East Asian Math. J.
Vol. 38 (2022), No. 3, pp. 347–355
http://dx.doi.org/10.7858/eamj.2022.022

ON THE EQUATIONS DEFINING SOME CURVES OF MAXIMAL REGULARITY IN $\mathbb{P}^5$

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Abstract. For a nondegenerate projective variety, it is a classical problem to study its defining equations with respect to a given embedding. In this paper, we precisely determine minimal sets of generators of the defining ideals of some curves of maximal regularity in $\mathbb{P}^5$.

1. Introduction

Throughout this paper, we work over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. Let $\mathbb{P}^r$ and $R := \mathbb{K}[X_0, X_1, \cdots, X_r]$ be respectively the projective r-space over $\mathbb{K}$ and the homogeneous coordinate ring of $\mathbb{P}^r$. Let $C \subset \mathbb{P}^r$ be a nondegenerate irreducible curve and I_C be the homogeneous ideal of C in R. To understand the curve C, it is natural to study the defining equations of I_C and the syzygies among them. To the authors' knowledge, it should be one of the most difficult problems in projective algebraic geometry and several results are known. As the simplest case, the problems to guarantee that I_C is generated by quadrics and that the first few syzygy modules are generated by linear syzygies are establishing (see [1], [3], [4], [5], [15], [16] and so on). Also the defining equations of Veronese varieties, rational normal scrolls and Segre varieties are well understood (see [7]). In [8], [10] and [11], one of the authors in the present paper provided a complete description of defining equations of non-normal del Pezzo varieties.

In this line, we continue the study begun in [13] and [12] to describe a minimal set of generators of the defining ideal I_{C_d} of rational curves $C_d \subset \mathbb{P}^r$, $d \geq r$ parameterized as

$$C_d = \{[s^d(P) : s^{d-1}t(P) : s^{r-2}t^{d-r+2}(P) : s^{r-3}t^{d-r+3}(P) : \cdots : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\} \tag{1}$$

Received April 9, 2022; Accepted April 30, 2022.
2010 *Mathematics Subject Classification.* 14C17, 14M20, 14Q10.
Key words and phrases. Castelnuovo-Mumford Regularity, rational normal surface scroll, rational curve, minimal generator.
This work was supported by a Research Grant of Pukyong National University(2021).
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(pISSN 1226-6973, eISSN 2287-2833)

where $T := \mathbb{K}[s,t]$ be the homogeneous coordinate ring of $\mathbb{P}^1$. In [13] and [12], the authors provide a complete description of defining equations for the cases where $r=3$ and $r=4$ in (1). The results are

Theorem 1.1 ([13]). *Let $C_d \subset \mathbb{P}^3$, $d \geq 3$ be a curve defined as the parametrization*

$$C_d = \{[s^d(P) : s^{d-1}t(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}.$$

Then C_d is a smooth rational curve of degree d and of maximal regularity $d-1$. In particular, the defining ideal I_{C_d} of C_d is minimally generated as follows:

$$I_{C_d} = \langle X_0X_3 - X_1X_2, F_1, F_2, \ldots, F_{d-1}\rangle$$

where $F_i = X_0^{d-i-1}X_2^i - X_1^{d-i}X_3^{i-1}$ for $1 \leq i \leq d-1$.

Theorem 1.2 ([12]). *Let $C_d \subset \mathbb{P}^4$, $d \geq 4$ be a curve defined as the parametrization*

$$C_d = \{[s^d(P) : s^{d-1}t(P) : s^2t^{d-2}(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}$$

Then C_d is a smooth rational curve of degree d and of maximal regularity $d-2$. In particular, the defining ideal I_{C_d} of C_d is minimally generated as follows: For $n \geq 2$,

(1) *If $d = 2n$, then*

$$I_{C_d} = \langle Q_1, Q_2, Q_3, G_1, G_2, H_n, H_{n+1}, \cdots, H_{2n-2}\rangle.$$

(2) *If $d = 2n+1$, then*

$$I_{C_d} = \langle Q_1, Q_2, Q_3, F_n, F_{n+1}, \cdots, F_{2n-1}\rangle$$

where

$$Q_1 = X_0X_3 - X_1X_2, \quad Q_2 = X_0X_4 - X_1X_3, \quad Q_3 = X_2X_4 - X_3^2 \quad \textit{and}$$
$$\begin{cases} G_i = X_1X_3^{i-1}X_4^{n-i} - X_2^{n+i-2}X_3^{2-i} & \textit{for } i = 1,2 \\ H_{n+j-1} = X_0^{2j-1}X_2^{n-j} - X_1^{2j}X_4^{n-j-1} & \textit{for } 1 \leq j \leq n-1 \\ F_{n+i-1} = X_0^{2i-2}X_2^{n-i+1} - X_1^{2i-1}X_4^{n-i} & \textit{for } 1 \leq i \leq n. \end{cases}$$

As a next case, the main purpose of this article is to determine a minimal generating set of the defining ideal of rational curves parameterized as (1) for $r = 5$.

First we show that C_d is a smooth rational curve of degree d which is contained in the rational normal surface scroll $S(1,3)$ as a divisor $H + (d-4)F$ where H and F are respectively the hyperplane divisor and a ruling line (see Proposition 2.2). This observation enables us to obtain the exact structure of minimal generators of I_{C_d} in terms of graded Betti numbers thanks to [9, Theorem 1.2]. We also compute several examples by means of the Computer Algebra System SINGULAR [2] which pose the concrete expressions of minimal generators of I_{C_d} in Theorem 2.4. In our main result, Theorem 2.4 provides an explicit description of a set of minimal generators of the ideal I_{C_d} according to the degree $d = 3n-1$, $d = 3n$ and $d = 3n+1$ for $n \geq 3$. For $d = 5, 6, 7$ we also obtain the minimal generating sets of the ideal I_{C_d} in Example 2.5.

2. Main Theorem

Notation and Remarks 2.1. (a) Let $T := \mathbb{K}[s,t]$ be the homogeneous coordinate ring of $\mathbb{P}^1$. For each $k \geq 1$, we denote by T_k the k-th graded component of T.
(b) A rational normal curve $\widetilde{C} \subset \mathbb{P}^d$ of degree d parameterized as

$$\widetilde{C} = \{[s^d(P) : s^{d-1}t(P) : \cdots : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\} \qquad (2)$$

is defined by the common zero locus of the polynomials $F_{i,j} = X_iX_j - X_{i-1}X_{j+1}$ for $1 \leq i \leq j \leq d-1$. In particular, the defining ideal $I_{\widetilde{C}}$ of $\widetilde{C}$ is minimally generated by the set $\{F_{i,j} \mid 1 \leq i \leq j \leq d-1\}$.
(c) A nondegenerate rational curve $C \subset \mathbb{P}^r$ of degree d is given by a linear projection $\pi_\Lambda : \widetilde{C} \to \mathbb{P}^r$ of $\widetilde{C} \subset \mathbb{P}^d$ from a linear subspace $\Lambda \cong \mathbb{P}^{d-r-1}$ of $\mathbb{P}^d$. Indeed, the normalization of C can be realized as the rational normal curve $\widetilde{C}$, hence it follows that there exists a subset $\{f_0, f_1, \ldots, f_r\} \subset T_d$ of $\mathbb{K}$-linearly independent forms of degree d in T such that C is a curve parameterized as

$$C = \{[f_0(P) : f_1(P)) : \cdots : f_r(P)] \mid P \in \mathbb{P}^1\}.$$

(d) A rational normal surface scroll $S := S(a_1, a_2) \subset \mathbb{P}^r$ of degree $a_1 + a_2$ parameterized as

$$S = \{[s^{a_1}(P) : s^{a_1-1}t(P) : \cdots : t^{a_1}(P) : s^{a_2}(P) : s^{a_2-1}t(P) : \cdots : t^{a_2}(P)] \mid P \in \mathbb{P}^1\}$$

is defined by (2×2)-minors of the matrix

$$\begin{bmatrix} X_0 & X_1 & \cdots & X_{a_1-1} & X_{a_1+1} & X_{a_1+2} & X_{a_1+a_2-1} \\ X_1 & X_2 & \cdots & X_{a_1} & X_{a_1+2} & X_{a_1+3} & X_{a_1+a_2} \end{bmatrix}.$$

For $a_1 = 1$ and $a_2 = 3$, the defining ideal I_S of S is minimally generated by the following set

$$\{X_0X_3 - X_1X_2, X_0X_4 - X_1X_3, X_0X_5 - X_1X_4, X_2X_4 - X_3^2, X_2X_5 - X_3X_4, X_3X_5 - X_4^2\}$$

In particular, a divisor of S is written by $aH + bF$ where H and F are respectively the hyperplane divisor and a ruling line of S for $a, b \in \mathbb{Z}$.
(e) Let $X \subset \mathbb{P}^r$ be a nondegenerate projective variety. The graded Betti numbers, denoted by $\beta_{i,j}(X)$, of X are defined as

$$\beta_{i,j}(X) := \dim_{\mathbb{K}} \mathrm{Tor}_i^R(I_X, \mathbb{K})_{i+j}.$$

Then we call $\beta(X)$ the Betti table of X consists of $\beta_{i,j}(X)$ as an entry in the i-th column and j-th row. In particular, $\beta_{1,j}(X)$ corresponds to the number of minimal generators of degree j in the defining ideal I_X of X.
(f) $X \subset \mathbb{P}^r$ is said to be m-regular if $\beta_{i,m+1}(X) = 0$ for every $i \geq 1$. In particular, I_X is generated by forms of degree $\leq m$. The Castelnuovo-Mumford regularity (or simply the regularity) of X, denoted by $\mathrm{reg}(X)$, is defined as the least integer m such that X is m-regular(cf. [14]). In [6], the authors proved that

$\mathrm{reg}(X) \le d-r+2$ for $\dim_{\mathbb{K}}(X)=1$. They also provided a complete classification theory about curves of maximal regularity $d-r+2$. It is interesting that if $d \ge r+2$ then X is a curve of maximal regularity if and only if it is a smooth rational curve which admits a $(d-r+2)$-secant line $\mathbb{L}$

Keeping the notation as above, let $C_d \subset \mathbb{P}^5$ for $d \ge 5$ be a curve described as

$$C_d := \{[s^d(P) : s^{d-1}t(P) : s^3t^{d-3}(P) : s^2t^{d-2}(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\} \tag{3}$$

Let $R := \mathbb{K}[X_0, X_1, X_2, X_3, X_4, X_5]$ be the homogeneous coordinate ring of $\mathbb{P}^5$.

Proposition 2.2. *Let C_d be as above. Then,*

(1) *C_d is a smooth rational curve of degree d.*
(2) *C_d is contained in the rational normal surface scroll $S(1,3)$ as a divisor linearly equivalent to $H+(d-4)F$ where H and F are the hyperplane divisor and a ruling line, respectively.*
(3) *C_d is a curve of maximal regularity $d-3$ with a $(d-3)$-secant line $\mathbb{L}$ to C_d. In particular, $\mathbb{L}$ is the minimal section $S(1)$ of $S(1,3)$.*

Proof. (1) If $d=5$, then it follows immediate from Notation and Remarks 2.1.(b). Now suppose that $d>5$ and let $\widetilde{C} \subset \mathbb{P}^d$ be the rational normal curve defined as in Notation and Remarks 2.1.(b). Let Λ be a $(d-6)$-dimensional linear subspace of $\mathbb{P}^d$ spanned by $(d-5)$ standard coordinate points

$$\{[0,0,1,0,\ldots,0,0],[0,0,0,1,0,\ldots,0,0],\ldots,[0,0,\cdots,0,1,0,0,0,0]\}$$

and consider the linear projection map $\pi_\Lambda : \widetilde{C} \to \mathbb{P}^5$ of $\widetilde{C}$ from the center Λ. Then it holds that the projection image $\pi_\Lambda(\widetilde{C})$ is the curve C_d by the construction. In particular, the map π_Λ is an isomorphism since $\Lambda \cap \widetilde{C}^2 = \emptyset$ where $\widetilde{C}^2$ is the second join of $\widetilde{C}$ with itself. For details, we refer to the reader to see [17] or [12, Notation and Remarks 2.2].
(2) It is easy to see that the curve C_d satisfies the following six quadratic equations

$$\{X_0X_3-X_1X_2, X_0X_4-X_1X_3, X_0X_5-X_1X_4, X_2X_4-X_3^2, X_2X_5-X_3X_4, X_3X_5-X_4^2\}$$

which define the rational normal surface scroll $S := S(1,3)$(see Notation and Remarks 2.1.(d)). Thus it holds that $C_d \subset S$, and hence C_d is linearly equivalent to a divisor $H+(d-4)F$ of S. Indeed, we may assume that $C_d \equiv aH+bF$ for some integer $a \ge 1$ and b since C_d is irreducible. Consider the long exact sequence

$$\to H^1(\mathbb{P}^r, \mathcal{I}_S(1)) \to H^1(\mathbb{P}^r, \mathcal{I}_{C_d}(1)) \to H^1(S, \mathcal{O}_S((1-a)H-bF)) \to H^2(\mathbb{P}^r, \mathcal{I}_S(1)) \to \cdots$$

which is obtained from the exact sequence

$$0 \to \mathcal{I}_S \to \mathcal{I}_{C_d} \to \mathcal{O}_S(-aH-bF) \to 0.$$

Then it follows that

$$H^1(\mathbb{P}^r, \mathcal{I}_{C_d}(1)) \cong H^1(S, \mathcal{O}_S((1-a)H-bF)) \tag{4}$$

since S is arithmetically Cohen-Macaulay. Now suppose that $a \geq 2$. Then it holds that $H^1(S, \mathcal{O}_S((1-a)H - bF)) = 0$ and hence C_d is linearly normal by the isomorphism (4). This is a contradiction, and hence $a = 1$ since the map π_Λ in the proof of (1) is an isomorphism. So C_d is linearly equivalent to a divisor $H + bF$ of degree d on $S(1,3)$. Thus we conclude that $b = d - 4$.
(3) It is well known that $\operatorname{reg}(C_d) \leq d-3$ by [6]. On the other hand, the minimal section $S(1)$ is a $(d-3)$-secant line to C_d since $\sharp(C_d \cap S(1)) = (H+(d-4)F).(H-3F) = d-3$. Thus $\operatorname{reg}(C_d) \geq d-3$. This completes the proof. □

Corollary 2.3. *Let C_d be as in Proposition 2.2. Then the Betti table $\beta(C_d)$ of C_d is described as following Table 1:*

i	1	2	3	4	5
$\beta_{i,d-3}$	1	4	6	4	1
$\beta_{i,d-4}$	0	0	0	0	0
$\beta_{i,d-5}$	1	4	6	4	1
$\beta_{i,d-6}$	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,n+3}$	0	0	0	0	0
$\beta_{i,n+2}$	1	4	6	4	1
$\beta_{i,n+1}$	0	0	0	0	0
$\beta_{i,n}$	4	12	12	4	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,3}$	0	0	0	0	0
$\beta_{i,2}$	6	8	3	0	0

i	1	2	3	4	5
$\beta_{i,d-3}$	1	4	6	4	1
$\beta_{i,d-4}$	0	0	0	0	0
$\beta_{i,d-5}$	1	4	6	4	1
$\beta_{i,d-6}$	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,n+3}$	1	4	6	4	1
$\beta_{i,n+2}$	0	0	0	0	0
$\beta_{i,n+1}$	1	4	10	6	1
$\beta_{i,n}$	2	4	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,3}$	0	0	0	0	0
$\beta_{i,2}$	6	8	3	0	0

i	1	2	3	4	5
$\beta_{i,d-3}$	1	4	6	4	1
$\beta_{i,d-4}$	0	0	0	0	0
$\beta_{i,d-5}$	1	4	6	4	1
$\beta_{i,d-6}$	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,n+3}$	0	0	0	0	0
$\beta_{i,n+2}$	1	4	6	4	1
$\beta_{i,n+1}$	0	6	8	3	0
$\beta_{i,n}$	1	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\beta_{i,3}$	0	0	0	0	0
$\beta_{i,2}$	6	8	3	0	0

TABLE 1. $\beta(C_d)$ where $d = 3n-1$, $d = 3n$ and $d = 3n+1$, respectively.

Proof. We may consider the curve C_d as a divisor $H + (d-4)F$ on the rational normal surface scorll $S(1.3)$ by Proposition 2.2.(2). Then we get the desired Betti tables by applying the curve C_d to [9, Theorem 1.3]. □

As in [13] and [12], we try to construct many examples of minimal generating sets of defining ideal I_{C_d} of C_d for small $d \geq 5$ by means of the Computer Algebra System "SINGULAR" [2]. Our many computational examples and intuitions enable us to expect general shapes of the minimal generators of defining ideals I_{C_d} in Theorem 2.4.

To state our main theorem, we fix some notations for $n \geq 3$ and $j = -1, 0, 1$ as followings:

$$(i) \begin{cases} F_{[j,i]} = X_1 X_4^i X_5^{n-i-1} - X_2^{n+i+j-1} X_3^{1-j-i} & \text{for } 0 \leq i \leq 1-j \\ G_{[j,k]} = X_0^{3k+2+j} X_2^{n-k-1} - X_1^{3(k+1)+j} X_5^{n-k-2} & \text{for } 0 \leq k \leq n-2 \end{cases}$$

$$(ii) \begin{cases} Q_1 = X_0X_3 - X_1X_2,\ Q_2 = X_0X_4 - X_1X_3,\ Q_3 = X_0X_5 - X_1X_4 \\ Q_4 = X_2X_4 - X_3^2,\ Q_5 = X_2X_5 - X_3X_4,\ Q_6 = X_3X_5 - X_4^2 \end{cases}$$

and the set $\Sigma = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$ is a minimal generating set of the defining ideal of $S(1,3)$

Theorem 2.4. *Let $C_d \subset \mathbb{P}^5$, $d \geq 8$ be stated as in (3). Then C_d is a smooth rational curve of degree d and of maximal regularity $d-3$. In particular, letting $d = 3n+j$ for $n \geq 3$ and $j = -1, 0, 1$, the following sets are minimal generating sets of defining ideal I_{C_d} of C_d:*

(1) *If $d = 3n-1$, then*
$$\Sigma_{-1} = \Sigma \cup \{F_{[-1,0]}, F_{[-1,1]}, F_{[-1,2]}\} \cup \{G_{[-1,k]} \mid 0 \leq k \leq n-2\}$$
is a minimal generating set of I_{C_d}.

(2) *If $d = 3n$, then*
$$\Sigma_0 = \Sigma \cup \{F_{[0,0]}, F_{[0,1]}\} \cup \{G_{[0,k]} \mid 0 \leq k \leq n-2\}$$
is a minimal generating set of I_{C_d}.

(3) *If $d = 3n+1$, then*
$$\Sigma_1 = \Sigma \cup \{F_{[1,0]}\} \cup \{G_{[1,k]} \mid 0 \leq k \leq n-2\}$$
is a minimal generating set of I_{C_d}.

Proof. The first part of the theorem follows from Proposition 2.2. For the remaining parts of the proof, we will describe a minimal set of generators of the defining ideal I_{C_d} according to the degree $d = 3n+j$ with $n \geq 3$ and $j = -1, 0, 1$ in turn. By construction of the set Σ_j, one can easily check that Σ_{-1} (resp. Σ_0 and Σ_1) is contained in $I(S)$ when $d = 3n-1$ (resp. $d = 3n$ and $d = 3n+1$). Thus by applying the sets $\sum_j$ to Corollary 2.3, it suffices to show the following statements:

(i) When $d = 3n-1$,
- $(i.1)$ six quadratic equations in Σ are $\mathbb{K}$-linearly independent,
- $(i.2)$ $F_{[-1,0]} \notin \langle \Sigma \rangle$,
- $(i.3)$ $F_{[-1,1]} \notin \langle \Sigma \cup \{F_{[-1,0]}\} \rangle$,
- $(i.4)$ $F_{[-1,2]} \notin \langle \Sigma \cup \{F_{[-1,0]}, F_{[-1,1]}\} \rangle$,
- $(i.5)$ $G_{[-1,0]} \notin \langle \Sigma \cup \{F_{[-1,0]}, F_{[-1,1]}, F_{[-1,2]}\} \rangle$,
- $(i.6)$ $G_{[-1,k]} \notin \langle \Sigma \cup \{F_{[-1,0]}, F_{[-1,1]}, F_{[-1,2]}, G_{[-1,0]}, \ldots, G_{[-1,k-1]}\} \rangle$ for $1 \leq k \leq n-2$.

(ii) When $d = 3n$,
- $(ii.1)$ six quadratic equations in Σ are $\mathbb{K}$-linearly independent,
- $(ii.2)$ $F_{[0,0]} \notin \langle \Sigma \rangle$,
- $(ii.3)$ $F_{[0,1]} \notin \langle \Sigma \cup \{F_{[0,0]}\} \rangle$,
- $(ii.4)$ $G_{[0,0]} \notin \langle \Sigma \cup \{F_{[0,0]}, F_{[0,1]}\} \rangle$,
- $(ii.5)$ $G_{[0,k]} \notin \langle \Sigma \cup \{F_{[0,0]}, F_{[0,1]}, G_{[0,0]}, \ldots, G_{[0,k-1]}\} \rangle$ for $1 \leq k \leq n-2$.

(iii) When $d = 3n+1$,
- $(iii.1)$ six quadratic equations in Σ are $\mathbb{K}$-linearly independent,
- $(iii.2)$ $F_{[1,0]} \notin \langle \Sigma \rangle$,
- $(iii.3)$ $G_{[1,0]} \notin \langle \Sigma \cup \{F_{[1,0]}\} \rangle$,

$(iii.4)$ $G_{[1,k]} \notin \langle \Sigma \cup \{F_{[1,0]}, G_{[1,0]}, \ldots, G_{[1,k-1]}\}\rangle$ for $1 \le k \le n-2$.

It is obvious that six quadratic equations in Σ are $\mathbb{K}$-linearly independent since they are minimal generators of the ideal $I_{S(1,3)}$ of $S(1,3)$. Thus $(i.1)$, $(ii.1)$ and $(iii,1)$ are proved. Note that the degrees of $F_{[j,0]}$, $F_{[j,0]}$, $F_{[j,2]}$ $G_{[j,1]}, \ldots, G_{[j,k]}$ are at least 3 and the degrees of $G_{[j,k]}$ for $0 \le k \le n-2$ are strictly increasing. We shall finish the proof through the following two steps.

Step1: Suppose that $F_{[j,0]}$ is in the ideal $\langle \Sigma \rangle$ and $F_{[j,i]}$ is in the ideal $\langle \Sigma \cup \{F_{[j,0]}, \ldots, F_{[j,i-1]}\}\rangle$ for $i = 1,2$. Then $F_{[j,0]}$ and $F_{[j,i]}$ can be written as followings:

$$\begin{cases} F_{[j,0]} = \sum\limits_{t=1}^{6} A_{[j,t]} Q_t \quad \text{and} \\ F_{[j,i]} = \sum\limits_{t=1}^{6} A_{[j,t]} Q_t + \sum\limits_{s=0,}^{i-1} B_{[j,s]} F_{[j,s]} \quad \text{for } i = 1,2 \end{cases} \tag{5}$$

where $\{A_{[j,t]}\}$ for $t = 1,2,3,4,5,6$ are homogeneous polynomials of degree $n-2$ in R and $\{B_{[j,s]}\}$ for $0 \le s \le i-1$ are constants. Then for every points in the set $\{[0,0,X_2,X_3,0,0]\} \subset \mathbb{P}^5$, the equations (5) yields respectively

$$\begin{cases} X_2^{n+j-1} X_3^{1-j} = 0 \quad \text{and} \\ X_2^{n+i+j-1} X_3^{1-j-i} = \sum\limits_{s=0}^{i-1} B_{[j,s]} X_2^{n+s+j-1} X_3^{1-j-s} \quad \text{for } i = 1,2 \end{cases}$$

which cannot occur. This shows that $(i.2) \sim (i.4)$, $(ii.3), (ii.4)$ and $(iii,2)$.

Step2: Suppose that $G_{[j,0]}$ is contained in the ideal $\langle \Sigma \cup \{F_{[j,i]} | 0 \le i \le j-1\}\rangle$ and $G_{[j,k]}$ is contained in the ideal

$$\langle \Sigma \cup \{F_{[j,i]} | 0 \le i \le j-1\} \cup \{G_{[j,0]}, \ldots, G_{[j,k-1]}\}\rangle$$

for $1 \le k \le n-2$. Then $G_{[j,k]}$ is written by the following combinations

$$\begin{cases} G_{[j,0]} = \sum\limits_{t=1}^{6} A_{[j,t]} Q_t + \sum\limits_{i=0,}^{1-j} B_{[j,i]} F_{[j,i]} \quad \text{and} \\ G_{[j,k]} = \sum\limits_{t=1}^{6} A_{[j,t]} Q_t + \sum\limits_{i=0,}^{1-j} B_{[j,i]} F_{[j,i]} + \sum\limits_{s=0}^{k-1} C_{[j,s]} G_{[j,s]} \quad \text{for } 1 \le k \le n-2 \end{cases} \tag{6}$$

where $A_{[j,t]}$, $B_{[j,i]}$ and $C_{[j,s]}$ are respectively the homogeneous polynomials of degree $n+2k+j-1$, $2k+j+1$ and $2(k-s)$ in R. For every points in the set $\{[0,X_1,0,0,0,X_5]\} \subset \mathbb{P}^5$, the formulas in (6) will be rewritten as

$$\begin{cases} X_1^{3+j} X_5^{n-2} & = -B_{[j,0]} X_1 X_5^{n-1} \quad \text{and} \\ X_1^{3(k+1)+j} X_5^{n-k-2} & = -B_{[j,0]} X_1 X_5^{n-1} \\ & \quad + \sum\limits_{s=0}^{k-1} C_{[j,s]} X_1^{3(s+1)+j} X_5^{n-s-2} \quad \text{for } 1 \le k \le n-2 \end{cases} \tag{7}$$

It is clear that the equality $X_1^{3+j} X_5^{n-2} + B_{[j,0]} X_1 X_5^{n-1} = 0$ cannot occur. Now consider the powers of X_5 in the second equality (7). The powers of X_5 on the right hand side are $n-1$ and $n-s-2$ for $0 \le s \le k-1$ and those are strictly bigger than its power $n-k-2$ on the left hand side. This is impossible. This complete the proof of $(i.5), (i.6), (ii.4), (ii.5), (iii.3)$ and $(iii.4)$. $\square$

Finally, we finish this section by providing minimal generating sets of the defining ideal of $C_d \subset \mathbb{P}^5$ for $5 \leq d \leq 7$ by means of the Computer Algebra System SINGULAR [2].

Example 2.5. Let $C_d \subset \mathbb{P}^5$, $5 \leq d \leq 7$ be a curve stated as in (3). Then

(i) $d = 5$ and $\Sigma \cup \{F_{[-1,0]}, F_{[-1,1]}, G_{[-1,0]}\} \cup \{X_1X_3 - X_2^2\}$ is a minimal generating set of I_{C_5},

(ii) $d = 6$ and $\Sigma \cup \{F_{[0,0]}, F_{[0,1]}, G_{[0,0]}\}$ is a minimal generating set of I_{C_6}, and

(iii) $d = 7$ and $\Sigma \cup \{F_{[1,0]}, G_{[1,0]}\}$ is a minimal generating set of I_{C_7}.

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