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OPTIMAL PORTFOLIO CHOICE IN A BINOMIAL-TREE AND ITS CONVERGENCE

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Abstract. This study investigates the convergence of the optimal consumption and investment policies in a binomial-tree model to those in the continuous-time model of Merton (1969). We provide the convergence in explicit form and show that the convergence rate is of order Δt, which is the length of time between consecutive time points. We also show by numerical solutions with realistic parameter values that the optimal policies in the binomial-tree model do not differ significantly from those in the continuous-time model for long-term portfolio management with a horizon over 30 years if rebalancing is done every 6 months.

1. Introduction

A binomial tree is a stochastic model that can be used to describe asset price movement as a discrete time process. In particular, it is well known that option prices are easily computed using a binomial model (Hull and Basu [15]). Also, it is often used to model price increments of risky assets in utility maximization problems (Ahn and Koo [2]).

In this study, we investigate the relationship between a binomial tree and a continuous time model in an optimal consumption and portfolio choice problem. We construct a binomial tree model corresponding to the continuous-time model of Merton [21] and compare the optimal policies. Specifically, we provide a proof of convergence of the optimal policies in the binomial model to those in the continuous time model and derive the convergence rate, as the time interval Δt between time nodes decreases. We also show by using numerical solutions that a binomial tree with time interval of 6 months can provide a good approximation to a continuous time model.

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Convergence from a utility maximization model in discrete time to that in continuous time has been studied in He [14] and Bayraktar et al. [7]. They investigate whether optimal policies and value functions converge if discrete-time price processes converge to those in continuous time in distribution, and propose necessary conditions for the convergence. He [14] studies weak convergence in a complete market and Bayraktar et al. [7] does in an incomplete market. In particular, He [14] shows that the optimal consumption and portfolio policies converge weakly and the value function converges to those in the continuous time model, investigating the convergence of a multi-nomial model to a multi-dimensional model. The previous research, however, establishes only abstract weak convergence based on the dual martingale approach and does not provide a guideline for practical application of a discrete-time model as an approximation to a continuous-time model.

We derive the explicit-form solutions of optimal choices in a financial market with asset prices following a binomial tree model, which converges to the Black and Scholes [8] model, and establish convergence of optimal policies. Our convergence proof is more explicit in that it establishes the convergence of explicit-form policies, not just weak convergence as in the previous literature. In addition, we provide a guideline for practical application of the binomial tree to portfolio choice problems by deriving the convergence rate of optimal policies. We show that the convergence rate is proportionate to the length Δt of the time interval between consecutive time points.[1] For an application to long-term portfolio management, by numerical solutions we provide evidence that the binomial model with time interval of 6 months provides a good approximation to optimal policies in the continuous-time model. This paper is the first to examine the convergence speed of the optimal policies of a binomial tree model in a utility optimization problem.

The rest of this paper is structured as follows. In Section 2, we provide a brief literature survey. Section 3 presents the utility maximization problem in a binomial model and derives its analytical solutions. Section 4 provides the proof of convergence of optimal policies and derive the convergence rate. Section 5 concludes.

2. Related Literature

Since Cox et al. [10], binomial tree models have been used in finance to model stochastic asset price processes. In particular, it has been widely used to solve challenging problems with elementary technique that are difficult to derive explicit solutions, e.g., pricing American options.

For convergence of asset prices in binomial models, Cox et al. [10] show that European vanilla option prices in a binomial model converge to the price in Black and Scholes [8] model and Omberg [22] investigate the convergence of several binomial-pricing parameters, including one used in Cox et al. [10], to

[1] That is, the convergence errors decline in proportion to Δt.

an appropriate lognormal diffusion process. As a natural next step, Amin and Khanna [3] and Jiang and Dai [17] prove that American vanilla option prices in a binomial tree model also converge to those in the corresponding continuous time model. Moreover, Jiang and Dai [18] investigate the convergence for European and American path-dependent options. Qian et al. [23] and Kwon and Kim [20] study the convergence with jump diffusion process in American and look-back options, respectively. All the studies mentioned above study the convergence in asset prices, but do not consider the convergence of optimal policies of an agent.

As other directions of taking advantage of simple binomial tree models, Boyle and Vorst [9] replicate options with transaction costs, Balzer [4] proposes a model dealing with an investment performance measure, and Bäuerle and Mundt [5] study a model of risk management with a binomial tree model. Again, the optimal policy of agents are not investigated in these studies.

There is a vast literature on utility optimization with a binomial tree. Dybvig and Koo [11] introduce a binomial tree method to analyze asset allocation with taxes and propose a numerical algorithm. Ahn and Koo [2] solve the optimization problem with Epastein and Zin [12]-type utility function, and Rizal et al. [24] do with a HARA utility function. Ahn et al. [1] analyze the equity premium puzzle in a general equilibrium framework. Jang et al. [16] use a binomial tree model to analyze a model with longevity risk. However, all these studies do not compare their optimal policies with those in a continuous time model.

For a utility optimization problem, in addition to He [14] and Bayraktar et al. [7] mentioned previously, Bayer and Veliyev [6] study the convergence of optimal policies in a discrete portfolio choice problem to those in a continuous model. They consider a terminal wealth maximization problem given proportional transaction costs and demonstrate the weak convergence of binomial model. However, they use the log utility function and only consider the terminal wealth, whereas we use a constant relative risk aversion (CRRA) utility function and consider utilities from intermediate consumptions as well as from the terminal wealth. Therefore, our study and their study are complementary to each other, since Bayer and Veliyev [6] consider transaction costs, not covered in this study.

3. A Binomial Tree Model and a Utility Maximization Problem

Before describing our binomial tree model, we review Merton [21]'s problem as a benchmark of the continuous time model. In his problem, an agent wishes to maximize the following lifetime utility function:

$$\mathbb{E}\left[\int_0^T e^{-\rho t}U(c_t)dt + e^{-\rho T}B(W_T)\right], \tag{1}$$

where c_t is the consumption rate at time t, W_t is the agent's wealth level at time t, $\rho > 0$ is the subjective discount rate for felicity of future consumption. He considers the felicity function U of the CRRA type and provides a solution in explicit form:

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

where γ is the coefficient of relative risk aversion. The bequest function $B(W)$ is given by

$$B(W) = \epsilon^{\gamma} \frac{W^{1-\gamma}}{1-\gamma},$$

where ϵ is a parameter denoting the strength of bequest motive.

Merton [21] assumes two financial assets in the market: a risky asset and a riskless asset.[2] The rate of return on the riskless asset is constant $r > 0$. The price dynamics of the risky asset follows a geometric Brownian motion as

$$\frac{dS}{S} = \mu dt + \sigma dZ_t, \tag{2}$$

where Z_t is the standard Brownian motion at time t, $\mu > 0$ is a constant expected rate of return, and σ is the constant volatility.

The agent's wealth evolves according to the dynamics:

$$dW_t = (rW_t - c_t + \pi_t(\mu - r))\, dt + \pi_t \sigma dZ_t$$

where π_t is the dollar amount of risky investment at time t. The following assumption is made for well-posedness of the optimization problem.

Assumption 3.1.

$$K \equiv r + \frac{\rho - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2 > 0. \tag{3}$$

Let θ denote the market price of risk as follows:

$$\theta \equiv \frac{\mu - r}{\sigma}.$$

Proposition 3.2. *(Merton) The optimal policies in the above optimization problem are given as follows:*

$$c^* = \frac{K}{1 + (K\epsilon - 1)\, e^{-KT}} W_0, \ \textit{for } K > 0 \tag{4}$$

$$\pi^* = \frac{\mu - r}{\gamma\sigma^2} W_0. \tag{5}$$

We now describe a counterpart of Merton [21]'s problem in discrete time. We consider the following objective function in discrete time:

$$\mathbb{E}\left[\sum_{i=0}^{N-1} \beta^i U(c_{t_i}) \Delta t + \beta^N B(W_T)\right]$$

[2] In section 4.2, the market portfolio is used as a proxy for this risky asset. Even if the risky asset is replaced by an individual stock, essentially the same result will be obtained in the economy where CAPM holds.

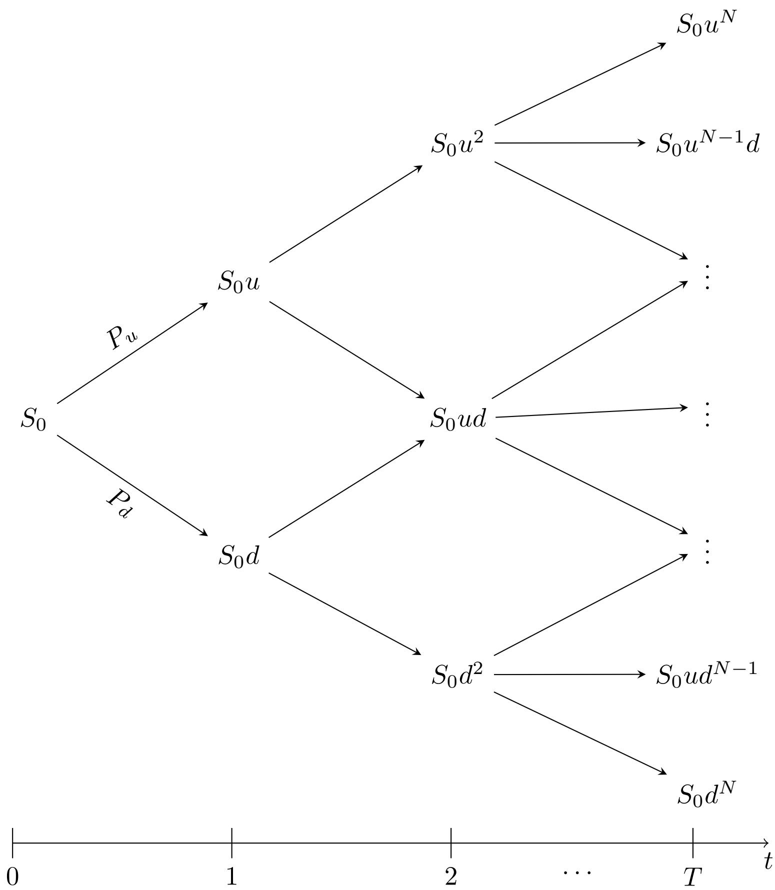


FIGURE 1. $N+1$-period Binomial tree

where β is the subjective discount factor for the future utility of consumption, which corresponds to $e^{-\rho\Delta t}$ in equation (1). The time interval between t_i and t_{i+1} is denoted by Δt and each t_i satisfies

$$0 = t_0 < t_1 < \cdots < t_N = T.$$

We consider a binomial tree model of the dynamics of the risky asset price $S(t)$.

As illustrated in Figure 1, there are two states of the gross return of the risky asset price at each step: *up* and *down*, and the gross returns are denoted by u and d, respectively. The probability of each state is also denoted P_u and P_d respectively. Of course, the sum of P_u and P_d is equal to 1. It is shown in

He [13] that this binomial tree model converges in distribution to the geometric Brownian motion in (2).

Note that if we denote $1+r$ as R, the agent's total wealth at time $t+1$, W_{t+1}, can be written as

$$\begin{aligned} W_{t+1,u} &= R\left(W_t - c_t\Delta t - \pi_t\right) + \pi_t u, \\ W_{t+1,d} &= R\left(W_t - c_t\Delta t - \pi_t\right) + \pi_t d. \end{aligned}$$

3.1. Two-period model

We first consider a two-period model. To derive the optimal policies of the model, we apply the dynamic programming approach. The agent's lifetime utility at time t can be represented as follows:

$$U(c_t)\Delta t + \beta\mathbb{E}_t\left[U(c_{t+1})\Delta t + \beta\mathbb{E}_{t+1}\left[B(W_{t+2})\right]\right], \tag{6}$$

where $\mathbb{E}_t$ is conditional expectation at time t. Equation (6) is different from other common 2-period utility optimization problems. As c_t in (6) is the consumption rate, the utility at time t should be $U(c_t)\Delta t$. In addition, even if (6) is a two-period model, it looks like a three-period model because the agent bequeaths wealth after consuming from $t+1$ to $t+2$. The agent's goal is finding the optimal combination of c_t and π_t in (6).

For simplicity, we take $t=0$. The value function $V_0(W_0)$ is written as follows:

$$V_0(W_0) = \max_{c,\pi}\left\{U(c_0)\Delta t + \beta\mathbb{E}\left[U(c_1)\Delta t + \beta\mathbb{E}_1\left[B(W_2)\right]\right]\right\}. \tag{7}$$

Equation (7) can also be rewritten as

$$V_0(W_0) = \max_{c_0,\pi_0}\left\{U(c_0)\Delta t + \beta\mathbb{E}\left[\max_{c_1,\pi_1}\left\{U(c_1)\Delta t + \beta\mathbb{E}_1\left[B(W_2)\right]\right\}\right]\right\}.$$

We define the following value function for $t=1$:

$$V_1(W_1) = \max_{c_1,\pi_1}\left\{U(c_1)\Delta t + \beta\mathbb{E}\left[B(W_2)\right]\right\}. \tag{8}$$

Proposition 3.3. *The optimal consumption and portfolio investment of the maximization problem in* (8) *are given as follows:*

$$\begin{aligned} c_1^* &= \frac{\beta^{-\frac{1}{\gamma}}R^{1-\frac{1}{\gamma}}\epsilon^{-1}}{\beta^{-\frac{1}{\gamma}}R^{1-\frac{1}{\gamma}}\epsilon^{-1}\Delta t + P_u^{\frac{1}{\gamma}}q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}}q_d^{1-\frac{1}{\gamma}}}W_1, \\ \pi_1^* &= \frac{R\left((q_u/P_u)^{-\frac{1}{\gamma}} - (q_d/P_d)^{-\frac{1}{\gamma}}\right)}{(u-d)\left(\beta^{-\frac{1}{\gamma}}R^{1-\frac{1}{\gamma}}\epsilon^{-1}\Delta t + P_u^{\frac{1}{\gamma}}q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}}q_d^{1-\frac{1}{\gamma}}\right)}W_1, \end{aligned}$$

where q_u and q_d are the risk neutral probabilities of up state and down state, respectively, as following:

$$q_u \equiv \frac{R-d}{u-d}, \quad q_d \equiv \frac{u-R}{R-d}. \tag{9}$$

The proof of Theorem 3.3 is provided in the appendix. Substituting optimal policies into the value function in (8), we obtain the following corollary.

Corollary 3.4. *The value function* (8) *is given by*

$$V_1(W_1) = A_1 \frac{W_1^{1-\gamma}}{1-\gamma}. \tag{10}$$

Here, A_1 *is defined as*

$$A_1 \equiv \beta R^{1-\gamma} \epsilon^{\gamma} \left(\beta^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}} \epsilon^{-1} \Delta t + P_u^{\frac{1}{\gamma}} q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}} q_d^{1-\frac{1}{\gamma}} \right)^{\gamma}. \tag{11}$$

By Corollary 3.4, we can easily derive the optimal policies for the two-period model.

Proposition 3.5. *The optimal policies for the two-period model are given as*

$$\begin{aligned}
c_0^* &= \frac{(\beta A_1)^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}}}{(\beta A_1)^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}} \Delta t + P_u^{\frac{1}{\gamma}} q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}} q_d^{1-\frac{1}{\gamma}}} W_0, \\
\pi_0^* &= \frac{R\left((q_u/P_u)^{-\frac{1}{\gamma}} - (q_d/P_d)^{-\frac{1}{\gamma}} \right)}{(u-d)\left((\beta A_1)^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}} \Delta t + P_u^{\frac{1}{\gamma}} q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}} q_d^{1-\frac{1}{\gamma}} \right)} W_0.
\end{aligned}$$

In addition, the value function (7) *can be written as*

$$V_0(W_0) = A_2 \frac{W_0^{1-\gamma}}{1-\gamma}$$

where

$$A_2 \equiv \beta R^{1-\gamma} A_1 \left((\beta A_1)^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}} \Delta t + P_u^{\frac{1}{\gamma}} q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}} q_d^{1-\frac{1}{\gamma}} \right)^{\gamma}.$$

3.2. N+1-period model

We now consider a general $N+1$-period model. By application of dynamic programming, we obtain the following result.

Theorem 3.6. *The optimal policies and value function of* $N+1$*-period model are given by*

$$c_0^* = \frac{(\beta A_N)^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}}}{(\beta A_N)^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}} \Delta t + P_u^{\frac{1}{\gamma}} q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}} q_d^{1-\frac{1}{\gamma}}} W_0, \tag{12}$$

$$\pi_0^* = \frac{R\left((q_u/P_u)^{-\frac{1}{\gamma}} - (q_d/P_d)^{-\frac{1}{\gamma}} \right)}{(u-d)\left((\beta A_N)^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}} \Delta t + P_u^{\frac{1}{\gamma}} q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}} q_d^{1-\frac{1}{\gamma}} \right)} W_0, \tag{13}$$

$$V_0(W_0) = A_{N+1} \frac{W_0^{1-\gamma}}{1-\gamma} \tag{14}$$

where the sequence $\{A_N\}$ *satisfies the following recursive relation:*

$$A_{N+1} = \beta R^{1-\gamma} A_N \left((\beta A_N)^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}} \Delta t + P_u^{\frac{1}{\gamma}} q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}} q_d^{1-\frac{1}{\gamma}} \right)^{\gamma}. \tag{15}$$

The solution to the recursive relation (15) can be represented as

$$A_{N+1}^{\frac{1}{\gamma}} = \Delta t \sum_{i=0}^{N} D^i + A_0^{\frac{1}{\gamma}} D^{N+1} \tag{16}$$

where

$$A_0 \equiv \epsilon^{\gamma}, \tag{17}$$

$$D \equiv \beta^{\frac{1}{\gamma}} R^{\frac{1-\gamma}{\gamma}} \left(P_u^{\frac{1}{\gamma}} q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}} q_d^{1-\frac{1}{\gamma}} \right). \tag{18}$$

4. Convergence to continuous time model

4.1. Convergence and convergence rate

We apply the parameters as proposed by Hull and Basu [15] as shown in the first and second columns of Table 1. In addition, the third column of Table 1 is the Taylor series approximation to the parameters in the second column.

Table 1. Parameters and approximation of binomial model

parameters	value	approximation
u	$e^{\sigma\sqrt{\Delta t}}$	$1 + \sigma\sqrt{\Delta t} + O(\Delta t)$
d	$e^{-\sigma\sqrt{\Delta t}}$	$1 - \sigma\sqrt{\Delta t} + O(\Delta t)$
R	$e^{r\Delta t}$	$1 + r\Delta t + O\left((\Delta t)^2\right)$
β	$e^{-\rho\Delta t}$	$1 - \rho\Delta t + O\left((\Delta t)^2\right)$
P_u	$\frac{e^{\mu\Delta t}-d}{u-d}$	$\frac{1}{2}\left(1 + \xi_\mu\sqrt{\Delta t} + O(\Delta t)\right)$
P_d	$\frac{u-e^{\mu\Delta t}}{u-d}$	$\frac{1}{2}\left(1 - \xi_\mu\sqrt{\Delta t} + O(\Delta t)\right)$
q_u	$\frac{e^{r\Delta t}-d}{u-d}$	$\frac{1}{2}\left(1 + \xi_r\sqrt{\Delta t} + O(\Delta t)\right)$
q_d	$\frac{u-e^{r\Delta t}}{u-d}$	$\frac{1}{2}\left(1 - \xi_r\sqrt{\Delta t} + O(\Delta t)\right)$

Here, we define ξ_μ and ξ_r in Table 1 as follows:

$$\xi_\mu = \frac{\mu}{\sigma}, \quad \xi_r = \frac{r}{\sigma}.$$

To prove the convergence of the optimal policies to those in the continuous time, it is sufficient to show that equations (12) and (13) converge to (4) and (5) respectively.

Lemma 4.1. *Applying the Taylor series expansion to D in* (18) *of the recurrence relation $\{A_N\}$, we obtain the following:*

$$D = 1 - \left(r + \frac{\rho - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2}\theta^2 \right) \Delta t + O\left((\Delta t)^2 \right).$$

By Assumption 3.1, $D < 1$. Then, the limit of D^N is given as

$$\lim_{N \to \infty} D^N = e^{-KT}.$$

The solution A_N to the recursive equation in (16) *is obtained as*

$$A_N^{\frac{1}{\gamma}} = \frac{1 - D^{N-1}}{1 - D} \Delta t + \epsilon D^N,$$

and the limit is

$$\lim_{N \to \infty} A_N^{\frac{1}{\gamma}} = \frac{1 + (K\epsilon - 1)\, e^{-KT}}{K}.$$

The following Theorem is the main result of this study. We show that the optimal consumption and portfolio policies in the binomial model converge to those in Merton's continuous time model, and the rate of convergence is proportional to Δt.

Theorem 4.2. *Let the convergence error of the optimal portfolio and consumption be l_π and l_c, respectively. When Δt is small enough, the errors satisfy the following:*

$$l_\pi = \frac{1}{\sigma} \left(M + \frac{\mu - r}{\gamma\sigma^2} \left(\frac{r}{\sigma} - \frac{K}{1 + (K\epsilon - 1)e^{-KT}} - \frac{1 - \gamma}{2\gamma^2}\theta^2 \right) \right) W_0 \Delta t + O\left((\Delta t)^2 \right), \quad (19)$$

$$l_c = \left(\frac{K}{1 + (K\epsilon - 1)e^{-KT}} \left(r + \frac{\rho - r}{\gamma} - 1 \right) - \frac{1 - \gamma}{2\gamma^2}\theta^2 \right) W_0 \Delta t + O\left((\Delta t)^2 \right) \quad (20)$$

where

$$M \equiv \frac{1}{3\gamma} \left(\frac{1}{\gamma} - 1 \right) \left(\frac{1}{\gamma} - 2 \right) \xi_\mu^3 - \frac{1}{\gamma^2} \left(\frac{1}{\gamma} - 1 \right) \xi_\mu^2 \xi_r + \frac{1}{\gamma^2} \left(\frac{1}{\gamma} + 1 \right) \xi_\mu \xi_r^2 - \frac{1}{3\gamma} \left(\frac{1}{\gamma} + 1 \right) \left(\frac{1}{\gamma} + 2 \right) \xi_r^3.$$

The proof is given in the appendix. Theorem 4.2 has a significance for an approach that approximates a continuous time model using a binomial tree model. We know that the convergence speed of risky asset to a continuous time model is proportional to $\sqrt{\Delta t}$, which is well represented in Table 1. However, the convergence rate of the optimal policies is proportional to Δt as in Theorem 4.2, and therefore, we can say that the convergence rate of the optimal strategy is faster than that of the asset price process. Consequently, it is efficient to approximate a continuous time model of consumption and portfolio choice by a binomial model.

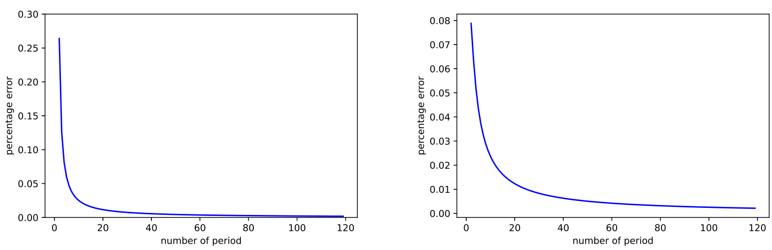
4.2. Numerical solutions and convergence

In this section we provide numerical solutions and illustrate the convergence of optimal policies. For the numerical solutions, we set the parameter values as follows:

$$\mu = 0.07, \quad r = 0.01, \quad \sigma = 0.2, \quad \gamma = 2, \quad \epsilon = 0.1, \quad \rho = 0.02, \quad T = 30. \tag{21}$$

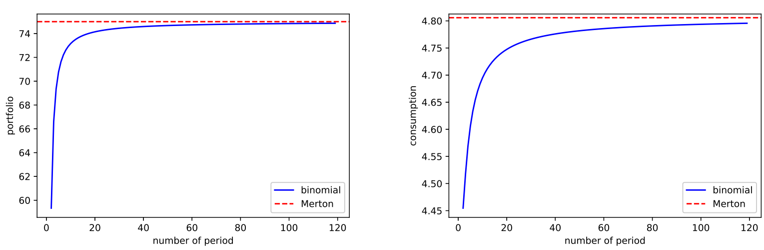
According to Jordà et al. [19], the average rate of return and the volatility of global stocks have been 6.88% and 21.79%, respectively, over the past 100 years. During the same period, the short-term bond yield has been 1.03%. Hence, the above parameters are consistent with the data from the global financial market. In addition, we set the maturity T is 30 years because we consider a long-term investor's problem.

Figure 2 shows convergence with respect to the number of period, and Figure 3 presents the percentage error of convergence. The left panels give the optimal share of investment in the risky asset, and the right panels give optimal consumption.



(a) Portfolio (b) Consumption

FIGURE 2. Percentage errors of optimal policies



(a) Portfolio (b) Consumption

FIGURE 3. Convergence of binomial model

Figures 2 and 3 show that as N increases, the optimal policies in the binomial model converge to those in the model of Merton [21].

In binomial tree models, fixing the expiration date and increasing the number of periods is equivalent to decreasing the length of time interval Δt. When consumption and investment decisions are made in discrete time,[3] we need to know how one year should be discretized to obtain results close enough to a continuous time model. For example, if an investor wants to reduce frequency of transactions to save transaction or other costs, we must know the minimum number of tradings that errors are within a certain level. The following table presents the percentage errors according to the number of periods on 30-year and 50-year maturity. The parameter values are the same as in (21) except for T.

TABLE 2. Percentage errors depending on maturity

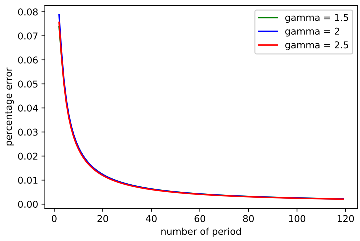
parameters	30-year maturity		50-year maturity	
(N)	portfolio	consumption	portfolio	consumption
10	0.0546	0.0236	0.0275	0.0295
30	0.0178	0.0083	0.0093	0.0103
50	0.0107	0.0050	0.0056	0.0062
100	0.0053	0.0025	0.0028	0.0031
300	0.0018	0.0009	0.0009	0.0010
500	0.0011	0.0005	0.0006	0.0006

The implication of Table 2 is as follows: with the maturity of 30 years, it is enough to model 50 discrete periods if the error between the policies in the discrete and continuous time is allowed to be within 1%. In other words, we have to make a decision only every 7 months to choose the optimal policies close to those in a continuous time. Likewise, if we consider a maturity of 50 years and the tolerance level is 0.01, then we may choose the number of period of 30 discrete times (i.e., the rebalancing takes place almost every 20 months). Note that the errors in Table 2 are the result from applying the parameter values in (21). They change with the level of γ, ϵ, ρ, reflecting the subjective preference.

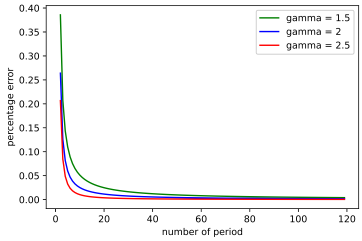
Figure 4 gives the percentage error according to a change in the risk aversion coefficient. All parameter values are the same as in (21) except for γ.

In Figure 4, the error of the consumption choice is consistent regardless of the level of γ, and however, the error of portfolio choices decreases with the coefficient of risk aversion. The error is also small enough to be ignored with large N regardless of γ, and thus the binomial model provides a good approximation to a continuous-time model if N is sufficiently large.

[3]Even with a high frequency trading, it is equivalent to a transaction which Δt is close to zero on discrete time



(a) Consumption



(b) Portfolio

FIGURE 4. Percentage errors depending on γ

5. Conclusion

In this study, we have shown that the optimal policies of a binomial model converge to those in a continuous time model, and the convergence rate is proportional to the length Δt between consecutive time points. In particular, We have obtained the solution of Merton [21] as Δt approaches zero.

We have also illustrated by numerical solutions that the approximation approach with a binomial tree model to a continuous model is very efficient. We have shown that the optimal policies in a long-term portfolio problem of more than 30 years are not significantly different from the those of the continuous model when rebalancing is made every six months.

It would be interesting to extend the research by considering richer economic features or more realistic market enviorments, e.g. the Epstein and Zin [12]-type utility function, time-varying investment opportunities, optimal stopping time problems, specific budget constraints, and transaction costs. It is also expected that many research problems, difficult to solve in a continuous time model, can be solved with elementary techniques in a discrete time model similar to the binomial model in this study.

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Appendix A. Proof of Theorem 3.3

The value function can be represented by following:

$$\begin{aligned} \frac{\partial V_1}{\partial c_1} &= c_1^{-\gamma} - R\beta\epsilon^{\gamma}[P_u\left(R\left(W_1 - c_1\Delta t - \pi_1\right) + u\pi_1\right)^{-\gamma} \quad (22) \\ &\quad + P_d\left(R\left(W_1 - c_1\Delta t - \pi_1\right) + d\pi_1\right)^{-\gamma}] \\ &= 0, \\ \frac{\partial V_1}{\partial \pi_1} &= \beta\epsilon^{\gamma}[(u-R)P_u\left(R\left(W_1 - c_1\Delta t - \pi_1\right) + u\pi_1\right)^{-\gamma} \quad (23) \\ &\quad + (R-d)P_d\left(R\left(W_1 - c_1\Delta t - \pi_1\right) + d\pi_1\right)^{-\gamma}] \\ &= 0. \end{aligned}$$

We can derive the following from equation (23) using risk neutral probability definition in (9).

$$\pi_1 = \frac{R\left((q_u/P_u)^{-\frac{1}{\gamma}} - (q_d/P_d)^{-\frac{1}{\gamma}}\right)}{(u-d)\left(P_u^{\frac{1}{\gamma}} q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}} q_d^{1-\frac{1}{\gamma}}\right)} \left(W_1 - c_1\Delta t\right). \quad (24)$$

And then the optimal consumption rate at time $t = 1$ (22) is calculated by the above equation (24) as follows:

$$c_1^* = \frac{\beta^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}} \epsilon^{-1}}{\beta^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}} \epsilon^{-1}\Delta t + P_u^{\frac{1}{\gamma}} q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}} q_d^{1-\frac{1}{\gamma}}} W_1.$$

Substituting this optimal consumption into Equation (24), we have the optimal portfolio as follows:

$$\pi_1^* = \frac{R\left((q_u/P_u)^{-\frac{1}{\gamma}} - (q_d/P_d)^{-\frac{1}{\gamma}}\right)}{(u-d)\left(\beta^{-\frac{1}{\gamma}} R^{1-\frac{1}{\gamma}} \epsilon^{-1}\Delta t + P_u^{\frac{1}{\gamma}} q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}} q_d^{1-\frac{1}{\gamma}}\right)} W_1.$$

Appendix B. Proof of Theorem 4.2

(1) The error of optimal portfolio is computed as follows:

$$\begin{aligned}
l_\pi &= \frac{\mu-r}{\gamma\sigma^2}W_0 - \frac{R\left((q_u/P_u)^{-\frac{1}{\gamma}} - (q_d/P_d)^{-\frac{1}{\gamma}}\right)}{(u-d)\left((\beta A_N)^{-\frac{1}{\gamma}}R^{1-\frac{1}{\gamma}}\epsilon^{-1}\Delta t + P_u^{\frac{1}{\gamma}}q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}}q_d^{1-\frac{1}{\gamma}}\right)}W_0 \\
&= \frac{\mu-r}{\gamma\sigma^2}W_0 - \frac{2(1+r\Delta t)\left(\frac{\xi_\mu-\xi_r}{\gamma}\sqrt{\Delta t} + M(\Delta t)^{\frac{3}{2}} + O\left((\Delta t)^{\frac{5}{2}}\right)\right)}{2\sigma\sqrt{\Delta t}\left[1+\left(\left(1+\left(r+\frac{\rho-r}{\gamma}\right)\Delta t\right)A_N^{-\frac{1}{\gamma}} + \frac{1-\gamma}{2\gamma^2}\theta^2\right)\Delta t + O\left((\Delta t)^2\right)\right]} \\
&= \frac{\mu-r}{\gamma\sigma^2}W_0 - \frac{\frac{\mu-r}{\gamma\sigma} + \left(M+\frac{r(\mu-r)}{\gamma\sigma}\right)\Delta t + O\left((\Delta t)^2\right)}{\sigma\left[1+\left(A_N^{-\frac{1}{\gamma}} + \frac{1-\gamma}{2\gamma^2}\theta^2\right)\Delta t + O\left((\Delta t)^2\right)\right]}W_0 \\
&= \frac{\mu-r}{\gamma\sigma^2}W_0 - \frac{1}{\sigma}\left(\frac{\mu-r}{\gamma\sigma} + \left(M+\frac{r(\mu-r)}{\gamma\sigma}\right)\Delta t + O\left((\Delta t)^2\right)\right) \\
&\quad \times\left(1-\left(A_N^{-\frac{1}{\gamma}} + \frac{1-\gamma}{2\gamma^2}\theta^2\right)\Delta t + O\left((\Delta t)^2\right)\right)W_0 \\
&= \frac{1}{\sigma}\left(M+\frac{r(\mu-r)}{\gamma\sigma} - \frac{\mu-r}{\gamma\sigma^2}\left(A_N^{-\frac{1}{\gamma}} + \frac{1-\gamma}{2\gamma^2}\theta^2\right)\right)W_0\Delta t + O\left((\Delta t)^2\right) \\
&= \frac{1}{\sigma}\left(M+\frac{\mu-r}{\gamma\sigma^2}\left(\frac{r}{\sigma} - A_N^{-\frac{1}{\gamma}} - \frac{1-\gamma}{2\gamma^2}\theta^2\right)\right)W_0\Delta t + O\left((\Delta t)^2\right).
\end{aligned}$$

Let $N\to\infty$ and we obtain (19).

$$l_\pi = \frac{1}{\sigma}\left(M+\frac{\mu-r}{\gamma\sigma^2}\left(\frac{r}{\sigma} - \frac{K}{1+(K\epsilon-1)e^{-KT}} - \frac{1-\gamma}{2\gamma^2}\theta^2\right)\right)W_0\Delta t + O\left((\Delta t)^2\right).$$

(2) The error of consumption is computed as follows:

$$\begin{aligned}
l_c &= \frac{K}{1+(K\epsilon-1)e^{-KT}}W_0 - \frac{(\beta A_N)^{-\frac{1}{\gamma}}R^{1-\frac{1}{\gamma}}}{(\beta A_N)^{-\frac{1}{\gamma}}R^{1-\frac{1}{\gamma}}\Delta t + P_u^{\frac{1}{\gamma}}q_u^{1-\frac{1}{\gamma}} + P_d^{\frac{1}{\gamma}}q_d^{1-\frac{1}{\gamma}}}W_0 \\
&= \frac{K}{1+(K\epsilon-1)e^{-KT}}W_0 - \frac{A_N^{-\frac{1}{\gamma}}\left(1+\left(r+\frac{\rho-r}{\gamma}\right)\Delta t\right) + O\left((\Delta t)^2\right)}{1+\left(A_N^{-\frac{1}{\gamma}} + \frac{1-\gamma}{2\gamma^2}\theta^2\right)\Delta t + O\left((\Delta t)^2\right)}W_0 \\
&= \frac{K}{1+(K\epsilon-1)e^{-KT}}W_0 - \left(A_N^{-\frac{1}{\gamma}} + A_N^{-\frac{1}{\gamma}}\left(r+\frac{\rho-r}{\gamma}\right)\Delta t + O\left((\Delta t)^2\right)\right) \\
&\quad \times\left(1-\left(A_N^{-\frac{1}{\gamma}} + \frac{1-\gamma}{2\gamma^2}\theta^2\right)\Delta t + O\left((\Delta t)^2\right)\right)W_0 \\
&= \frac{K}{1+(K\epsilon-1)e^{-KT}}W_0 - \left(A_N^{-\frac{1}{\gamma}} + \left(A_N^{-\frac{1}{\gamma}}\left(r+\frac{\rho-r}{\gamma}-1\right) - \frac{1-\gamma}{2\gamma^2}\theta^2\right)\Delta t + O\left((\Delta t)^2\right)\right)W_0.
\end{aligned}$$

In a similar way, we obtain (20) as $N\to\infty$ as follows:

$$l_c = \left(\frac{K}{1+(K\epsilon-1)e^{-KT}}\left(r+\frac{\rho-r}{\gamma}-1\right) - \frac{1-\gamma}{2\gamma^2}\theta^2\right)W_0\Delta t + O\left((\Delta t)^2\right).$$

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