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A NEW CLASS OF INTERPOLATORY HERMITE SUBDIVISION SCHEMES REPRODUCING POLYNOMIALS

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Abstract. In this paper, we present a new class of interpolatory Hermite subdivision schemes of order 2 reproducing polynomials. Each member in this class, denoted by H_n for $n \geq 1$, preserves polynomials of degree up to $4n+1$ admitting the approximation order of $4n+2$. Furthermore, it has free parameters which provide flexibility in designing curves/surfaces. H_1, the simplest and the most attractive scheme in this class, achieves C^4 smoothness with the parameters in certain ranges, and its performance is demonstrated with numerical examples.

1. Introduction

Interpolatory subdivision schemes are recursive algorithms generating smooth curves/surfaces which pass through the initially given discrete data. In applications such as motion control, Hermite interpolatory subdivision schemes are very desirable since they simultaneously refine additional quantities such as velocities, acceleration, and even higher derivatives associated with the initial positions. Hermite interpolatory subdivision schemes are first introduced and analyzed in [10] and in [5, 6]. Afterwards, many researchers have investigated their properties [8, 11], found tools to analyze them [1, 7, 12, 13], and constructed some with useful properties [3, 9, 15, 16]. As it is still young in the literature, Hermite interpolatory subdivision schemes possess promising potential yet to be investigated.

This paper presents a rich class of interpolatory Hermite subdivision schemes reproducing polynomials by acting on a sequence of vectors consisting of two elements corresponding to values of a function and its first derivative. The reproducing property guarantees the convergence of the scheme as well as the smoothness and the approximation order of its limit curves. Each subdivision scheme, dented by H_n for $n \geq 1$, in this class provides the approximation order

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$4n + 2$. Moreover, it has free parameters which provide flexibility in designing curves/surfaces in that we can edit the shape while not deviating from the initially given control points, and retaining all important properties including smoothness and approximation order. For instance, H_1, the simplest and the most attractive scheme in this new class, achieves C^4 with parameters in certain ranges.

We start by introducing some notations and definitions related to Hermite subdivision schemes in Section 2, and then construct the proposed interpolatory Hermite subdivision schemes in Section 3. In Section 4, focusing on H_1, the smoothness of its limit function is analyzed. In Section 5 is analyzed the approximation order of members in this new class. Finally, the performance of the proposed scheme is demonstrated with numerical examples in Section 6.

2. Preliminaries

Let $\ell^2(\mathbb{Z})$ be the linear space of all sequences of 2-vectors. Given an initial sequence $\mathbf{f}^{[0]} \in \ell^2(\mathbb{Z})$, a Hermite subdivision scheme of order 2 generates new data $\mathbf{f}^{[k]} \in \ell^2(\mathbb{Z})$ by iteratively applying the refinement rule

$$\mathbf{D}^{k+1}\mathbf{f}^{[k+1]}(i) = \sum_{j\in\mathbb{Z}} \mathbf{A}(i-2j)\mathbf{D}^k\mathbf{f}^{[k]}(j), \tag{1}$$

where $\mathbf{D} := \operatorname{diag}(1, 2^{-1})$. We call the sequence of coefficient matrices $\mathbf{A} := \{\mathbf{A}(i) \in \mathbb{R}^{2\times 2} : i \in \mathbb{Z}\}$ the subdivision mask. Only finitely many elements of the subdivision mask are assumed to be nonzero matrices. Using the notation $H_{\mathbf{A}}$ for the Hermite subdivision operator, we can formally express the relation (1) as

$$\mathbf{D}^{k+1}\mathbf{f}^{[k+1]} = H_{\mathbf{A}}\mathbf{D}^k\mathbf{f}^{[k]}.$$

The i-th vector $\mathbf{f}^{[k]}(i)$ of the sequence $\mathbf{f}^{[k]}$ is usually assigned to the value $\tau_i^k = 2^{-k}i$.

Definition 1. A Hermite subdivision scheme $H_{\mathbf{A}}$ is *convergent* if for any initial sequence $\mathbf{f}^{[0]}$, there exists a uniformly continuous function $\mathbf{f} = [\mathbf{f}_0\ \mathbf{f}_1]^T \in C(\mathbb{R}, \mathbb{R}^2)$ such that for an arbitrary compact subset $\Omega \in \mathbb{R}$,

$$\lim_{k\to\infty} \sup_{i\in 2^k\mathbb{Z}\cap\Omega} \|\mathbf{f}^{[k]}(i) - \mathbf{f}(\tau_i^k)\|_\infty = 0,$$

and $\mathbf{f} \neq \mathbf{0}$ for an initial sequence $\mathbf{f}^{[0]}$. Furthermore, $H_{\mathbf{A}}$ is said to be C^N-*convergent* for $N \geq 1$ if $\mathbf{f}_0 \in C^N(\mathbb{R})$ and $\mathbf{f}_0' = \mathbf{f}_1$.

Definition 2. A Hermite subdivision scheme $H_{\mathbf{A}}$ *reproduces* a function f if for the initial data $\mathbf{f}^{[0]} = \{\mathbf{f}^{[0]}(i) = [f(\tau_i^0)\ f'(\tau_i^0)]^T : i \in \mathbb{Z}\}$, it generates the refined data $\mathbf{f}^{[k]}$ satisfying $\mathbf{f}^{[k]}(i) = [f(\tau_i^k)\ f'(\tau_i^k)]^T$ for $i \in \mathbb{Z}$.

3. Interpolatory Hermite subdivision schemes

In this section, we construct the proposed Hermite subdivision schemes, denoted by H_n for $n \geq 1$. To this end, we employ the same construction technique used in [15]. Let $q_j(x) = x^j/j!$ for $j = 0, 1, \ldots, 4n+1$, which spans π_{4n+1}, the space of polynomials of degree less than or equal to $4n+1$. Then we find the mask $\mathbf{A}$ of H_n such that for the initial data $\mathbf{f}^{[0]} := \{\mathbf{f}^{[0[}(i) = [q_j(\tau_i^0)\ q_j'(\tau_i^0)]^T : i \in \mathbb{Z}\}$, H_n produces the refined data $\mathbf{f}^{[k]}$ satisfying $\mathbf{f}^{[k]}(i) = [q_j(\tau_i^k)\ q_j'(\tau_i^k)]^T$ for $k \in \mathbb{Z}_+$ and $i \in \mathbb{Z}$, i.e., H_n reproduces polynomials in π_{4n+1}. From the refinement equation (1), this property can be formulated as follows. For $\nu = 1, 2$, the entries in the ν-th row of the odd mask $\{\mathbf{A}(1-2i) \in \mathbb{R}^{2\times 2} : i = -n, \ldots, n+1\}$ are obtained by solving the system

$$2^{1-\nu} q_j^{(\nu-1)}(\tfrac{1}{2}) = \sum_{i=-n}^{n+1} \mathbf{A}_{\nu 1}(1-2i) q_j(i) + \sum_{i=-n}^{n+1} \mathbf{A}_{\nu 2}(1-2i) q_j'(i) \tag{2}$$

for $j = 0, 1, \ldots, 4n+1$. Since this is an underdetermined system of $4n+2$ linear equations with $4n+4$ unknowns for each ν, we set two free parameters λ and μ as

$$\mathbf{A}_{11}(\pm(2n+1)) = 2^{-4n+2}\lambda, \quad \mathbf{A}_{22}(\pm(2n+1)) = 2^{-4n+1}\mu. \tag{3}$$

Similarly, the even mask $\{\mathbf{A}(2i) \in \mathbb{R}^{2\times 2} : i = -n, \ldots, n\}$ is determined by solving the system

$$2^{1-\nu} q_j^{(\nu-1)}(0) = \sum_{i=-n}^{n} \mathbf{A}_{\nu 1}(-2i) q_j(i) + \sum_{i=-n}^{n} \mathbf{A}_{\nu 2}(-2i) q_j'(i) \tag{4}$$

with $j = 0, 1, \ldots, 4n+1$ for $\nu = 1, 2$. This system has a unique solution since there are $4n+2$ equations with $4n+2$ unknowns for each ν. In order to derive the solutions of the above systems explicitly, we use the following notations. Let F_i and G_i, $i = -n, \ldots, n$, be the fundamental Hermite interpolating polynomials given by

$$F_i(x) := \ell_i(x)^2(1 - 2\ell_i'(i)(x-i)), \quad G_i(x) := \ell_i(x)^2(x-i) \tag{5}$$

for the Lagrange basis polynomials

$$\ell_i(x) := \prod_{\substack{\alpha=-n \\ \alpha \neq i}}^{n} \frac{x-\alpha}{i-\alpha}. \tag{6}$$

Then we have for $i = -n, \ldots, n$,

$$F_i(0) = \delta_{i,0}, \quad F_i'(0) = 0, \quad G_i(0) = 0, \quad G_i'(0) = \delta_{i,0} \tag{7}$$

The following properties of the Lagrange basis polynomial are useful for derivation of the mask of the proposed Hermite schemes.

Lemma 3.1. *The Lagrange basis polynomial ℓ_{-n} given in* (6) *satisfies the following properties:*
(a) $\ell_{-n}(n+1) = 1$

(b) $\ell'_{-n}(n+1) = -\ell'_{-n}(-n) = \sum_{\alpha=1}^{2n} \frac{1}{\alpha}$
(c) $\ell_{-n}(\frac{1}{2}) = \frac{(-1)^n}{2^{4n}}\binom{2n}{n}$
(d) $\ell'_{-n}(\frac{1}{2}) = 0$

Proof. A direct calculation yields the first identity $\ell_{-n}(n+1) = 1$. Next, since

$$\ell'_{-n}(x) = \ell_{-n}(x) \sum_{\alpha=-n+1}^{n} \frac{1}{x-\alpha}, \tag{8}$$

it follows that

$$\ell'_{-n}(n+1) = \sum_{\alpha=-n+1}^{n} \frac{1}{n+1-\alpha} = \sum_{\alpha=1}^{2n} \frac{1}{\alpha}$$

and

$$\ell'_{-n}(-n) = \sum_{\alpha=-n+1}^{n} \frac{1}{-n-\alpha} = -\sum_{\alpha=1}^{2n} \frac{1}{\alpha}.$$

Thirdly, using the relation $\prod_{\alpha=1}^{n}(2\alpha-1) = \frac{(2n)!}{2^n n!}$, we derive

$$\ell_{-n}(\tfrac{1}{2}) = \frac{1}{2^{2n}} \prod_{\alpha=-n+1}^{n} \frac{1-2\alpha}{-n-\alpha} = \frac{(-1)^n}{2^{2n}(2n)!}\left(\prod_{\alpha=1}^{n}(2\alpha-1)\right)^2 = \frac{(-1)^n}{2^{4n}}\binom{2n}{n}.$$

Lastly, since the integer set $\{-n+1, ..., n\}$ is symmetric about $\frac{1}{2}$, we see that $\sum_{\alpha=-n+1}^{n} \frac{1}{1/2-\alpha} = 0$. It follows from (8) that $\ell'_{-n}(\frac{1}{2}) = 0$. $\square$

Now, we derive the explicit form of our mask $\mathbf{A}$ in terms of the matrix-valued function $\mathbf{U}^{[i]}$ given by

$$\mathbf{U}^{[i]}(x) := \begin{bmatrix} F_i(x) & G_i(x) \\ F'_i(x) & G'_i(x) \end{bmatrix}$$

with F_i and G_i in (5).

Theorem 3.2. *Let $\{\mathbf{A}(\alpha) : \alpha = -2n-1, \dots, 2n+1\}$ be the coefficient matrices in the systems (2) and (4). Suppose that λ and μ are the free parameters given in (3). For $s_m := \sum_{\alpha=1}^{m} \frac{1}{\alpha}$, define a matrix $\mathbf{W}$ by*

$$\mathbf{W} := 2^{-4n+2} \begin{bmatrix} \lambda & \frac{\binom{2n}{n}^2}{2^{4n+4}s_{2n}} - \frac{\lambda}{2s_{2n}} \\ \frac{\binom{2n}{n}^2}{2^{4n+3}(2n+1)} - s_{2n+1}\mu & \frac{\mu}{2} \end{bmatrix}. \tag{9}$$

Then the mask of the proposed Hermite subdivision scheme H_n can be expressed as

$$\begin{aligned} &\mathbf{A}(1-2i) = \mathbf{D}\mathbf{U}^{[i]}(\tfrac{1}{2}) - \mathbf{W}\mathbf{U}^{[i]}(n+1) \quad \textit{and} \quad \mathbf{A}(-2n-1) = \mathbf{W}, \\ &\mathbf{A}(2i) = \delta_{i,0}\mathbf{D}, \end{aligned} \tag{10}$$

for $i = -n, \dots, n$.

Proof. For the derivation of the odd mask, denoting by $\mathbf{A}_j(i)$ the j-th column of the coefficient matrix $\mathbf{A}(i)$, let

$$\bar{\mathbf{A}} := \begin{bmatrix}\mathbf{A}_1(2n+1)\dots\mathbf{A}_1(-2n+1) & \mathbf{A}_2(2n+1)\dots\mathbf{A}_2(-2n+1)\end{bmatrix} \in \mathbb{R}^{2\times(4n+2)}.$$

Then the system (2) can be written in the matrix form

$$\mathbf{D}\begin{bmatrix}\mathbf{q}^{(0)}(\frac{1}{2})\\ \mathbf{q}^{(1)}(\frac{1}{2})\end{bmatrix} = \bar{\mathbf{A}}\begin{bmatrix}\mathbf{Q}^{(0)}\\ \mathbf{Q}^{(1)}\end{bmatrix} + \mathbf{A}(-2n-1)\begin{bmatrix}\mathbf{q}^{(0)}(n+1)\\ \mathbf{q}^{(1)}(n+1)\end{bmatrix} \tag{11}$$

where the vector-valued function $\mathbf{q}^{(\gamma)}$ is defined as

$$\mathbf{q}^{(\gamma)}(x) = \begin{bmatrix}q_j^{(\gamma)}(x) : j = 0,\dots,4n+1\end{bmatrix} \in \mathbb{R}^{1\times(4n+2)}, \tag{12}$$

and the matrix $\mathbf{Q}^{(\gamma)}$ is given by

$$\mathbf{Q}^{(\gamma)} := \begin{bmatrix}q_j^{(\gamma)}(i) : i = -n,\dots,n,\ j = 0,\dots,4n+1\end{bmatrix} \in \mathbb{R}^{(2n+1)\times(4n+2)}.$$

The Hermite interpolation of polynomials yields the identity

$$\mathbf{U}(x) = \begin{bmatrix}\mathbf{q}^{(0)}(x)\\ \mathbf{q}^{(1)}(x)\end{bmatrix}\begin{bmatrix}\mathbf{Q}^{(0)}\\ \mathbf{Q}^{(1)}\end{bmatrix}^{-1} \tag{13}$$

where

$$\mathbf{U}(x) := \begin{bmatrix}\mathbf{U}_1^{[-n]}(x)\dots\mathbf{U}_1^{[n]}(x) & \mathbf{U}_2^{[-n]}(x)\dots\mathbf{U}_2^{[n]}(x)\end{bmatrix} \in \mathbb{R}^{2\times(4n+2)}.$$

Plugging (13) into the system (11), we have

$$\bar{\mathbf{A}} = \mathbf{D}\mathbf{U}(\tfrac{1}{2}) - \mathbf{A}(-2n-1)\mathbf{U}(n+1). \tag{14}$$

For simplicity, put

$$\mathbf{A}(-2n-1) = \begin{bmatrix}\theta_1 & \eta_1\\ \theta_2 & \eta_2\end{bmatrix}.$$

Then by comparing both sides of (14), we obtain the relation

$$\begin{aligned}\mathbf{A}_{11}(2n+1) &= F_{-n}(\tfrac{1}{2}) - \theta_1 F_{-n}(n+1) - \eta_1 F'_{-n}(n+1),\\ \mathbf{A}_{22}(2n+1) &= 2^{-1}G'_{-n}(\tfrac{1}{2}) - \theta_2 G_{-n}(n+1) - \eta_2 G'_{-n}(n+1).\end{aligned} \tag{15}$$

Due to our setting in (3), we have

$$\mathbf{A}_{11}(2n+1) = \theta_1, \quad \mathbf{A}_{22}(2n+1) = \eta_2. \tag{16}$$

Using Lemma 3.1, it can be easily checked that $F'_{-n}(n+1) \neq 0$ and $G_{-n}(n+1) \neq 0$. Thus, the identities (15) are equivalent to

$$\begin{aligned}\eta_1 &= \frac{F_{-n}(\frac{1}{2})}{F'_{-n}(n+1)} - \frac{1+F_{-n}(n+1)}{F'_{-n}(n+1)}\theta_1,\\ \theta_2 &= \frac{2^{-1}G'_{-n}(\frac{1}{2})}{G_{-n}(n+1)} - \frac{1+G'_{-n}(n+1)}{G_{-n}(n+1)}\eta_2.\end{aligned}$$

A direct calculation with Lemma 3.1 yields

$$\eta_1 = -\frac{\ell_{-n}(\frac{1}{2})^2}{4\ell'_{-n}(-n)} + \frac{1}{2\ell'_{-n}(-n)}\theta_1,$$

$$\theta_2 = \frac{\ell_{-n}(\frac{1}{2})^2}{4n+2} + \frac{(4n+2)\ell'_{-n}(-n) - 2}{2n+1}\eta_2.$$

Moreover, using Lemma 3.1(b) and (c) together with the setting (3), we obtain the explicit formula (9) for $\mathbf{A}(-2n-1)$. This together with (14) completes the derivation of the odd mask in (10). The even mask is uniquely determined by solving the system (4), which formulates the evaluation of the Hermite interpolation at $x = 0$. Thus, it follows from the property (7) that

$$\mathbf{A}(-2i) = \delta_{i,0}\mathbf{D}, \quad i = -n, \dots, n,$$

which completes the proof. □

Remark 1. Using the same method in the proof of [15, Theorem 3.5], we can easily show that the mask $\mathbf{A}$ constructed by solving the systems (2) and (4) fulfills the symmetry $\mathbf{A}(-i) = \mathbf{S}\mathbf{A}(i)\mathbf{S}$, $i = 1, \dots, 2n+1$, where $\mathbf{S} := \operatorname{diag}(1, -1)$.

For specific choices of the free parameters, the proposed scheme H_n reproduces polynomials of higher degrees.

Theorem 3.3. *Let H_n be the proposed Hermite subdivision scheme with the mask in* (10)*. Choose the free parameters λ and μ of H_n as*

$$\lambda = -\big(2 + (2n+1)s_{2n}\big)\mu \quad \textit{and} \quad \mu = -2^{-4n-4}\binom{2n}{n}^2. \tag{17}$$

Then H_n reproduces polynomials of degree up to $4n+3$.

Proof. We can add two more equations for $j = 4n+2, 4n+3$ to the system (2). Then the system is uniquely solvable. The solution is obtained by evaluating the Hermite interpolating polynomial for the given points $\{-n, \dots, n+1\}$ at $x = 1/2$. This induces the identity

$$\mathbf{A}(-2n-1) = \mathbf{D}\begin{bmatrix} \tilde{F}_{n+1}(\frac{1}{2}) & \tilde{G}_{n+1}(\frac{1}{2}) \\ \tilde{F}'_{n+1}(\frac{1}{2}) & \tilde{G}'_{n+1}(\frac{1}{2}) \end{bmatrix},$$

where $\tilde{F}_i$ and $\tilde{G}_i$, the fundamental Hermite interpolating polynomials, are defined as the same manner in (5) with the following Lagrange basis polynomial

$$\tilde{\ell}_i(x) := \prod_{\substack{\alpha=-n \\ \alpha\neq i}}^{n+1} \frac{x-\alpha}{i-\alpha}.$$

Now we determine the parameters λ and μ such that

$$\mathbf{W} = \mathbf{D}\begin{bmatrix} \tilde{F}_{n+1}(\frac{1}{2}) & \tilde{G}_{n+1}(\frac{1}{2}) \\ \tilde{F}'_{n+1}(\frac{1}{2}) & \tilde{G}'_{n+1}(\frac{1}{2}) \end{bmatrix},$$

for the matrix $\mathbf{W}$ given in (9). This can be done by setting

$$\lambda = 2^{4n-2}\tilde{F}_{n+1}(\tfrac{1}{2}), \quad \mu = 2^{4n-2}\tilde{G}'_{n+1}(\tfrac{1}{2}), \tag{18}$$

and showing that

$$\mathbf{W}_{12} = \tilde{G}_{n+1}(\tfrac{1}{2}) \quad \text{and} \quad \mathbf{W}_{21} = 2^{-1}\tilde{F}'_{n+1}(\tfrac{1}{2}). \tag{19}$$

As in the proof of Lemma 3.1, it can be easily shown that

$$\tilde{\ell}'_{n+1}(n+1) = \frac{1}{2n+1} + s_{2n}, \quad \tilde{\ell}_{n+1}(\tfrac{1}{2}) = \frac{(-1)^n}{2^{4n+1}}\binom{2n}{n}, \quad \tilde{\ell}'_{n+1}(\tfrac{1}{2}) = \frac{2\tilde{\ell}_{n+1}(\frac{1}{2})}{2n+1}.$$

Then a direct computation with these identities induces the formulas (17), by which the relations (19) can be easily verified. The uniqueness of the solution to the system (2) completes the proof. □

Remark 2. In [15], the authors introduced a parametric family of primal Hermite schemes reproducing polynomials in π_{4m-1} for $m \geq 1$. Each member of this family becomes interpolatory when its parameters are chosen to be zero. In fact, for $m \geq 2$, it is a special case of the proposed Hermite scheme H_{m-1} with the choice (17).

Example 3.4. For practical applications, H_1 may be the most attractive scheme among all the interpolatory Hermite subdivision schemes in the proposed class. It reproduces polynomials in π_5 for any choice of free parameters. The mask $\mathbf{A}$ of H_1 is supported in $[-3,3]\cap\mathbb{Z}$, and the explicit form of the odd mask is given as

$$\mathbf{A}(1) = \begin{bmatrix} \frac{1}{2} - \frac{\lambda}{4} & \frac{17}{128} + \frac{\lambda}{4} \\ -\frac{99}{128} + \frac{9\mu}{8} & -\frac{9}{64} + \frac{9\mu}{8} \end{bmatrix}, \mathbf{A}(3) = \begin{bmatrix} \frac{\lambda}{4} & -\frac{1}{384} + \frac{\lambda}{12} \\ -\frac{1}{384} + \frac{11\mu}{24} & \frac{\mu}{8} \end{bmatrix},$$
$$\mathbf{A}(-i) = \mathbf{S}\mathbf{A}(i)\mathbf{S}, \quad i = 1, 3,$$

where $\mathbf{S} := \operatorname{diag}(1,-1)$. According to Theorem 3.3, if we choose $\lambda = \frac{3}{128}$ and $\mu = -\frac{1}{64}$, then H_1 reproduces polynomials in π_7.

4. Analysis of smoothness

In this section, we investigate the regularity of the limit function generated by the proposed Hermite subdivision scheme. The primary target of our analysis is the scheme H_1. We adopt the analysis framework based on the factorization of subdivision operators given in [1]. For the detailed description, readers are referred to [1, 2]. The factorization is performed in terms of the symbol

$$\mathcal{A}(z) := \sum_{\alpha\in\mathbb{Z}} \mathbf{A}(\alpha)z^{\alpha}, \qquad z \in \mathbb{C}$$

associated with the mask $\mathbf{A}$. We start by factoring the symbol of H_1. As shown in [2], the polynomial reproducing property of H_1 induces the factorization of its associated symbol $\mathcal{A}(z)$ as

$$\mathcal{A}(z) = \frac{1}{2}(\mathcal{T}(z))^{-1}\mathcal{B}(z)\mathcal{T}(z^2),$$

where

$$\mathcal{T}(z) = \begin{bmatrix} z^{-1}-1 & -1 \\ 0 & 1 \end{bmatrix},$$

and the entries of $\mathcal{B}(z)$, so-called the Taylor symbol of H_1, are

$$\begin{aligned}
\mathcal{B}_{11}(z) &= \big((96\lambda + 176\mu - 1)z^3 - 96\lambda z^2 + (608\mu - 106)z + 192 + (176\mu - 96\lambda - 1)z^{-1} \\
&\quad + 96\lambda z^{-2}\big)/192, \\
\mathcal{B}_{12}(z) &= \big((64\lambda + 128\mu)z^3 - (64\lambda + 1)z^2 + (176\mu - 96\lambda - 103)z + (96\lambda + 51) \\
&\quad + (104 - 256\mu)z^{-1} - 51z^{-2} + (32\lambda - 48\mu - 1)z^{-3} + (1 - 32\lambda)^{-4}\big)/192, \\
\mathcal{B}_{21}(z) &= \big((1 - 176\mu)z^3 + (298 - 608\mu)z + (1 - 176\mu)z^{-1}\big)/192, \\
\mathcal{B}_{22}(z) &= \big((1 - 128\mu)z^3 + (244 - 176\mu)z + 192 + (256\mu - 53)z^{-1} + 48\mu z^{-3}\big)/192.
\end{aligned}$$

In order to factor the Taylor symbol $\mathcal{B}(z)$, we compute the joint-1 eigenvector $\mathbf{v}$ of two matrices $B_e := \sum_{j\in\mathbb{Z}} B(2j)$ and $B_o := \sum_{j\in\mathbb{Z}} B(2j+1)$, i.e., $B_e\mathbf{v} = B_o\mathbf{v} = \mathbf{v}$. In our case, we obtain $\mathbf{v} = [0\ 1]^T$. For $\mathbf{e} := [1\ 0]^T$, let

$$\mathcal{V}(z) := \operatorname{diag}(z^{-1} - 1, 1)[\mathbf{v}\ \mathbf{e}]^{-1}. \tag{20}$$

Then $\mathcal{B}(z)$ is factored as

$$\mathcal{B}(z) = \frac{1}{2}(\mathcal{V}(z))^{-1}\mathcal{B}^{(1)}(z)\mathcal{V}(z^2),$$

with symbol $\mathcal{B}^{(1)}(z)$ consisting of the following components

$$\begin{aligned}
\mathcal{B}^{(1)}_{11}(z) &= \big((1 - 128\mu)z^3 + (128\mu - 1)z^2 + (245 - 304\mu)z + (304\mu - 53) - 48\mu z^{-1} \\
&\quad + 48\mu z^{-2}\big)/96, \\
\mathcal{B}^{(1)}_{12}(z) &= \big((176\mu - 1)z^3 + (1 - 176\mu)z^2 + (608\mu - 298)z + (298 - 608\mu) \\
&\quad + (176\mu - 1)z^{-1} + (1 - 176\mu)z^{-2}\big)/96, \\
\mathcal{B}^{(1)}_{21}(z) &= \big(-(64\lambda + 128\mu)z^3 + (64\lambda + 1)z^2 + (32\lambda - 304\mu + 103)z - (32\lambda + 50) \\
&\quad + (32\lambda - 48\mu - 1)z^{-1} + (1 - 32\lambda)z^{-2}\big)/96, \\
\mathcal{B}^{(1)}_{22}(z) &= \big((96\lambda + 176\mu - 1)z^3 - 96\lambda z^2 + (608\mu - 106)z + 192 + (176\mu - 96\lambda - 1)z^{-1} \\
&\quad + 96\lambda z^{-2}\big)/96.
\end{aligned}$$

Similarly, the factorization of $\mathcal{B}^{(1)}(z)$ with the joint-1 eigenvector $\mathbf{v} = [2\ 1]^T$ of $B_e^{(1)}$ and $B_o^{(1)}$ yields $\mathcal{B}^{(2)}(z)$ whose entries are

$$\begin{aligned}
\mathcal{B}_{11}^{(2)}(z) &= \big(-(32\lambda+80\mu+1)z^3+(64\lambda+80\mu+3)z^2+(97-80\mu)z+(80\mu-64\lambda-5)\\
&\quad +(32\lambda+2)z^{-1}\big)/48\\
\mathcal{B}_{12}^{(2)}(z) &= \big((64\lambda+128\mu)z^3-(128\lambda+128\mu+1)z^2+(32\lambda+304\mu-102)z+(64\lambda-304\mu\\
&\quad +153)+(48\mu-64\lambda-49)z^{-1}+(64\lambda-48\mu-2)z^{-2}+(1-32\lambda)z^{-3}\big)/48\\
\mathcal{B}_{21}^{(2)}(z) &= \big(-(64\lambda+80\mu+3)z^3+(64\lambda-80\mu+5)z^2+(64\lambda-80\mu+5)z-64\lambda-80\mu\\
&\quad -3\big)/48\\
\mathcal{B}_{22}^{(2)}(z) &= \big((128\lambda+128\mu+1)z^3+(128\mu-128\lambda-3)z^2+(304\mu-64\lambda+39)z+(64\lambda\\
&\quad +304\mu+47)+(48\mu-64\lambda+2)z^{-1}+(64\lambda+48\mu-2)z^{-2}\big)/48
\end{aligned}$$

The joint-1 eigenvector $\mathbf{v} = [3\ 1]^T$ of $B_e^{(2)}$ and $B_o^{(2)}$ leads us to obtain $\mathcal{B}^{(3)}(z)$ with entries

$$\begin{aligned}
\mathcal{B}_{11}^{(3)}(z) &= \big(-(32\lambda+56\mu+4)z^3+(64\lambda+10)z^2+(32\mu+17)z+(2-64\lambda)\\
&\quad +(32\lambda+24\mu-1)z^{-1}\big)/12\\
\mathcal{B}_{12}^{(3)}(z) &= \big((64\lambda+80\mu+3)z^3-(128\lambda+8)z^2+(128\lambda+8)-(64\lambda+80\mu+3)z^{-1}\big)/24\\
\mathcal{B}_{21}^{(3)}(z) &= (-(160\lambda+224\mu+21)z^3+(128\lambda-448\mu+28)z^2+(192\lambda-96\mu-48)z\\
&\quad +(4-192\mu-128\lambda)+(1-32\lambda)z^{-1})/24\\
\mathcal{B}_{22}^{(3)}(z) &= ((80\lambda+80\mu+4)z^3+(160\mu-64\lambda-6)z^2+(80\mu-96\lambda+41)z+(64\lambda+160\mu+2)\\
&\quad +(16\lambda+1)z^{-1})/12
\end{aligned}$$

Finally, utilizing the joint-1 eigenvector $\mathbf{v} = [3\ 1]^T$ of $B_e^{(3)}$ and $B_o^{(3)}$, we get the $\mathcal{B}^{(4)}(z)$ whose components are

$$\begin{aligned}
\mathcal{B}_{11}^{(4)}(z) &= \big(-(64\mu+13)z^3+(29-64\mu)z^2+(128\mu+5)z+3\big)/12\\
\mathcal{B}_{12}^{(4)}(z) &= \big((160\lambda+224\mu+21)z^3+(224\mu-288\lambda-49)z^2+(76-352\mu-64\lambda)z\\
&\quad +(320\lambda+96\mu-52)+(3-192\mu-96\lambda)z^{-1}+(1-32\lambda)z^{-2}\big)/12\\
\mathcal{B}_{21}^{(4)}(z) &= \big(-(8\mu+2)z^3+(1-32\mu)z^2-(8\mu+2)z\big)/3\\
\mathcal{B}_{22}^{(4)}(z) &= \big((96\lambda+112\mu+13)z^3+(448\mu-8)z^2+(160\mu-192\lambda+82)z+192\mu\\
&\quad +(96\lambda+48\mu-3)z^{-1}\big)/12
\end{aligned}$$

According to the theory in [1, 2], if the N-th difference scheme $\frac{1}{2}H_{\mathbf{B}^{(N)}}$ associated with the symbol $\mathcal{B}^{(N)}(z)$ is contractive, i.e.,

$$\|(\tfrac{1}{2}H_{\mathbf{B}^{(N)}})^L\|_\infty < 1$$

for some number $L \geq 1$, then the corresponding Hermite scheme is C^N. In our case, with the aid of MATLAB software, we see that $\|(\frac{1}{2}H_{\mathbf{B}^{(4)}})^{15}\|_\infty < 1$ for $\lambda \in [0.135, 0.145]$ and $\mu \in [-0.08, -0.06]$. Therefore, the proposed Hermite scheme is C^4 for this range of parameter values.

5. Order of approximation

The approximation capability of a Hermite subdivision scheme is usually expressed in terms of the approximation order. It measures how much the limit function obtained by the subdivision scheme with the initial data sampled from a given function, is close to the original function. In this paper, we consider the approximation of functions in the Sobolev space

$$\mathcal{W}_{\infty}^{N}(\Omega)=\left\{f:\mathbb{R}\to\mathbb{R}:\|f\|_{N,\Omega}:=\sum_{r=0}^{N}\|f^{(r)}\|_{L_\infty(\Omega)}<\infty\right\},\quad N\in\mathbb{Z}_+.$$

Then the following estimation given in [15] holds for Hermite subdivision schemes reproducing polynomials.

Theorem 5.1. [15] *A C^γ-convergent Hermite subdivision scheme $H_{\mathbf{A}}$ of order $\gamma+1$ reproducing polynomials in π_m for $m\geq\gamma$ satisfies the estimate*

$$\|f^{(\nu)}-f_{\infty}^{(\nu)}\|_{L_\infty(\Omega)}\leq C\|f\|_{m+1,\Omega}2^{-k(m+1-\nu)},\quad \nu=0,\dots,\gamma,$$

for a compact subset $\Omega\subset\mathbb{R}$ and a constant $C>0$ independent of k and f.

According to this theorem, our Hermite scheme is $(4n+2)$-th order accurate.

Corollary 5.2. *The proposed interpolatory Hermite subdivision scheme H_n achieves the approximation order $4n+2$.*

6. Numerical examples

In this section, we present some numerical examples of the proposed scheme H_1 whose mask is given in Example 3.4. Figure 1 shows the limit functions f_∞ and g_∞ generated by H_1 with the initial data $\mathbf{f}^{[0]}=\delta_{i,0}[1\ 0]^T$ and $\mathbf{g}^{[0]}=\delta_{i,0}[0\ 1]^T$ for $i\in\mathbb{Z}$, respectively. For this example, we chose $\lambda=0.13775$ and $\mu=-0.06725$. The derivatives of f_∞ and g_∞ are presented in Figures 2 and 3, which verify that the limit curve of H_1 is C^4. Figure 4 illustrates the limit curves produced from the same initial polygon with different parameter values. The arrow indicates the gradient at each vertex. In this case, we set $(\lambda,\mu)=(0.13775,-0.06725)+\beta(0.1,0.05)$ for $\beta=-2,\dots,2$.

References

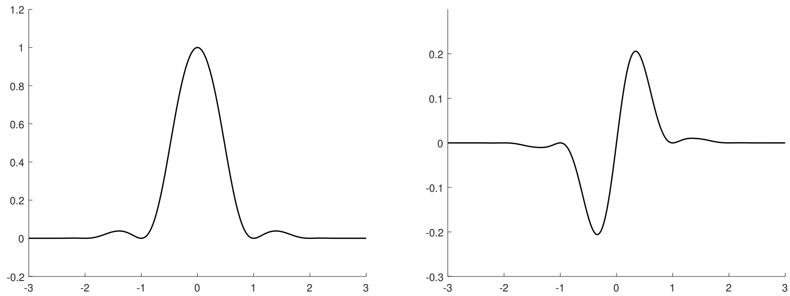
[1] M. Charina, C. Conti, T. Sauer, *Regularity of multivariate vector subdivision schemes*, Numer. Algorithms **39** (2005), 97–113.

[2] C. Conti, J.-L. Merrien, L. Romani, *Dual Hermite subdivision schemes of de Rham-type*, BIT Numer. Math. **54** (2014), 955–977.

[3] C. Conti, L. Romani, M. Unser, *Ellipse-preserving Hermite interpolation and subdivision*, J. Math. Anal. Appl. **426** (2015), 211–227.

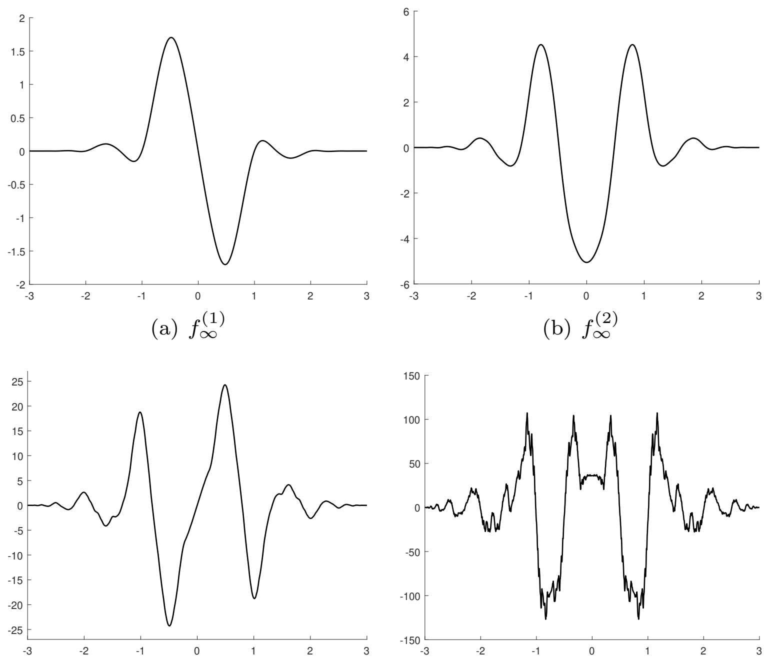
[4] M. Cotronei, C. Moosmüller, T. Sauer, N. Sissouno, *Level-dependent interpolatory Hermite subdivision schemes and wavelets*, Constr. Approx. **50** (2019), 341–366.

[5] N. Dyn, D. Levin, *Analysis of Hermite-type subdivision schemes*, in: C.K. Chui, L.L. Schumaker (Eds.), Approximation Theory VIII, Wavelets and Multilevel Approximation, vol.2, College Station, TX, World Sci., River Edge, NJ, (1995), 117–124.



(a) f_∞ (b) g_∞

FIGURE 1. Limit functions generated by H_1 with different initial data: (a) $\mathbf{f}^{[0]} = \delta_{i,0}[1\ 0]^T$, (b) $\mathbf{g}^{[0]} = \delta_{i,0}[0\ 1]^T$.

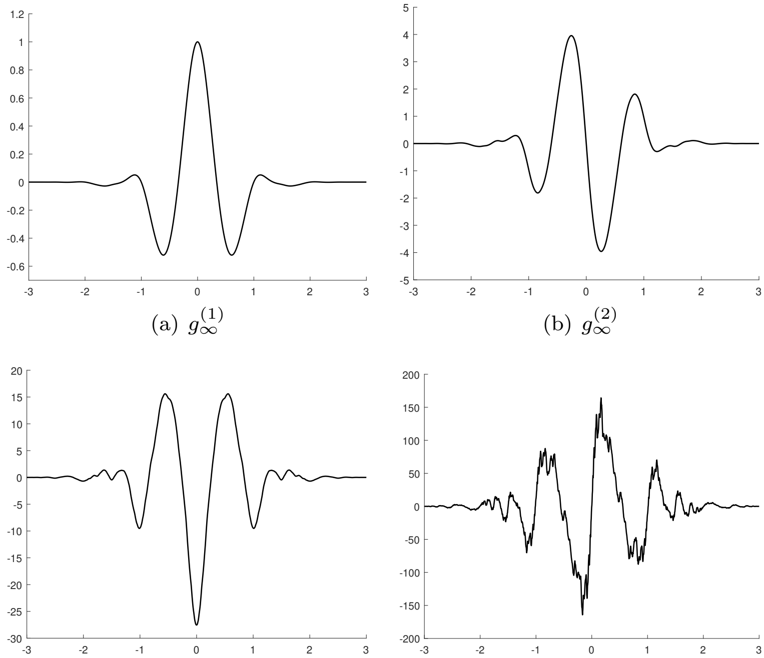


(a) $f_\infty^{(1)}$ (b) $f_\infty^{(2)}$

(c) $f_\infty^{(3)}$ (d) $f_\infty^{(4)}$

FIGURE 2. Derivatives of the limit function f_∞ in Figure 1(a)

[6] N. Dyn, D. Levin *Analysis of Hermite-interpolatory subdivision schemes*, in: S. Dubuc, G. Deslauriers (Eds.), Spline Functions and the Theory of Wavelets, Am. Math. Soc., Providence, RI, (1999), 105–113.



(a) $g_\infty^{(1)}$ (b) $g_\infty^{(2)}$

(c) $g_\infty^{(3)}$ (d) $g_\infty^{(4)}$

FIGURE 3. Derivatives of the limit function g_∞ in Figure 1(b)

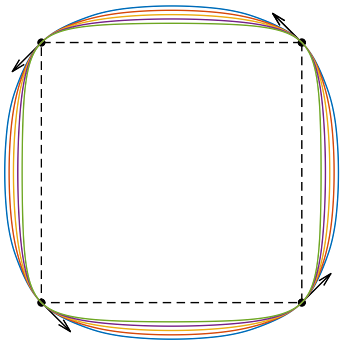


FIGURE 4. Limit curves generated by H_1 with different choices of λ and μ: $(\lambda, \mu) = (0.13775, -0.06725) + \beta(0.1, 0.05)$ for $\beta = -2(\text{outermost}), \ldots, 2(\text{innermost})$.

[7] S. Dubuc, J.-L. Merrien, *Hermite subdivision schemes and Taylor polynomials*, Constr. Approx. **29** (2009), 219–245.

[8] N. Guglielmi, C. Manni, D. Vitale, *Convergence analysis of C^2 Hermite interpolatory subdivision schemes by explicit joint spectral radius formulas*, Linear Algebra Appl. **434** (2011), 884–902.

[9] B. Han, *Approximation properties and construction of Hermite interpolants and biorthogonal multiwavelets*, J. Approx. Theory **110** (2001), no. 1, 18–53.

[10] J.-L. Merrien, *A family of Hermite interpolants by bisection algorithms*, Numer. Algorithms **2** (1992), 187–200.

[11] C. Manni, M.-L. Mazure, *Shape constraints and optimal bases for C^1 Hermite interpolatory subdivision schemes*, SIAM J. Numer. Anal. **48** (2010), no. 4, 1254–1280.

[12] J.-L. Merrien, T. Sauer, *A generalized Taylor factorization for Hermite subdivision schemes*, J. Comput. Appl. Math. **236** (2011), no. 4, 565–574.

[13] J.-L. Merrien, T. Sauer, *From Hermite to stationary subdivision schemes in one or several variables*, Adv. Comput. Math. **36** (2012), 547–579.

[14] M. K. Jena, *A Hermite interpolatory subdivision scheme constructed from quadratic rational BernsteinBezier spline*, Math. Comput. Simulat. **187** (2021), 433–448.

[15] B. Jeong and J. Yoon, *Construction of Hermite subdivision schemes reproducing polynomials*, J. Math. Anal. Appl. **451** (2017), 565–582.

[16] L. Romani, *A circle-preserving C^2 Hermite interpolatory subdivision scheme with tension control*, Comput. Aided Geom. D. **27** (2010), 36–47.

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