The series $\mathfrak{S}(n)$ is convergent and satisfies
$$0 < c^* \leq \mathfrak{S}(n) \ll d(n),$$

where positive constant $c^*$ is defined in Lemma 4.6 in Xue, Zhang and Li [9].

Proof. See (2.4) in Xue, Zhang and Li [9]. □

Lemma 2.6. For $(a, q) = 1$, we have
$$|S^*_k(q, a)| \ll q^{\frac{1}{2}+\varepsilon}.$$  

Proof. See Vinogradov [6, Chapter VI]. □

3. Auxiliary estimates

We initiate our proof by recalling the Farey dissection (2.1) that
$$R(n) = \int_{\mathbb{Q}_0} g_2(\alpha)g^*_2(\alpha)e(-\alpha n)d\alpha + \int_m g_2(\alpha)g^*_2(\alpha)e(-\alpha n)d\alpha.$$
3.1. The evaluation of the integral over \( \mathcal{M}_0 \)

**Proposition.** For \( \frac{N}{2} < n \leq N \), we have

\[
\int_{\mathcal{M}_0} g_2(\alpha)g_3^5(\alpha)e(-\alpha n)d\alpha = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})}{\Gamma(\frac{13}{6})} \Theta(n) n^{\frac{2}{3}} + O \left( \frac{n^{\frac{2}{3}} \log \log n}{\log n} \right).
\]

**Proof.** For \( \alpha = \frac{a}{q} + \lambda \), let

\[
S_k^*(q,a) = S_k^*(q,a)/\varphi(q) \quad \text{and} \quad S(n) = \sum_{q=1}^{\infty} A(q,n).
\]

Then it follows from Lemma 2.4 that

\[
\int_{\mathcal{M}_0} g_2(\alpha)g_3^5(\alpha)e(-\alpha n)d\alpha = \int_{\mathcal{M}_0} f_2(\alpha)f_3^5(\alpha)e(-\alpha n)d\alpha + O \left( \frac{n^{\frac{2}{3}} \log \log n}{\log n} \right).
\]

It is easy to see that

\[
\int_{\mathcal{M}_0} f_2(\alpha)f_3^5(\alpha)e(-\alpha n)d\alpha = \sum_{q \leq Q_{100}} A(q,n) \int_{|\lambda| \leq \frac{Q_{100}}{\sqrt{N}}} u_2(\lambda)u_3^5(\lambda)e(-\lambda n)d\lambda.
\]

Moreover, it follows from [2, Lemma 7.16] that

\[
\int_{|\lambda| \leq \frac{Q_{100}}{\sqrt{N}}} u_2(\lambda)u_3^5(\lambda)e(-\lambda n)d\lambda = \int_{0}^{1} u_2(\lambda)u_3^5(\lambda)e(-\lambda n)d\lambda + O \left( \int_{\frac{Q_{100}}{\sqrt{N}}}^{1} \frac{1}{\lambda^{\frac{7}{6}} n^{\frac{1}{6}}} d\lambda \right)
\]

\[
= \int_{0}^{1} u_2(\lambda)u_3^5(\lambda)e(-\lambda n)d\lambda + O(N^{\frac{2}{3}} Q_{100}^{-100}).
\]

Similar to [2, Lemma 7.19], we have

\[
\int_{0}^{1} u_2(\lambda)u_3^5(\lambda)e(-\lambda n)d\lambda = \frac{\Gamma(\frac{3}{2})\Gamma^5(\frac{4}{3})}{\Gamma(\frac{13}{6})} n^{\frac{2}{3}} + O \left( \frac{n^{\frac{2}{3}} \log \log n}{\log n} \right).
\]

By (3.3) and (3.4), we obtain

\[
\int_{|\lambda| \leq \frac{Q_{100}}{\sqrt{N}}} u_2(\lambda)u_3^5(\lambda)e(-\lambda n)d\lambda = \frac{\Gamma(\frac{3}{2})\Gamma^5(\frac{4}{3})}{\Gamma(\frac{13}{6})} n^{\frac{2}{3}} + O \left( \frac{n^{\frac{2}{3}} \log \log n}{\log n} \right).
\]
From Lemma 2.6 and the inequality \( \varphi(q) \gg \frac{q}{\log q} \), we get
\[
\sum_{q \leq Q_0^{100}} A(q, n) = \mathbb{S}(n) - \sum_{q > Q_0^{100}} A(q, n)
\]
\[
= \mathbb{S}(n) + O\left( \sum_{q > Q_0^{100}} q \left( \frac{q^{1+\varepsilon}}{\log q} \right)^{5} \right)
\]
\[
= \mathbb{S}(n) + O\left( \sum_{q > Q_0^{100}} q^{-2+\varepsilon} \right)
\]
(3.6)
\[
= \mathbb{S}(n) + O(Q_0^{-100+\varepsilon}).
\]

On combining (3.1), (3.2), (3.5) and (3.6), we get
\[
\int_{\mathbb{M}} g_2(\alpha) g_3^{5}(\alpha) e(-\alpha n) d\alpha = \frac{\Gamma(\frac{3}{2}) \Gamma^5(\frac{4}{3})}{\Gamma(\frac{13}{6}) \log \frac{6}{n}} \mathbb{S}(n) n^{\frac{7}{6}} + O\left( n^{\frac{7}{6}} \log \log n \log \frac{7}{n} \right).
\]

This completes the proof of the Proposition. \( \square \)

3.2. The estimation of the integrals over \( m_j (j = 1, 2) \)

We divide \( m \) into two parts \( m_1 \) and \( m_2 \), where
\[
\mathfrak{M} = \bigcup_{1 \leq q \leq Q_1} \bigcup_{a,q=1}^{q} \left( \frac{a}{q} - \frac{Q_1}{q^{N}} \frac{a}{q} + \frac{Q_1}{q^{N}} \right),
\]
\[
m_1 = \mathfrak{M} \setminus \mathfrak{M}_0, \quad m_2 = \mathfrak{M} \setminus \mathfrak{M}.
\]

And we simply divide the range of \( n \) into dyadic intervals, and denote by \( Z_j(N) \) the set of integers with \( \frac{N}{2} < n \leq N \) for which the inequality
\[
\left| \int_{m_j} g_2(\alpha) g_3^{5}(\alpha) e(-\alpha n) d\alpha \right| > \frac{n^{\frac{7}{6}}}{\log \frac{7}{n}}
\]
holds. For simplicity, we abbreviate the cardinality of \( Z_j(N) \) to \( Z_j \). Next, define the complex number \( \eta_n \) by taking \( \eta_n = 0 \) for \( n \notin Z_j(N) \), and for \( n \in Z_j(N) \) by means of the equation
\[
\left| \int_{m_j} g_2(\alpha) g_3^{5}(\alpha) e(-\alpha n) d\alpha \right| = \eta_n \int_{m_j} g_2(\alpha) g_3^{5}(\alpha) e(-\alpha n) d\alpha.
\]

Clearly, one has \( |\eta_n| = 1 \) whenever \( \eta_n \) is non-zero. Thus, we have
\[
\sum_{n \in Z_j(N)} \eta_n \int_{m_j} g_2(\alpha) g_3^{5}(\alpha) e(-\alpha n) d\alpha = \int_{m_j} g_2(\alpha) g_3^{5}(\alpha) K_j(\alpha) d\alpha,
\]
where the exponential sum $K_j(\alpha)$ is defined by

$$K_j(\alpha) = \sum_{n \in Z_j(N)} \eta_n e(-\alpha n).$$

Let

$$I_j = \int_{m_j} g_2(\alpha)g_3^5(\alpha)K_j(\alpha) d\alpha \quad (j = 1, 2).$$

By (3.7)-(3.9), we get

$$I_j > \sum_{n \in Z_j(N)} \frac{n^\frac{7}{6}}{\log n} \gg Z_j N^\frac{7}{6},$$

By (3.10),

$$I_j \leq \sum_{n \in Z_j(N)} \frac{n^\frac{7}{6}}{\log n} \gg Z_j N^\frac{7}{6}.$$

### 3.2.1. The estimation of $Z_1$. We now establish our estimate for $Z_1$. It is easy to see that

$$I_1 \ll (\log N) \max_{N^{\frac{7}{24}} \leq X \leq N^{\frac{2}{3}}} \left| \int_{m_1} g_3(\alpha, X)g_2(\alpha)g_3^4(\alpha)K_1(\alpha) d\alpha \right|$$

$$+ N^{\frac{7}{24}} \int_{m_1} |g_2(\alpha)g_3^4(\alpha)K_1(\alpha)| d\alpha$$

$$= (\log N) \max_{N^{\frac{7}{24}} \leq X \leq N^{\frac{2}{3}}} |I_{11}| + I_{12}, \text{ say.}$$

It follows from Cauchy’s inequality, Hua’s inequality and Wooley [8, Lemma 2.1] with $k = 2$ that

$$I_{12} \ll N^{\frac{7}{24}} \left( \int_{0}^{1} |g_3^2(\alpha)K_1(\alpha)| d\alpha \right)^{\frac{1}{2}} \left( \int_{0}^{1} |g_3^8(\alpha)| d\alpha \right)^{\frac{1}{2}}$$

$$\ll Z_j N^{\frac{7}{5} + \varepsilon} + Z_j^\frac{7}{5} N^{\frac{7}{5} + \varepsilon}.$$ 

Furthermore, by taking $h(\alpha) = g_3(\alpha, X)$ and $G(\alpha) = g_2(\alpha)g_3^4(\alpha)K_1(\alpha)$ in Lemma 2.1, we have

$$I_{11} = \int_{m_1} |g(\alpha)G(\alpha)h(\alpha)| d\alpha$$

$$\ll X^J \left( \int_{m_1} |G(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} J^{\frac{1}{2}} + X^{\frac{7}{8} + \varepsilon} J,$$

where $J = \int_{m_1} |g_2(\alpha)g_3^4(\alpha)K_1(\alpha)| d\alpha$.

Similar to (3.12), we have

$$J \ll Z_j N^{\frac{7}{8} + \varepsilon} + Z_j^\frac{7}{8} N^{\frac{7}{8} + \varepsilon}.$$ 

We deduce from Lemma 2.2 and Hua’s inequality that

$$\int_{m_1} |G(\alpha)|^2 d\alpha \ll Z_j^2 \max_{\alpha \in m_1} |g_3^2(\alpha)| \times \left( \int_{0}^{1} |g_2^4(\alpha)| d\alpha \right)^{\frac{1}{2}} \left( \int_{0}^{1} |g_3^8(\alpha)| d\alpha \right)^{\frac{1}{2}}$$
On combining Lemma 2.1, (3.13)-(3.15), we have
\[ I_{11} \ll Z_1 X^{1/2+\varepsilon} N^{3/4+\varepsilon} + Z_1 X^{1/2+\varepsilon} N^{1/2+\varepsilon}, \]
(3.16)
Therefore, we obtain from (3.11)-(3.12) and (3.16) that
\[ I_1 \ll Z_1 N^{3/4+\varepsilon} + Z_1^{3/4} N^{3/4+\varepsilon} + Z_1^{1/2} N^{1/4+\varepsilon}. \]
(3.17)
It follows from (3.10) and (3.17) that
\[ Z_1 \ll N^{1/2+\varepsilon}. \]
(3.18)
\[ 3.2.2. \textit{The estimation of } Z_2. \] Define \( V_k(\alpha) = \frac{N^{4/3}(\log N)^{\varepsilon}}{q^{2-\varepsilon}(1+N|\lambda|)^{1/2}}. \) On making use of Lemma 2.3, it is readily seen that
\[ g_2(\alpha) \ll N^{4/3+\varepsilon} + (\log N) \max_{N^{4/3} \leq X \leq N} |g_2(\alpha, X)| \]
(3.19)
\[ \ll N^{4/3+\varepsilon} + V_2(\alpha). \]
Similarly, we have
\[ g_3(\alpha) \ll N^{4/3+\varepsilon} + (\log N) \max_{N^{4/3} \leq X \leq N} |g_3(\alpha, X)| \]
(3.20)
\[ \ll N^{4/3+\varepsilon} + V_3(\alpha). \]
It follows from (3.19)-(3.20) that
\[ I_2 \ll N^{4/3+\varepsilon} \int_{m_2} |g_3^5(\alpha)K_2(\alpha)|d\alpha + \int_{m_2} |V_2(\alpha)g_3^5(\alpha)K_2(\alpha)|d\alpha \]
\[ \ll N^{4/3+\varepsilon} \int_{m_2} |g_3^5(\alpha)K_2(\alpha)|d\alpha + N^{4/3+\varepsilon} \int_{m_2} |g_3^5(\alpha)V_3^3(\alpha)K_2(\alpha)|d\alpha \]
\[ + N^{4/3+\varepsilon} \int_{m_3} |V_2(\alpha)g_3^5(\alpha)K_2(\alpha)|d\alpha + \int_{m_3} |V_2(\alpha)V_3^3(\alpha)g_3^5(\alpha)K_2(\alpha)|d\alpha \]
(3.21)
\[ \ll I_{21} + I_{22} + I_{23} + I_{24}. \]
We have
\[ I_{21} \ll Z_2 N^{4/3+\varepsilon} \left( \int_{m_2} 1d\alpha \right)^{3/2} \left( \int_0^1 |g_3(\alpha)|^6d\alpha \right)^{1/2} \]
(3.22)
\[ \ll Z_2 N^{4/3+\varepsilon}, \]
where \(|K_2(\alpha)| \ll Z_2, \) H"older's inequality and Hua's inequality are employed. Moreover, by the obvious estimate \(|V_2(\alpha)| \leq N^{1/2}Q_0^{-A}, \) \(|V_3(\alpha)| \ll N^{1/2} \) and the
arguments similar to that leading to (2.2), we get

\[ I_{22} \ll Z_2 N \frac{41}{50} + \epsilon (N^\frac{1}{2})^3 \sum_{q \leq N^\frac{1}{2}} \sum_{a \equiv 1 (q)} \int_{|\alpha - \frac{a}{q}| \leq \frac{1}{\sqrt{q}}} \left| \sum_{p \leq N^\frac{1}{2}} c(p^3 \alpha)^2 (\log N)^c \right| q(1 + N|\alpha - \frac{a}{q}|) d\alpha \]

(3.23) \quad \ll Z_2 N^{\frac{41}{50}} + \epsilon ,

\[ I_{23} \ll Z_2 N^\frac{2}{5} + \epsilon (N^\frac{1}{2}) \sum_{q \leq N^\frac{1}{2}} \sum_{a \equiv 1 (q)} \int_{|\alpha - \frac{a}{q}| \leq \frac{1}{\sqrt{q}}} \left| \sum_{p \leq N^\frac{1}{2}} c(p^3 \alpha)^2 (\log N)^c \right| q(1 + N|\alpha - \frac{a}{q}|) d\alpha \]

(3.24) \quad \ll Z_2 N^{\frac{2}{5}} + \epsilon ,

and

\[ I_{24} \ll \max_{\alpha \in m_2} |V_2(\alpha)|(N^\frac{1}{2})^3 \sum_{q \leq N^\frac{1}{2}} \sum_{a \equiv 1 (q)} \int_{|\alpha - \frac{a}{q}| \leq \frac{1}{\sqrt{q}}} \left| \sum_{p \leq N^\frac{1}{2}} c(p^3 \alpha)^2 (\log N)^c \right| q(1 + N|\alpha - \frac{a}{q}|) d\alpha \]

(3.25) \quad \ll Z_2 N^\frac{2}{5} Q_0^{-A} .

A combination of (3.21)-(3.25) then yields

(3.26) \quad I_2 \ll \frac{Z_2 N^\frac{2}{5}}{\log A N} ,

4. Proof of Theorem 1

Let \( Z(N) \) denote the number of integers \( n \) in the interval \([N^\frac{3}{4}, N]\) such that the asymptotic formula (1.6) fails to hold. On recalling (3.18) and (3.26), we arrive at the conclusion that

\[ Z(N) \leq Z_1 + Z_2 \ll N^\frac{3}{4} + \epsilon , \]

and

\[ E(N) \leq N^\frac{3}{4} + \epsilon \sum_{0 \leq j \leq J} Z \left( \frac{N}{2^j} \right) \ll N^\frac{3}{4} + \epsilon , \]

where \( J \) is chosen in such a way that \( 2^{J-1} < N^\frac{3}{4} \leq 2^J \). Now the proof of Theorem 1 is completed.

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