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ON QUANTITATIVE TWO WEIGHT ESTIMATES FOR SOME DYADIC OPERATORS

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Abstract. In this paper, a comparison of two types of quantitative two weight conditions for the boundedness of the dyadic paraproduct and the commutator of the Hilbert transform is provided. In the case of the commutator $[b, H]$, the conditions of the well-known Bloom's inequality [2] and the slightly different types of two weight inequality introduced in [1] are compared around the A_2-conditions on weights and the novel conditions on the function b.

1. Introduction

Let u be a weight on $\mathbb{R}$, i.e. an almost everywhere positive locally integrable function. Then we define $L^2(\mathbb{R}, u) = L^2(u)$ to be the space of functions that are square integrable with respect to the measure $u(x)dx$, namely

$$\|f\|_{L^2(u)} := \left(\int_{\mathbb{R}} |f(x)|^2 u(x)dx\right)^{1/2}.$$

For a given weight u and an interval I, let $u(I) = \int_I u(x)dx$ and $m_I u = u(I)/|I|$. We say that a weight u satisfies the A_p condition if and only if u is a weight, so u^{-1} is also a weight, and the supremum over intervals below is finite.

$$[u]_{A_p} := \sup_I m_I u \left(m_I(u^{-\frac{1}{p-1}})\right)^{p-1} < \infty .$$

In 1985, Bloom characterized the boundedness of the commutator of the Hilbert transform $[b, H]$ from $L^p(u)$ into $L^p(v)$ when both weights u and v are in A_p. When the weights $u = v \in A_2$ then it is well-known that the characterization is $b \in \text{BMO}$. However, Bloom provides boundedness in case $u \neq v$ and $u, v \in A_p$. The boundedness is characterized in terms of a BMO space adapted to the weight $\rho = (u/v)^{1/p}$, namely

$$\|b\|_{BMO_\rho} := \sup_I \left(\frac{1}{\rho(I)} \int_I |b(x) - m_I b|^2 dx\right)^{1/2}.$$

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Theorem 1.1 ([2], $p = 2$). *Let $u, v \in A_2$ and put $\rho = (u/v)^{1/2}$ and suppose that $b \in L^1$. Then*

(i) *If $b \in BMO_\rho$, the commutator $[b, H]$ is a bounded map from $L^2(u)$ into $L^2(v)$, with*
$$\|[b, H]f\|_{L^2(v)} \leq C\|f\|_{L^2(u)}\,.$$
(ii) *Conversely, if $[b, H] : L^2(u) \to L^2(v)$ is bounded, then $b \in BMO_\rho$.*

Let us denote $\mathcal{D}$ and $\mathcal{D}(J)$ the collection of all dyadic intervals and the collection of all dyadic subintervals of J respectively. For $I \in \mathcal{D}$ the Haar function associated with I is
$$h_I = |I|^{-1/2}(\mathbb{1}_{I_+} - \mathbb{1}_{I_-})$$
where $I_\pm$ are the left and right dyadic children of I. In [5] and [6], the authors present the modern proof of the Theorem 1.1 and also provide the boundedness of the dyadic paraproduct in the spirit of the Theorem 1.1. Here the dyadic paraproduct is defined as
$$\pi_b f := \sum_{I \in \mathcal{D}} m_I f b_I h_I\,,$$
where $b_I = \langle b, h_I \rangle$.

Theorem 1.2 ([5], $p = 2$). *Let $u, v \in A_2$ and suppose that $\mathbf{B}_2[u, v]$ is finite where*
$$\mathbf{B}_2[u, v] := \sup_{J \in \mathcal{D}} u^{-1}(J)^{-1/2} \left\| \sum_{I \in \mathcal{D}(J)} b_I m_I(u^{-1}) h_I \right\|_{L^2(v)}\,.$$
Then
$$\|\pi_b f\|_{L^2(v)} \leq C\mathbf{B}_2[u, v]\|f\|_{L^2(u)}\,.$$

Note that by the boundedness of the square function in one weight case, if $v \in A_2$ one can easily characterize $\mathbf{B}_2[u, v]$ by:
$$\mathbf{B}_2[u, v]^2 := \sup_{J \in \mathcal{D}} \frac{1}{u^{-1}(J)} \sum_{I \in \mathcal{D}(J)} b_I^2 (m_I(u^{-1}))^2 m_I v\,. \tag{1.1}$$
We also say that a pair of weight (u, v) satisfies the joint A_2 condition if and only if both v and u are weights and
$$[u, v]_{A_2} := \sup_{I \in \mathcal{D}} m_I(u^{-1}) m_I v < \infty\,.$$
A positive sequence $\{\alpha_I\}_{I \in \mathcal{D}}$ is a v-Carleson sequence if there is a constant $C > 0$ such that for all dyadic intervals J
$$\sum_{I \in \mathcal{D}(J)} \alpha_I \leq Cv(J)\,.$$
The infimum among all C's that satisfy the inequality is called the intensity of the v-Carleson sequence $\{\alpha_I\}_{I \in \mathcal{D}}$.

In [4], the author provides quantitative estimates with a slightly different condition on b, namely the two weight Carleson class denoted by $Carl_{u,v}$, and also on the weights u and v, namely the joint A_2 condition. The function class $Carl_{u,v}$ is introduced and studied in many papers, such as [1] , [9], and [10]. Given a pair of weights (u, v), we say that a locally integrable function b belongs to the two weight Carleson class, $Carl_{u,v}$, if

$$\mathcal{B}_{u,v} := \sup_{J\in\mathcal{D}} \frac{1}{u^{-1}(J)} \sum_{I\in\mathcal{D}(J)} \frac{b_I^2}{m_I v} < \infty\,. \tag{1.2}$$

Theorem 1.3 ([4]). *Let $(u, v) \in A_2$. If $u, v \in A_2$ and $b \in Carl_{u,v}$ then the commutator $[b, H]$ is bounded from $L^2(u)$ into $L^2(v)$ with*

$$\|[b, H]f\|_{L^2(v)} \le C\|f\|_{L^2(u)}\,.$$

For the dyadic paraproduct, the authors in [1] provide the following quantitative two weight estimates.

Theorem 1.4 ([1]). *Let (u, v) be a pair of weights such that*

(i) $(u, v) \in A_2$
(ii) *there is a constant $\mathcal{D}_{u,v} > 0$ such that*

$$\sup_{J\in\mathcal{D}} \frac{1}{v(J)} \sum_{I\in\mathcal{D}(J)} |\Delta_I v|^2 |I| m_I(u^{-1}) \le \mathcal{D}_{u,v} \tag{1.3}$$

where $\Delta_I v := m_{I_+}v - m_{I_-}v$

Assume that $b \in Carl_{u,v}$, that is $\mathcal{B}_{u,v} < \infty$, then π_b is bounded from $L^2(u)$ into $L^2(v)$.

Since the joint A_2-condition is a condition for the relationship between the two weights, and A_2-condition on each weight is a condition for each function, we can easily see that the two conditions are independent of each other. However, it is a very strong assumption to assume both conditions. Comparisons of the conditions on b such as BMO_ρ, $\mathbf{B}_2[u, v]$, and $\mathcal{B}_{u,v}$ are discussed in Section 2. We provide the discussion about the joint A_2-conditions as a necessary condition for the boundedness of the dyadic operators in Section 3. Then in Section 4 we restate the theorems which are introduced in this section.

2. $\mathbf{BMO}_\rho$, $\mathbf{B}_2[u, v]$, and $\mathcal{B}_{u,v}$

In this section we will compare conditions on b. Throughout the paper a constant C will be a numerical constant that may chage from line to line. First, we will compare the conditions on b under the joint-A_2 conditions.

Proposition 2.1. *For $(u, v) \in A_2$, there holds*

$$\mathbf{B}_2[u, v] \le [u, v]_{A_2} \sqrt{\mathcal{B}_{u,v}}$$

Proof. The result follows immediately from the joint-A_2 condition, $m_I(u^{-1})m_I v < [u,v]_{A_2}$ for all $I \in \mathcal{D}$.

$$\begin{aligned}
\mathbf{B}_2[u,v]^2 &= \sup_{J\in\mathcal{D}} \frac{1}{u^{-1}(J)} \sum_{I\in\mathcal{D}(J)} b_I^2 (m_I(u^{-1}))^2 m_I v \\
&\le [u,v]_{A_2} \sup_{J\in\mathcal{D}} \frac{1}{u^{-1}(J)} \sum_{I\in\mathcal{D}(J)} b_I^2 (m_I(u^{-1})) \\
&\le [u,v]_{A_2}^2 \sup_{J\in\mathcal{D}} \frac{1}{u^{-1}(J)} \sum_{I\in\mathcal{D}(J)} \frac{b_I^2}{m_I v} \\
&= [u,v]_{A_2}^2 \mathcal{B}_{u,v}\,.
\end{aligned}$$

□

However, we can't get the inequality in opposite way for Proposition 2.1. Let us assume that $(u,v) \in A_2$, $\{|I||\Delta_I v|^2 m_I(u^{-1})\}_I$ is a v-Carleson sequence. Then, by using the fact that the dyadic square function S^d is bounded from $L^2(v^{-1})$ into $L^2(u^{-1})$ and $1 \le m_I v m_I v^{-1}$, we have the followings

$$\begin{aligned}
\mathcal{B}_{u,v} &= \sup_{J\in\mathcal{D}} \frac{1}{u^{-1}(J)} \sum_{I\in\mathcal{D}(J)} \frac{b_I^2}{m_I v} \\
&\le \sup_{J\in\mathcal{D}} \frac{1}{u^{-1}(J)} \sum_{I\in\mathcal{D}(J)} b_I^2 m_I(v^{-1}) \\
&\le \sup_{J\in\mathcal{D}} \frac{1}{u^{-1}(J)} \|S^d((b - m_J b)\mathbb{1}_J)\|_{L^2(v^{-1})}^2 \\
&\le \|S^d\|_{L^2(v^{-1})\to L^2(u^{-1})} \sup_{J\in\mathcal{D}} \frac{1}{u^{-1}(J)} \|((b - m_J b)\mathbb{1}_J\|_{L^2(u^{-1})}^2 \\
&= \|S^d\|_{L^2(v^{-1})\to L^2(u^{-1})} \sup_{J\in\mathcal{D}} \frac{1}{u^{-1}(J)} \sum_{I\in\mathcal{D}(J)} b_I^2 m_I(u^{-1})\,.
\end{aligned}$$

In order to compare the last quantity in the inequality with $\mathbf{B}_2[u,v]^2$, we need an extra assumption that there is a constant $q > 0$ such that $m_I(u^{-1})m_I v \ge q$ for all $I \in \mathcal{D}$. But this extra assumption essentially reduces the problem to the one weight case [8]. Assuming the opposite case, that is $u, v \in A_2$, the following results are obtained. In [6], it is shown that the following are equivalent conditions

(i) $\displaystyle \sup_{I\in\mathcal{D}} \frac{1}{\rho(I)} \int_I |b(x) - m_I b|\,dx < \infty\,.$

(ii) $\displaystyle \sup_{I\in\mathcal{D}} \frac{1}{\rho(I)} \int_I |b(x) - m_I b|^2 \rho^{-1}(x)\,dx < \infty\,.$

(iii) $\displaystyle \sup_{I\in\mathcal{D}} \frac{1}{u(I)} \int_I |b(x) - m_I b|^2 v(x)\,dx < \infty\,.$

(iv) $\sup_{I\in\mathcal{D}} \frac{1}{v^{-1}(I)} \int_I |b(x) - m_I b|^2 u^{-1}(x)dx < \infty\,.$

Proposition 2.2. *For $u, v \in A_2$ and $\rho = (u/v)^{1/2}$ there holds*

$$Carl_{v,u} = BMO_\rho\,.$$

Proof. First we assume that $b \in Carl_{v,u}$, that is, there is a constant C such that $\sum_{I\in\mathcal{D}(J)} b_I^2/m_I u \le m_J(v^{-1})$ for all $J \in \mathcal{D}$. When $w \in A_2$ the dyadic square function S^d obeys a lower bound $\|f\|_{L^2(w)} \le C[w]_{A_2}^{1/2}\|S^d f\|_{L^2(w)}$. Using the lower bounds for S^d, for all $J \in \mathcal{D}$, we get the estimate

$$\begin{aligned}\|(b - m_J b)\mathbb{1}_J\|^2_{L^2(u^{-1})} &\le C[u]_{A_2}\|S^d((b - m_J b)\mathbb{1}_J)\|^2_{L^2(u^{-1})}\\ &= C[u]_{A_2}\sum_{I\in\mathcal{D}(J)} b_I^2 m_I(u^{-1})\\ &\le C[u]^2_{A_2}\sum_{I\in\mathcal{D}(J)}\frac{b_I^2}{m_I u}\\ &\le C[u]^2_{A_2} v^{-1}(J)\mathcal{B}_{v,u}\,.\end{aligned}$$

Hence we have that

$$\sup_{J\in\mathcal{D}}\frac{1}{v^{-1}(J)}\int_J |b(x) - m_I b|^2 u^{-1}(x)dx < C\mathcal{B}_{v,u}\,.$$

Assume now that $b \in \mathrm{BMO}_\rho$, that is, there is a constant C such that

$$\|(b - m_J b)\mathbb{1}_J\|^2_{L^2(u^{-1})} \le C\mathbf{B}_2[u, v]v^{-1}(J)\,.$$

We can conclude that

$$\begin{aligned}\sum_{I\in\mathcal{D}(J)}\frac{b_I^2}{m_I u} &\le \sum_{I\in\mathcal{D}(J)} b_I^2 m_I(u^{-1}) = \|S^d((b - m_J b)\mathbb{1}_J)\|^2_{L^2(u^{-1})}\\ &\le [u]_{A_2}\|(b - m_J b)\mathbb{1}_J\|^2_{L^2(u^{-1})}\\ &\le C[u]_{A_2} v^{-1}(J)\,.\end{aligned}$$

□

3. The joint A_2-condition as a necessary condition

In this section we will discuss about the joint-A_2 condition which is a necessary condition for the boundedness of the dyadic paraproduct and the commutator. For fixed $I \in \mathcal{D}$, let us choose $b = \sqrt{|I|}h_I = \mathbb{1}_{I_+} - \mathbb{1}_{I_-}$ and $f = u^{-1}\mathbb{1}_I$. Since $\langle\sqrt{|I|}h_I, h_J\rangle = \sqrt{|I|}$ for $I = J$ only and $\langle\sqrt{|I|}h_I, h_J\rangle = 0$ for others,

$$\|\pi_b\|_{L^2(u)\to L^2(v)} = \sup_{f\in L^2(u)}\frac{\|\pi_b f\|_{L^2(v)}}{\|f\|_{L^2(u)}} \ge \frac{\|\pi_{h_I} u^{-1}\mathbb{1}_I\|_{L^2(v)}}{\|u^{-1}\mathbb{1}_I\|_{L^2(u)}}$$

$$= \frac{\left(\int_{\mathbb{R}} \left|\sqrt{|I|} m_I(u^{-1}\mathbb{1}_I) h_I\right|^2 v dx\right)^{1/2}}{\left(\int_{\mathbb{R}} |u^{-1}\mathbb{1}_I|^2 u dx\right)^{1/2}}$$

$$= \frac{m_I(u^{-1})(m_I v)^{1/2}}{(m_I(u^{-1}))^{1/2}}.$$

We, therefore, have

$$m_I(u^{-1}) m_I v \leq \|\pi_b\|_{L^2(u)\to L^2(v)}.$$

To see the necessary condition for the commutator of the Hilbert transform, it suffices to check them for the commutator of the Haar shift operator, which has proven to be useful proof technique. With the decomposition of the commutator of the Haar shift operator from [3],

$$[b, Ш]f = [Ш(\pi_b f) - \pi_b(Шf)] + [Ш(\pi_b^* f) - \pi_b^*(Шf)] + [\pi_{Шf} b - Ш(\pi_f b)] \tag{3.1}$$

In [3], it also has presented that the first two terms have more singularities than the last term. Indeed, the last term is well localized due to some cancellations, but the others lost the localized property. Thus, we will only observe the necessary conditions by dealing with the last term of the decomposition (3.1). Similar to the calculation of the dyadic paraproduct, we choose $b = \sqrt{|I|} h_I$ and $f = u^{-1}\mathbb{1}_{I_\pm}$. By direct calculation,

$$\pi_{Шf} b - Ш(\pi_f b) = \sum_{I\in\mathcal{D}} \frac{b_I f_I}{\sqrt{|I|}} (h_{I_+} - h_{I_-}).$$

Thus, we get

$$\left\|\pi_{Ш(u^{-1}\mathbb{1}_{I_\pm})} \sqrt{|I|} h_I - Ш(\pi_{(u^{-1}\mathbb{1}_{I_\pm})} \sqrt{|I|} h_I)\right\|_{L^2(v)}$$

$$= \left(\int_{\mathbb{R}} \left|\langle u^{-1}\mathbb{1}_{I_\pm}, h_I\rangle^2 (h_{I_+} - h_{I_-})\right|^2 v dx\right)^{1/2}$$

$$= \frac{1}{4} m_{I_\pm}(u^{-1}) |I|^{1/2} \left(\int_I \left|h_{I_+} - h_{I_-}\right|^2 v dx\right)^{1/2}$$

$$= \frac{\sqrt{2}}{4} m_{I_\pm}(u^{-1}) |I|^{1/2} (m_I v)^{1/2},$$

and

$$\|u^{-1}\mathbb{1}_{I_\pm}\|_{L^2(u)} = \left(\int_{I_\pm} u^{-1} dx\right)^{1/2} = u^{-1}(I_\pm)^{1/2}.$$

Similar to the case of the dyadic paraproduct, the following two inequalities can be obtained:

$$\|\pi_{Шf} b - Ш(\pi_f b)\|_{L^2(u)\to L^2(v)} \geq C\sqrt{m_{I_+}(u^{-1})(m_I v)}$$

and

$$\|\pi_{\text{Ш} f} b - \text{Ш}(\pi_f b)\|_{L^2(u)\to L^2(v)} \geq C\sqrt{m_{I_-}(u^{-1})(m_I v)}\,.$$

By adding these inequalities, we get

$$\|\pi_{\text{Ш} f} b - \text{Ш}(\pi_f b)\|_{L^2(u)\to L^2(v)} \geq C\sqrt{m_I(u^{-1})(m_I v)}$$

Therefore, we can see in both cases of the dyadic paraproduct and the commutators of the Hilbert transform that $[u,v]_{A_2}$ is bounded by the operator norm from $L^2(u)$ into $L^2(v)$ and so $(u,v) \in A_2$.

4. Two weight estimate for some dyadic operators

In this section, we introduce the new versions of Theorem 1.1, 1.2, and 1.3 which are stated in Section 1. First using Proposition 2.2, we can replace BMO_ρ with $Carl_{u,v}$ as follows.

Theorem 4.1. *Let $u, v \in A_2$ and suppose that $b \in L^1$. Then*

(i) *If $b \in Carl_{v,u}$, the commutator $[b,H]$ is a bounded map from $L^2(u)$ into $L^2(v)$.*

(ii) *Conversely, if $[b,H] : L^2(u) \to L^2(v)$ is bounded, then $b \in Carl_{v,u}$.*

Using the observation in Section 3, we place the joint A_2-conditions as a necessary and sufficient condition for the boundedness of the given dyadic operators. Then we have the following theorems

Theorem 4.2. *Let $u, v \in A_2$ and $b \in Carl_{u,v}$. Then*

(i) *If $(u,v) \in A_2$ then the commutator $[b,H]$ is a bounded map from $L^2(u)$ into $L^2(v)$.*

(ii) *Conversely, if $[b,H] : L^2(u) \to L^2(v)$ is bounded, then $(u,v) \in A_2$.*

Theorem 4.3. *Let (u,v) be a pair of weights such that*

$$\sup_{J\in\mathcal{D}} \frac{1}{v(J)} \sum_{I\in\mathcal{D}(J)} |\Delta_I v|^2 |I| m_I(u^{-1}) < \infty$$

and $b \in Carl_{u,v}$. Then

(i) *If $(u,v) \in A_2$ then the dyadic paraproduct π_b is a bounded map from $L^2(u)$ into $L^2(v)$.*

(ii) *Conversely, if π_b is bounded from $L^2(u)$ into $L^2(v)$ then $(u,v) \in A_2$.*

References

[1] O. Beznosova, D. Chung, J.C. Moraes, and M.C. Pereyra, *On two weight estimates for dyadic operators*, Harmonic Analysis, Partial Differential Equations, Complex Analysis, Banach Spaces, and Operator Theory (Volume 2) AWM Series 5, Springer International Publishing (2017) 135–169

[2] S. Bloom, *A commutator theorem and weighted BMO*, Trans. Amer. Math. Soc. **292** (1985), no. 1, 103–122.

[3] D. Chung, *Sharp estimates for the commutators of the Hilbert, Riesz transforms and the Beurling-Ahlfors operator on weighted Lebsgue spaces*, Indiana Univ. Math. Journal, **60** (2011) no. 5, 1543–1588.
[4] D. Chung, *On a two weights estimate for the commutator*, East Asian Math. J. **33** (2017), no.1, 103-113.
[5] I. Holmes, M.T. Lacey and B.D. Wick, *Blooms inequality: commutators in a two-weight setting*, Arch. Math. **106** (2016), no. 1, 53–63.
[6] I. Holmes, M.T. Lacey and B.D. Wick, *Commutators in the two-weight setting*, Math. Ann. (2016). doi:10.1007/s00208-016-1378-1.
[7] T. Hytönen and A. Kairema, *Systems of dyadic cubes in a doubling metric space*, Colloq. Math. **126** (2012), 1–33.
[8] J. C. Moraes, *Weighted estimates for dyadic operators with complexity* PhD Dissertation, University of New Mexico, (2011)
[9] J. C. Moraes and M.C. Pereyra, *Weighted estimates for dyadic Paraproduct and t-Haar multiplies with complexity* (m, n), Publ. Mat. **57** (2013), 265–294.
[10] F. Nazarov, S. Treil and A. Volberg, *The Bellman functions and the two-weight inequalities for Haar Multipliers*, Journal of the AMS, **12** (1992), 909–928.

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