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THE MONOTONE PROPERTY OF THE FIRST NONZERO EIGENVALUE OF THE P-LAPLACIAN ALONG THE INVERSE MEAN CURVATURE FLOW WITH FORCED TERM

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Abstract. In this paper, we prove that the first nonzero eigenvalues λ_1 of the Laplacian and the p-Laplacian are decreasing along the inverse mean curvature flow with forced term in Euclidean space.

1. Introduction

Laplace-Beltrami operator is a second order differential operator on a Riemannian manifold. It occurs in many places that describes physical phenomena, such as the diffusions of heat and fluid flow, wave propagation, and quantum mechanics. The p-Laplace operator is a nonlinear generalization of the Laplace-Beltrami operator, where p is allowed to range over $1 < p < \infty$. It is written as

$$\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u).$$

In the special case when $p = 2$, p-Laplace operator reduces to the usual Laplace-Beltrami operator.

Geometric flows also called geometric evolution equations were originally conceived as an approach to solve geometric and topological problems. They are divided into two categories: extrinsic geometric flows are flows on embedded submanifolds, or more generally on immersed submanifolds; intrinsic geometric flows are flows on the Riemannian metric, independent of any embedding or immersion. A well-known intrinsic flow is the Ricci flow. Ricci flow was introduced by Hamilton [9]. Given a Riemannian manifold (M, g_0), Ricci flow is defined as the evolution equation of the metric:

$$\frac{\partial}{\partial t} g = -2Ric_g,$$

with the initial condition

$$g|_{t=0} = g_0.$$

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Here, Ric_g is the Ricci curvature of the Riemannian metric g. Using Ricci flow, Perelman proved the 3-dimensional Poincaré conjecture. Ricci flow has many other applications. See [2, 8] for example.

The mean curvature flow is one of the extrinsic geometric flows. Let $\{M_t\}_{t\in[0,\infty)}$ be a one parameter family of smooth hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$, with a given initial hypersurface M_0. Let these hypersurfaces be locally represented by a diffeomorphism

$$X : U \to X(U) \subset \mathbb{R}^{n+1},$$

where U is an open subset in $\mathbb{R}^n$. We say that M_t is evolving by the mean curvature flow if it satisfies the following equation

$$\frac{\partial X}{\partial t} = \vec{H}(x,t),$$

where $\vec{H}(x,t)$ is the mean curvature vector of the hypersurface M_t at $x \in U$ and time t. See [3, 4, 11, 16] and the references therein for the results related to the mean curvature flow.

Another extrinsic flow which is well studied and has many applications is the inverse mean curvature flow (IMCF). It is the evolution of hypersurfaces in the direction of the unit normal vector with speed equal to the reciprocal of the mean curvature. In contrast to the mean curvature flow, the hypersurface is expanding by the IMCF. One of classical results for IMCF is given by Gerhardt [6]: Under IMCF, any star-shaped compact without boundary, hypersurface with strictly positive mean curvature evolves for all time and converges to a round sphere after rescaling (see also [5]). For non-star-shaped hypersurfaces, singularities may occur in finite time. Using level-set approach and developing the notion of weak solutions for IMCF, Huisken and Ilmanen [12] overcame these difficulties and proved the Riemmanian Penrose inequality (see also Bray [1]).

In this paper, we consider the IMCF with forced term in Euclidean space. We consider an n-dimensional compact hypersurface M_0 without boundary, which is smoothly embedded in Euclidean space $\mathbb{R}^{n+1}$. If M_0 is locally represented by a diffeomorphism

$$X_0 : U \to X_0(U) \subset M_0 \subset \mathbb{R}^{n+1}$$

where U is an open set in $\mathbb{R}^n$, then the inverse mean curvature flow is defined as

$$\frac{\partial X}{\partial t} = \frac{1}{H}\vec{N}, \quad X|_{t=0} = X_0.$$

Here H is the mean curvature and $\vec{N}$ is the outward unit normal of $M = M_t$.

We consider the IMCF with forced term in $\mathbb{R}^{n+1}$. We consider embedded hypersurfaces $M_t = X_t(M)$ in $\mathbb{R}^{n+1}$ that move with IMCF with forced term $(c < 0)$:

$$\frac{\partial X}{\partial t} = \left(\frac{1}{H} - c\right)\vec{N}, \quad X|_{t=0} = X_0. \tag{1}$$

In a local coordinate $\{x_1, \dots, x_n\}$ of X for some given point, we denote the metric of X by $g = g_{ij}$, the second fundamental form $h_{ij} = -\left\langle \frac{\partial^2 X}{\partial x_i \partial x_j}, \vec{N} \right\rangle$, and the mean curvature $H = \sum_{i,j=1}^n g^{ij} h_{ij}$. We will prove the following theorem related to the IMCF with forced term in Euclidean space.

Theorem 1.1. *Let $M = M_0 \subset \mathbb{R}^{n+1}$ be a closed hypersurface satisfies*

$$h_{ij} \geq \epsilon g_{ij} H$$

and the mean curvature of M satisfies the pinching condition

$$(1 - p\epsilon) \max_M H \leq \min_M H$$

along the flow (1) *for some $0 < \epsilon \leq \dfrac{1}{n}$. Then the first nonzero eigenvalue of the p-Laplacian with respect to the metric g is monotonically decreasing along the flow* (1)*.*

We emphasize that we do not assume the differentiability of the eigenvalue along the flow, since in general we only know that the first nonzero eigenvalue (that is the second eigenvalue since the first eigenvalue is always zero) is Lipschitz continuous, which makes things complicated. But we can overcome this difficulty by following the method in [15], which does not depend on the differentiability of the eigenvalue and the corresponding eigenfunction.

We note that the monotonically decreasing property of the first eigenvalue of Laplacian to the IMCF without forced term had shown by Guo *et al*. [7] and Pak-Tung Ho and the author [10].

2. Monotonically decreasing property

We denote by A the second fundamental form of (X, g) and dV_g the volume form. We start with the evolution equations to the IMCF with forced term.

Proposition 2.1. *The following evolution equations hold:*

(1) $\dfrac{\partial}{\partial t} g_{ij} = 2\left(\dfrac{1}{H} - c\right) h_{ij},$

(2) $\dfrac{\partial}{\partial t} dV_g = H\left(\dfrac{1}{H} - c\right) dV_g,$

(3) $\dfrac{\partial}{\partial t} h_{ij} = \dfrac{1}{H^2}\Delta_g h_{ij} - \dfrac{2}{H^3}\nabla_i H \nabla_j H - c\displaystyle\sum_{l,m=1}^n g^{lm} h_{jl} h_{im} + \dfrac{1}{H^2}|A|^2 h_{ij},$

(4) $\dfrac{\partial}{\partial t} H = \dfrac{1}{H^2}\Delta_g H - \dfrac{2}{H^3}|\nabla_g H|^2 - \left(\dfrac{1}{H} - c\right)|A|^2.$

Proof. For (1), the induced metric g_{ij} is given by

$$g_{ij} = \left\langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j} \right\rangle.$$

Differentiate it with respect to t, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= \Big\langle \frac{\partial}{\partial x_i}\left(\frac{\partial X}{\partial t}\right), \frac{\partial X}{\partial x_j}\Big\rangle + \Big\langle \frac{\partial X}{\partial x_i}, \frac{\partial}{\partial x_j}\left(\frac{\partial X}{\partial t}\right)\Big\rangle \\
&= \Big\langle \frac{\partial}{\partial x_i}\left(\left(\frac{1}{H}-c\right)\vec{N}\right), \frac{\partial X}{\partial x_j}\Big\rangle + \Big\langle \frac{\partial X}{\partial x_i}, \frac{\partial}{\partial x_j}\left(\left(\frac{1}{H}-c\right)\vec{N}\right)\Big\rangle \\
&= \left(\frac{1}{H}-c\right)\Big\langle \frac{\partial \vec{N}}{\partial x_i}, \frac{\partial X}{\partial x_j}\Big\rangle + \left(\frac{1}{H}-c\right)\Big\langle \frac{\partial X}{\partial x_i}, \frac{\partial \vec{N}}{\partial x_j}\Big\rangle \\
&= -\left(\frac{1}{H}-c\right)\Big\langle \vec{N}, \frac{\partial^2 X}{\partial x_i \partial x_j}\Big\rangle - \left(\frac{1}{H}-c\right)\Big\langle \frac{\partial^2 X}{\partial x_i \partial x_j}, \vec{N}\Big\rangle = 2\left(\frac{1}{H}-c\right) h_{ij}
\end{aligned}$$

where the second equality follows from (1), and the third equality follows from $\Big\langle \frac{\partial X}{\partial x_i}, \vec{N}\Big\rangle = 0$. This proves (1).

For (2), we note that

$$\frac{\partial}{\partial t}\det(g) = \det(g)\,\mathrm{tr}\Big(g^{-1}\frac{\partial g}{\partial t}\Big)$$

where g^{-1} is the inverse of the matrix g, and $\frac{\partial g}{\partial t}$ is the $n \times n$ matrix whose (i, j)-entry is $\frac{\partial g_{ij}}{\partial t}$. Hence, by (1), we have

$$\frac{\partial}{\partial t}\det(g) = 2\left(\frac{1}{H}-c\right)\det(g)\,\mathrm{tr}(g^{-1}h) = 2\left(\frac{1}{H}-c\right)\det(g)H. \tag{2}$$

In local coordinates $(x_1, x_2, ..., x_n)$, we have $dV_g = \sqrt{\det(g)}\,dx_1 \cdots dx_n$. Therefore, we have

$$\frac{\partial}{\partial t} dV_g = \frac{1}{2\sqrt{\det(g)}}\frac{\partial}{\partial t}\det(g)\,dx_1 \cdots dx_n = H\left(\frac{1}{H}-c\right) dV_g$$

by (2). This proves the assertion.

For (3), using the proof of Lemma 3.1(i) in [5], we get

$$\begin{aligned}
\frac{\partial}{\partial t} h_{ij} &= -\nabla_i \nabla_j \left(\frac{1}{H}-c\right) + \left(\frac{1}{H}-c\right)\sum_{l,m=1}^{n} g^{lm} h_{jl} h_{im} - \left(\frac{1}{H}-c\right)\overline{R}_{i0j0} \\
&= \frac{1}{H^2}\nabla_i \nabla_j H - \frac{2}{H^3}\nabla_i H \nabla_j H + \left(\frac{1}{H}-c\right)\sum_{l,m=1}^{n} g^{lm} h_{jl} h_{im},
\end{aligned} \tag{3}$$

where $\overline{R}_{ijkl}$ is the Riemannian tensors of the ambient space. Now (3) follows from (3) and the Simons' identity (see [14]),

$$\Delta_g h_{ij} = \nabla_i \nabla_j H + H \sum_{l,m=1}^{n} g^{lm} h_{jl} h_{im} - |A|^2 h_{ij}$$

where $|A|^2 = \sum_{i,j=1}^{n} h^{ij}h_{ij}$ denotes the squared norm of the second fundamental form.

For (4), we have

$$H = \sum_{i,j=1}^{n} g^{ij}h_{ij}$$

by definition. Differentiate it with respect to t, it follows by (1) and (3). This proves (4). □

In [13], Liu proved that the flow (1) preserves the convexity:

Theorem 2.2 (Liu). *If the initial hypersurface M_0 of the flow (1) is strictly convex, then M_t is also strictly convex for $t > 0$.*

Using the Liu's result and evolution equations in Proposition 2.1, we have

Lemma 2.3. *For any $0 \le \epsilon \le \frac{1}{n}$, the condition*

$$h_{ij} \ge \epsilon H g_{ij}$$

is preserved along the flow (1).

Theorem 2.4 (=Theorem1.1)**.** *Suppose that the second fundamental form satisfies*

$$h_{ij} \ge \epsilon g_{ij} H \tag{4}$$

and the mean curvature of M satisfies the pinching condition

$$(1 - p\epsilon) \max_M H \le \min_M H \tag{5}$$

along the flow (1) for some $0 < \epsilon \le \dfrac{1}{n}$. Then the first nonzero eigenvalue of the p-Laplacian with respect to the metric g is monotonically decreasing along the flow (1).

Proof. For any fixed time t_1, we can construct a smooth function $f(t)$ along the flow (1) satisfying

$$\int_M |f(t)|^p dV_g = 1 \quad \text{and} \quad \int_M |f(t)|^{p-2} f(t) dV_g = 0 \tag{6}$$

and $f(t_1)$ is the first eigenfunction of the p-Laplacian with respect to $g(t_1)$. Indeed, if f_1 is the first eigenfunction of the p-Laplacian with respect to $g(t_1)$, then one can show that

$$f(t) = \frac{h(t)}{\left(\int_M |h(t)|^p dV_g\right)^{\frac{1}{p}}}$$

satisfies (6), where

$$h(t) = f_1 \left[\frac{\det(g_{ij}(t_1))}{\det(g_{ij}(t))}\right]^{\frac{1}{2(p-1)}}$$

and $g(t)$ is the induced metric of M along the flow (1).

Differentiating the first equation in (6) and using Proposition 2.1, we obtain

$$p\int_M |f(t)|^{p-2} f(t)\frac{\partial f(t)}{\partial t} dV_g + \int_M |f(t)|^p H\left(\frac{1}{H} - c\right) dV_g = 0. \tag{7}$$

If we let

$$G(g(t), f(t)) = \int_M |\nabla_g f(t)|^p dV_g,$$

then, by (7), Proposition 2.1 and integration by parts, we have

$$\begin{aligned}
&(8)\\
&\mathcal{G}(g(t), f(t))\\
&:= \frac{d}{dt} G(g(t), f(t))\\
&= \int_M \frac{\partial}{\partial t}(|\nabla_g f(t)|^p) dV_g + \int_M |\nabla_g f(t)|^p \frac{\partial}{\partial t} dV_g\\
&= \int_M p|\nabla_g f(t)|^{p-2}\Big(-\Big(\frac{1}{H} - c\Big)\sum_{i,j=1}^n h^{ij}\frac{\partial f(t)}{\partial x_i}\frac{\partial f(t)}{\partial x_j} + \Big\langle \nabla_g f(t), \nabla_g \frac{\partial f(t)}{\partial t}\Big\rangle\Big) dV_g\\
&\quad + \int_M |\nabla_g f(t)|^p H\Big(\frac{1}{H} - c\Big) dV_g\\
&= -p\int_M |\nabla_g f(t)|^{p-2}\Big(\frac{1}{H} - c\Big)\sum_{i,j=1}^n h^{ij}\frac{\partial f(t)}{\partial x_i}\frac{\partial f(t)}{\partial x_j} dV_g\\
&\quad - p\int_M \operatorname{div}_g\Big(|\nabla_g f(t)|^{p-2}\nabla_g f(t)\Big)\frac{\partial f(t)}{\partial t} dV_g + \int_M |\nabla_g f(t)|^p H\Big(\frac{1}{H} - c\Big) dV_g\\
&= -p\int_M |\nabla_g f(t)|^{p-2}\Big(\frac{1}{H} - c\Big)\sum_{i,j=1}^n h^{ij}\frac{\partial f(t)}{\partial x_i}\frac{\partial f(t)}{\partial x_j} dV_g\\
&\quad + p\lambda_p(t)\int_M |f(t)|^{p-2} f(t)\frac{\partial f(t)}{\partial t} dV_g + \int_M |\nabla_g f(t)|^p H\Big(\frac{1}{H} - c\Big) dV_g\\
&= -p\int_M |\nabla_g f(t)|^{p-2}\Big(\frac{1}{H} - c\Big)\sum_{i,j=1}^n h^{ij}\frac{\partial f(t)}{\partial x_i}\frac{\partial f(t)}{\partial x_j} dV_g\\
&\quad - \lambda_p(t)\int_M |f(t)|^p H\Big(\frac{1}{H} - c\Big) dV_g + \int_M |\nabla_g f(t)|^p H\Big(\frac{1}{H} - c\Big) dV_g.
\end{aligned}$$

Combining (4) and (8), we obtain

(9)
$$
\begin{aligned}
&\mathcal{G}(g(t), f(t)) \\
&\le -p\epsilon \int_M |\nabla_g f(t)|^{p-2} H\Big(\frac{1}{H} - c\Big) \sum_{i,j=1}^{n} g^{ij} \frac{\partial f(t)}{\partial x_i}\frac{\partial f(t)}{\partial x_j} dV_g \\
&\quad - \lambda_p(t) \int_M |f(t)|^p H\Big(\frac{1}{H} - c\Big) dV_g + \int_M |\nabla_g f(t)|^p H\Big(\frac{1}{H} - c\Big) dV_g \\
&= -\lambda_p(t) \int_M |f(t)|^p H\Big(\frac{1}{H} - c\Big) dV_g + (1 - p\epsilon) \int_M |\nabla_g f(t)|^p H\Big(\frac{1}{H} - c\Big) dV_g.
\end{aligned}
$$

In particular, at $t = t_1$, we have

(10)
$$
\begin{aligned}
&- \lambda_p(t_1) \int_M |f(t_1)|^p H\Big(\frac{1}{H} - c\Big) dV_g + (1 - p\epsilon) \int_M |\nabla_g f(t_1)|^p H\Big(\frac{1}{H} - c\Big) dV_g \\
&= -\lambda_p(t_1) + c\lambda_p(t_1) \int_M |f(t_1)|^p H dV_g + (1 - p\epsilon)\lambda_p(t_1) - c(1 - p\epsilon) \int_M |\nabla_g f(t_1)|^p H dV_g \\
&\le -p\epsilon\lambda_p(t_1) + c\lambda_p(t_1)\Big(\min_M H\Big) \int_M |f(t_1)|^p dV_g - c(1 - p\epsilon)\Big(\max_M H\Big) \int_M |\nabla_g f(t_1)|^p dV_g \\
&= \lambda_p(t_1) \Big[-p\epsilon - c\Big((1 - p\epsilon) \max_M H - \min_M H\Big)\Big] < 0,
\end{aligned}
$$

where we have used (6), (5), and the fact that $c < 0$ and $f(t_1)$ is the first eigenfunction of the p-Laplacian with respect to $g(t_1)$. Hence, by (9), (10) and the assumption that f is smooth, we can conclude that

$$\mathcal{G}(g(t), f(t)) \le 0$$

when t is sufficiently closed to t_1. Since $G(g(t_1), f(t_1)) = \lambda_p(t_1)$ and the first eigenvalue is the infimum of Rayleigh quotient, we conclude that

$$\lambda_p(t_2) - \lambda_p(t_1) \le 0$$

when $t_2 \ge t_1$ and t_2 is sufficiently closed to t_1. Since t_1 is arbitrary, we prove the assertion. □

Theorem 2.5. *The first nonzero eigenvalue of the Laplacian with respect to g is monotonically decreasing along the flow* (1).

Proof. Since the initial surface M is compact and uniformly convex, we can find a small $\epsilon > 0$ such that (4) is satisfied and $h_{ij} \ge \epsilon g_{ij} H$ on M. It follows from Lemma 2.3 that (4) is preserved along the flow (1). Now Theorem 2.5 follows from Theorem 2.4 with $p = 2$. □

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