Honam Mathematical J. **44** (2022), No. 2, pp. 259–270
https://doi.org/10.5831/HMJ.2022.44.2.259

A GEOMETRIC APPROACH TO TIMELIKE FLOWS IN TERMS OF ANHOLONOMIC COORDINATES

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Abstract. This paper is devoted to the geometry of vector fields and timelike flows in terms of anholonomic coordinates in three dimensional Lorentzian space. We discuss eight parameters which are related by three partial differential equations. Then, it is seen that the curl of tangent vector field does not include any component in the direction of principal normal vector field. This implies the existence of a surface which contains both $s-lines$ and $b-lines$. Moreover, we examine a normal congruence of timelike surfaces containing the $s-lines$ and $b-lines$. Considering the compatibility conditions, we obtain the Gauss-Mainardi-Codazzi equations for this normal congruence of timelike surfaces in the case of the abnormality of normal vector field is zero. Intrinsic geometric properties of these normal congruence of timelike surfaces are obtained. We have dealt with important results on these geometric properties.

1. Introduction

Differential geometry of surfaces deals with the smooth surfaces, which includes a variety of different structures, usually a Riemann metric. Mostly, surfaces have been investigated from two mainly perspectives. The first one is extrinsically that is relating to their embedding in Euclidean or non-Euclidean space. The second one is intrinsically which is reflecting their properties determined by the distance within the surface as measured along curves on the surface. The most well known concepts investigated is the Gaussian curvature which is first introduced by Carl Friedrich Gauss [4, 5, 6]. Carl Friedrich Gauss proved that curvature was an intrinsic property of a surface, independent of its isometric embedding in Euclidean space. Moreover Gauss discussed the properties which are obtained by the geodesic distances between points on the surface independently of the particular way in which the surface is embedding in the Euclidean space.

Received January 28, 2022. Accepted February 16, 2022.
2020 Mathematics Subject Classification. 53A35, 53A04, 53Z05.
Key words and phrases. Anholonomic coordinates, normal congruence, timelike surfaces.
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We are familiar with the concept of curves lying on a surface by investigating intrinsic property of surfaces. From this point of view, it is the most preferred way of examining the local differential geometric structure of the curve. In many studies dealing with differential geometric properties of curves, some methods and tools of differential calculus are used. This review makes use of the well-known Frenet-Serret frame $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$. Analyzing the geometric structures of curves with the help of vector analysis is very important in this context. Considering that $\sigma = \sigma(s, n, b)$ is a space curve in three dimensional Euclidean space, where s, n and b are the distance along $s - lines$, $n - lines$ and $b - lines$, respectively. The main object is the system obtaining by the directional derivatives of moving frame $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ which is deeply discussed in [10]. The quantities, the normal deformations of the vector-tube in the directions $\overrightarrow{n}$ and $\overrightarrow{b}$,

$$\beta_{ns} = g(\overrightarrow{n}, \frac{\partial}{\partial n}\overrightarrow{t}), \quad \beta_{bs} = g(\overrightarrow{b}, \frac{\partial}{\partial n}\overrightarrow{t})$$

are firstly introduced in [1], respectively.

Among the non-Euclidean geometries, Lorentzian geometry has the most well known applications [16, 2, 17, 14, 18, 9, 3]. Then, Lorentzian geometry is a very common research area of differential geometry with physical problems on integrable systems, soliton theory, fluid dynamics, field theories, etc. [13, 10, 11, 12]. Moreover, Lorentzian geometry has been the mathematical theory which is used by general relativity. Since Lorentz-Minkowski spacetime was extended to a curved spacetime by A. Einstein in order to model nonzero gravitational fields, Lorentzian geometry has been the mathematical theory which is used by general relativity. This situation was a great incentive for the development and advance of new techniques in the study of cosmological models more and more adapted to the physical reality.

Our main aim with this paper is to give an extraordinary view of the timelike curve flow on Lorentzian space. Let us introduce the metric for three dimensional Lorentzian space

$$g_L(\overrightarrow{x}, \overrightarrow{y}) = \langle \overrightarrow{x}, \overrightarrow{y} \rangle_L = -x_1y_1 + x_2y_2 + x_3y_3$$

where $\overrightarrow{x}$, $\overrightarrow{y} \in \mathbb{R}^3$. From now on, we denote Euclidean space equipped with Lorentzian metric by $\mathbb{E}^3_1$. In section two, we investigate the three dimensional vector field and the differential geometric aspects of curvature and torsion of vector lines by means of anholonomic coordinates. Important examples of anholonomic coordinates are the orthogonal frames constructed from a metric on a manifold, the null frames used in general relativity, the left (or right) invariant vector fields on a Lie group, the moving frames adapted to a free group action on a manifold. We describe Frenet Serret frame $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ of given a timelike space curve in $\mathbb{E}^3_1$ in terms of anholonomic coordinates which includes eight parameters, related by three partial differential equations. We prove that the curl of tangent vector field does not include any component in the direction of principal normal vector field. This shows that there exists a surface which contains both $s-lines$ and $b-lines$. Therefore, description of a normal congruence of timelike surfaces containing the $s-lines$ and $b-lines$ is given in the last section. Intrinsic geometric properties of this normal congruence of timelike surfaces are obtained. We have dealt with important results on these geometric properties.

2. Geometric Constraints on Timelike Space Curve in Three Dimensional Lorentzian Space

In this section, the formulas, which will be used to investigate the three dimensional vector field and the differential geometric aspects of curvature and torsion of vector lines, are given by means of anholonomic coordinates. In this context, we consider that $\sigma = \sigma(s, n, b)$ is a given timelike space curve lying in three dimensional Lorentzian space. As known, s denotes the distance along $s-lines$ of the curve in tangential direction so that the unit timelike tangent vector of $s-lines$ is defined by

$$\overrightarrow{t} = \frac{\partial\sigma}{\partial s}.$$

Then the distance along $n-lines$ of the curve σ in principal normal direction is denoted by n. This means that the unit spacelike tangent vector of $n-lines$ is defined by

$$\overrightarrow{n} = \frac{\partial\sigma}{\partial n}.$$

Moreover, b denotes the distance along $b-lines$ of the curve σ in binormal direction. So that the unit spacelike tangent vector of $b-lines$ is defined by

$$\overrightarrow{b} = \frac{\partial\sigma}{\partial b}$$

[7, 8]. Therefore, the moving trihedron of orthonormal unit vectors $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ provides a platform for investigating intrinsic features of the curve σ, where $\overrightarrow{t}$ is the tangential vector, $\overrightarrow{n}$ is the principal normal vector, and $\overrightarrow{b}$ is the binormal vector of the curve σ. To explain the intrinsic differential geometric structure of a timelike curve in three-dimensional Lorentzian space, it is needed to know the arc length on the curve, curvature and torsion which are two independent parameters. On the other hand, intrinsic description of a vector field is more complicated in three-dimensional Lorentzian space. The existence of a field of basis vectors such as $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ to vector lines in three-dimensional Lorentzian space does not imply the existence of a corresponding coordinate system in general. But as is known that a three-dimensional vector field can be described in terms of anholonomic coordinates which includes eight parameters, related by three partial differential equations [15]. The following gives the aforementioned description for the Frenet Serret frame $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ of given

a timelike space curve $\sigma = \sigma(s, n, b)$ in three dimensional Lorentzian space. These relations are also called extended Frenet Serret formulas.

Let $\sigma = \sigma(s, n, b)$ be a timelike space curve lying in three dimensional Lorentzian space. Directional derivative with respect to arclength parameter of the moving trihedron of orthonormal unit vectors $\{\overrightarrow{t}, \overrightarrow{n}, \overrightarrow{b}\}$ is given in the following form:

$$\frac{\partial}{\partial s}\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix} \tag{1}$$

which is derived directly from Frenet Serret equation for unit speed timelike curve. We know that for $i = 1, 2, 3$ there exist smooth functions; α_i and β_i such that

$$\frac{\partial}{\partial n}\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix} = \begin{bmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_1 & 0 & \alpha_3 \\ \alpha_2 & -\alpha_3 & 0 \end{bmatrix}\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix},$$

$$\frac{\partial}{\partial b}\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix} = \begin{bmatrix} 0 & \beta_1 & \beta_2 \\ \beta_1 & 0 & \beta_3 \\ \beta_2 & -\beta_3 & 0 \end{bmatrix}\begin{bmatrix} \overrightarrow{t} \\ \overrightarrow{n} \\ \overrightarrow{b} \end{bmatrix}.$$

First of all, we have

$$\alpha_1 = \left\langle \frac{\partial \overrightarrow{t}}{\partial n}, \overrightarrow{n} \right\rangle_L = \xi_{ns}, \quad \beta_2 = \left\langle \frac{\partial \overrightarrow{t}}{\partial b}, \overrightarrow{b} \right\rangle_L = \xi_{bs}$$

by our assumptions. Then, other geometric quantities are computed by the vector analysis formulae in the following manner. We obtain the followings:

$$\operatorname{div} \overrightarrow{t} = \left\langle \overrightarrow{t}, \kappa \overrightarrow{n} \right\rangle_L + \left\langle \overrightarrow{n}, \xi_{ns} \overrightarrow{n} + \alpha_2 \overrightarrow{b} \right\rangle_L + \left\langle \overrightarrow{b}, \beta_1 \overrightarrow{n} + \xi_{bs} \overrightarrow{b} \right\rangle_L = \xi_{ns} + \beta_1,$$

$$\begin{aligned}\operatorname{div} \overrightarrow{n} &= \left\langle \overrightarrow{t}, \kappa \overrightarrow{t} + \tau \overrightarrow{b} \right\rangle_L + \left\langle \overrightarrow{n}, \xi_{ns} \overrightarrow{t} + \alpha_3 \overrightarrow{b} \right\rangle_L + \left\langle \overrightarrow{b}, \beta_1 \overrightarrow{t} + \beta_3 \overrightarrow{b} \right\rangle_L \\ &= -\kappa + \beta_3,\end{aligned}$$

$$\operatorname{div} \overrightarrow{b} = \left\langle \overrightarrow{t}, -\tau \overrightarrow{n} \right\rangle_L + \left\langle \overrightarrow{n}, \alpha_2 \overrightarrow{t} - \alpha_3 \overrightarrow{n} \right\rangle_L + \left\langle \overrightarrow{b}, \xi_{bs} \overrightarrow{t} - \beta_3 \overrightarrow{n} \right\rangle_L = -\alpha_3.$$

Thus we obtain

$$\beta_3 = \operatorname{div} \overrightarrow{n} + \kappa, \quad \alpha_3 = -\operatorname{div} b.$$

On the other hand, we also obtain

$$\begin{aligned}\operatorname{curl} \overrightarrow{t} &= \overrightarrow{t} \times_L (\kappa \overrightarrow{n}) + \overrightarrow{n} \times_L (\xi_{ns} \overrightarrow{n} + \alpha_2 \overrightarrow{b}) + \overrightarrow{b} \times_L (\beta_1 \overrightarrow{n} + \xi_{bs} \overrightarrow{b}) \\ &= \kappa \overrightarrow{b} - \alpha_2 \overrightarrow{t} + \beta_1 \overrightarrow{t} = (\beta_1 - \alpha_2) \overrightarrow{t} + \kappa \overrightarrow{b},\end{aligned}$$

$$\begin{aligned}\operatorname{curl}\overrightarrow{n} &= \overrightarrow{t}\times_L(\kappa\overrightarrow{t}+\tau\overrightarrow{b})+\overrightarrow{n}\times_L(\xi_{ns}\overrightarrow{t}-\operatorname{div}\overrightarrow{b}\overrightarrow{b})\\ &+\overrightarrow{b}\times_L(\beta_1\overrightarrow{t}+(\operatorname{div}\overrightarrow{n}+\kappa)\overrightarrow{b})\\ &=\tau\overrightarrow{n}-\xi_{ns}\overrightarrow{b}+\operatorname{div}b\overrightarrow{t}-\beta_1\overrightarrow{n}=\operatorname{div}\overrightarrow{b}\overrightarrow{t}+(\tau-\beta_1)\overrightarrow{n}-\xi_{ns}\overrightarrow{b}\end{aligned}$$

and

$$\begin{aligned}\operatorname{curl}\overrightarrow{b} &= \overrightarrow{t}\times_L(-\tau\overrightarrow{n})+\overrightarrow{n}\times_L(\alpha_2\overrightarrow{t}+\operatorname{div}\overrightarrow{b}\overrightarrow{n})\\ &+\overrightarrow{b}\times_L(\xi_{bs}\overrightarrow{t}-(\operatorname{div}\overrightarrow{n}+\kappa)\overrightarrow{n})\\ &=-\tau\overrightarrow{b}-\alpha_2\overrightarrow{b}-\xi_{bs}\overrightarrow{n}-(\operatorname{div}\overrightarrow{n}+\kappa)\overrightarrow{t}\\ &=-(\operatorname{div}\overrightarrow{n}+\kappa)\overrightarrow{t}-\xi_{bs}\overrightarrow{n}+(-\tau-\alpha_2)\overrightarrow{b}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\mu_s &= \left\langle \operatorname{curl}\overrightarrow{t},\overrightarrow{t}\right\rangle_L=\alpha_2-\beta_1,\\ \mu_n &= \langle \operatorname{curl}\overrightarrow{n},\overrightarrow{n}\rangle_L=\tau-\beta_1,\\ \mu_b &= \left\langle \operatorname{curl}\overrightarrow{b},\overrightarrow{b}\right\rangle_L=-\tau-\alpha_2.\end{aligned}$$

This implies

$$\beta_1=\tau-\mu_n,\quad \alpha_2=-\tau-\mu_b.$$

Finally, if we substitute the obtained values of the smooth functions; α_i and β_i for $i=1,2,3$, then we get

$$\frac{\partial}{\partial n}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix}=\begin{bmatrix}0 & \xi_{ns} & -\tau-\mu_b\\ \xi_{ns} & 0 & -\operatorname{div}b\\ -\tau-\mu_b & \operatorname{div}b & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix},$$

$$\text{(2)}\qquad \frac{\partial}{\partial b}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix}=\begin{bmatrix}0 & \tau-\mu_n & \xi_{bs}\\ \tau-\mu_n & 0 & \operatorname{div}\overrightarrow{n}+\kappa\\ \xi_{bs} & -(\operatorname{div}\overrightarrow{n}+\kappa) & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{t}\\ \overrightarrow{n}\\ \overrightarrow{b}\end{bmatrix},$$

where

$$\xi_{ns}=\left\langle\frac{\partial\overrightarrow{t}}{\partial n},\overrightarrow{n}\right\rangle_L,\quad \xi_{bs}=\left\langle\frac{\partial\overrightarrow{t}}{\partial b},\overrightarrow{b}\right\rangle_L$$

are the normal deformations of the vector-tube in the directions $\overrightarrow{n}$ and $\overrightarrow{b}$, respectively. Here $\kappa=\kappa(s,n,b)$ is the curvature function and $\tau=\tau(s,n,b)$ is the torsion function of the unit speed timelike curve $\sigma=\sigma(s,n,b)$.

As a result, a relationship between abnormalities of $\overrightarrow{t}$, $\overrightarrow{n}$ and $\overrightarrow{b}$ is obtained as follows:

$$\mu_s+\tau=\frac{1}{2}(\mu_s+\mu_n-\mu_b).$$

This relation shows that important results involving the Dupin theorem which states that in a triply orthogonal coordinate system all coordinate surfaces intersect along common curvature lines. We also see that

$$\operatorname{curl}\overrightarrow{t} = -\mu_s \overrightarrow{t} + \kappa \overrightarrow{b}.$$

This formula yields many interesting results, which we will discuss in the next sections. Since $\operatorname{curl}\overrightarrow{t}$ does not include any component in the direction of principal normal $\overrightarrow{n}$, then there exists a surface which contains both $s-lines$ and $b-lines$. This idea will be the most important motivation of last section of the paper.

Now, the identity, $\operatorname{curl}\operatorname{grad} f = 0$ yields

$$\begin{aligned}
&\operatorname{curl}\operatorname{grad} f\\
&= \overrightarrow{t} \times_L (\frac{\partial \overrightarrow{t}}{\partial s}\frac{\partial f}{\partial s} + \overrightarrow{t}\frac{\partial^2 f}{\partial s^2} + \frac{\partial \overrightarrow{n}}{\partial s}\frac{\partial f}{\partial n} + \overrightarrow{n}\frac{\partial^2 f}{\partial s \partial n} + \frac{\partial \overrightarrow{b}}{\partial s}\frac{\partial f}{\partial b} + \overrightarrow{b}\frac{\partial^2 f}{\partial s \partial b})\\
&+ \overrightarrow{n} \times_L (\frac{\partial \overrightarrow{t}}{\partial n}\frac{\partial f}{\partial s} + \overrightarrow{t}\frac{\partial^2 f}{\partial n \partial s} + \frac{\partial \overrightarrow{n}}{\partial n}\frac{\partial f}{\partial n} + \overrightarrow{n}\frac{\partial^2 f}{\partial n^2} + \frac{\partial \overrightarrow{b}}{\partial n}\frac{\partial f}{\partial b} + \overrightarrow{b}\frac{\partial^2 f}{\partial n \partial b})\\
&+ \overrightarrow{b} \times_L (\frac{\partial \overrightarrow{t}}{\partial b}\frac{\partial f}{\partial s} + \overrightarrow{t}\frac{\partial^2 f}{\partial b \partial s} + \frac{\partial \overrightarrow{n}}{\partial n}\frac{\partial f}{\partial n} + \overrightarrow{n}\frac{\partial^2 f}{\partial b \partial n} + \frac{\partial \overrightarrow{b}}{\partial b}\frac{\partial f}{\partial b} + \overrightarrow{b}\frac{\partial^2 f}{\partial b^2})\\
&= \frac{\partial f}{\partial s}\operatorname{curl}\overrightarrow{t} + \frac{\partial f}{\partial n}\operatorname{curl}\overrightarrow{n} + \frac{\partial f}{\partial b}\operatorname{curl}\overrightarrow{b}\\
&+ \overrightarrow{t} \times_L (\overrightarrow{t}\frac{\partial^2 f}{\partial s^2} + \overrightarrow{n}\frac{\partial^2 f}{\partial s \partial n} + \overrightarrow{b}\frac{\partial^2 f}{\partial s \partial b})\\
&+ \overrightarrow{n} \times_L (\overrightarrow{t}\frac{\partial^2 f}{\partial n \partial s} + \overrightarrow{n}\frac{\partial^2 f}{\partial n^2} + \overrightarrow{b}\frac{\partial^2 f}{\partial n \partial b})\\
&+ \overrightarrow{b} \times_L (\overrightarrow{t}\frac{\partial^2 f}{\partial b \partial s} + \overrightarrow{n}\frac{\partial^2 f}{\partial b \partial n} + \overrightarrow{b}\frac{\partial^2 f}{\partial b^2}),
\end{aligned}$$

$$\begin{aligned}
\operatorname{curl}\operatorname{grad} f = {}& \frac{\partial f}{\partial s}\operatorname{curl}\overrightarrow{t} + \frac{\partial f}{\partial n}\operatorname{curl}\overrightarrow{n} + \frac{\partial f}{\partial b}\operatorname{curl}\overrightarrow{b} + (\frac{\partial^2 f}{\partial b \partial n} - \frac{\partial^2 f}{\partial n \partial b})\overrightarrow{t}\\
&+ (\frac{\partial^2 f}{\partial s \partial b} - \frac{\partial^2 f}{\partial b \partial s})\overrightarrow{n} + (\frac{\partial^2 f}{\partial s \partial n} - \frac{\partial^2 f}{\partial n \partial s}) = \overrightarrow{0}.
\end{aligned}$$

By using of obtained relations, we get

$$\begin{aligned}\overrightarrow{0} &= \frac{\partial f}{\partial s}[-\mu_s \overrightarrow{t} + \kappa \overrightarrow{b}] + \frac{\partial f}{\partial n}[\operatorname{div} \overrightarrow{b}\, \overrightarrow{t} + \mu_n \overrightarrow{n} - \xi_{ns} \overrightarrow{b}] \\ &+ \frac{\partial f}{\partial b}[(\operatorname{div} \overrightarrow{n} + \kappa) \overrightarrow{t} - \xi_{bs} \overrightarrow{n} + \mu_b \overrightarrow{b}] \\ &+ (\frac{\partial^2 f}{\partial b \partial n} - \frac{\partial^2 f}{\partial n \partial b}) \overrightarrow{t} + (\frac{\partial^2 f}{\partial s \partial b} - \frac{\partial^2 f}{\partial b \partial s}) \overrightarrow{n} + (\frac{\partial^2 f}{\partial s \partial n} - \frac{\partial^2 f}{\partial n \partial s}) \overrightarrow{b}, \\ \overrightarrow{0} &= (\frac{\partial^2 f}{\partial b \partial n} - \frac{\partial^2 f}{\partial n \partial b} - \frac{\partial f}{\partial s}\mu_s + \frac{\partial f}{\partial n} \operatorname{div} \overrightarrow{b} + \frac{\partial f}{\partial b}(\operatorname{div} \overrightarrow{n} + \kappa)) \overrightarrow{t} \\ &+ (\frac{\partial^2 f}{\partial s \partial b} - \frac{\partial^2 f}{\partial b \partial s} + \frac{\partial f}{\partial n}\mu_n - \frac{\partial f}{\partial b}\xi_{bs}) \overrightarrow{n} \\ &+ (\frac{\partial^2 f}{\partial s \partial n} - \frac{\partial^2 f}{\partial n \partial s} + \frac{\partial f}{\partial s}\kappa - \frac{\partial f}{\partial n}\xi_{ns} + \frac{\partial f}{\partial b}\mu_b) \overrightarrow{b}.\end{aligned}$$

This gives the following relations:

$$\begin{aligned}\frac{\partial^2 f}{\partial b \partial n} - \frac{\partial^2 f}{\partial n \partial b} &= \frac{\partial f}{\partial s}\mu_s - \frac{\partial f}{\partial n} \operatorname{div} \overrightarrow{b} - \frac{\partial f}{\partial b}(\operatorname{div} \overrightarrow{n} + \kappa), \\ \frac{\partial^2 f}{\partial s \partial b} - \frac{\partial^2 f}{\partial b \partial s} &= -\frac{\partial f}{\partial n}\mu_n + \frac{\partial f}{\partial b}\xi_{bs}, \\ \frac{\partial^2 f}{\partial s \partial n} - \frac{\partial^2 f}{\partial n \partial s} &= -\frac{\partial f}{\partial s}\kappa + \frac{\partial f}{\partial n}\xi_{ns} - \frac{\partial f}{\partial b}\mu_b.\end{aligned}$$

Therefore, we see that the second-order mixed intrinsic derivatives do not commute in general. That is the quantities s, n and b represent anholonomic coordinates. The following nine conditions on the eight geometric parameters κ, τ, μ_s, μ_b, $\operatorname{div} \overrightarrow{n}$, $\operatorname{div} \overrightarrow{b}$, ξ_{ns}, ξ_{bs} occurring in the intrinsic representations of grad $\overrightarrow{t}$, grad $\overrightarrow{n}$ and grad $\overrightarrow{b}$ can be given by the compatibility of the linear system

$$\frac{\partial \xi_{ns}}{\partial b} + \frac{\partial(\mu_n - \tau)}{\partial n} = [-(\mu_b + \tau)\kappa\mu_s - (\tau - \mu_n)](\operatorname{div} \overrightarrow{n} + \kappa) + (\xi_{bs} - \xi_{ns}) \operatorname{div} \overrightarrow{b},$$

$$\frac{\partial(\mu_b + \tau)}{\partial b} + \frac{\partial \xi_{bs}}{\partial n} = (\xi_{ns} + \xi_{bs})(\operatorname{div} \overrightarrow{n} + \kappa) + (-\mu_b - \mu_n) \operatorname{div} \overrightarrow{b},$$

$$\frac{\partial(\tau - \mu_n)}{\partial s} - \frac{\partial \kappa}{\partial b} = -\xi_{ns}\mu_n + \xi_{bs}(2\tau - \mu_n), \tag{3}$$

$$
\begin{aligned}
(4) \qquad \frac{\partial \xi_{bs}}{\partial s} &= (\mu_b + 2\tau)\mu_n + \xi_{bs}^2 - \tau^2 + \kappa(\operatorname{div}\overrightarrow{n} + \kappa), \\
\frac{\partial \xi_{ns}}{\partial s} - \frac{\partial \kappa}{\partial n} &= -(2\mu_b + \tau)\tau - \kappa^2 + \xi_{ns}^2 + \mu_n\mu_b, \\
\frac{\partial(\mu_b + \tau)}{\partial s} &= \kappa \operatorname{div}\overrightarrow{b} + (\mu_b + 2\tau)\xi_{ns} + \xi_{bs}\mu_b, \\
\frac{\partial \operatorname{div}\overrightarrow{b}}{\partial s} + \frac{\partial \tau}{\partial n} &= \operatorname{div}\overrightarrow{b}\,\xi_{ns} + (\operatorname{div}\overrightarrow{n} + \kappa)\mu_b + (\mu_b + 2\tau)\kappa, \\
\frac{\partial \operatorname{div}\overrightarrow{b}}{\partial b} + \frac{\partial(\operatorname{div}\overrightarrow{n} + \kappa)}{\partial n} &= \xi_{bs}\xi_{ns} + (\mu_b + \tau)(\tau - \mu_n) - (\operatorname{div}\overrightarrow{b})^2 \\
&\quad - (\operatorname{div}\overrightarrow{n} + \kappa)^2 - \mu_s\tau,
\end{aligned}
$$

$$
(5) \qquad \frac{\partial(\operatorname{div}\overrightarrow{n} + \kappa)}{\partial s} - \frac{\partial \tau}{\partial b} = (\operatorname{div}\overrightarrow{n} + 2\kappa)\xi_{bs} + \operatorname{div}\overrightarrow{b}\,\mu_n.
$$

3. Normal Congruence of Timelike Surfaces Containing $s - lines$ and $b - lines$

There exists a normal congruence of surfaces containing the $s - lines$ and $b - lines$ if and only if the abnormality of $\overrightarrow{n}$ is zero, i.e.

$$\mu_n = 0.$$

This condition represents the necessary and sufficient condition for the existence of a one-parameter family of surfaces containing the $s - lines$ and $b - lines$. Now, we consider the compatibility conditions Equation 3, 4 and 5. In the case of $\mu_n = 0$, these equations reduces to the nonlinear system

$$
\begin{aligned}
\frac{\partial \tau}{\partial s} - \frac{\partial \kappa}{\partial b} &= 2\tau\xi_{bs}, \\
\frac{\partial \xi_{bs}}{\partial s} &= \xi_{bs}^2 - \tau^2 + \kappa(\operatorname{div}\overrightarrow{n} + \kappa), \\
\frac{\partial(\operatorname{div}\overrightarrow{n} + \kappa)}{\partial s} - \frac{\partial \tau}{\partial b} &= \xi_{bs}(\operatorname{div}\overrightarrow{n} + 2\kappa).
\end{aligned}
$$

which is called Gauss-Mainardi-Codazzi equations for this normal congruence of surfaces. In the case ($\mu_n = 0$) of the above foliation, there exist the constituent surfaces Ψ. Since the unit timelike tangent vector of $s - lines$ is defined by $\overrightarrow{t} = \frac{\partial \sigma}{\partial s}$ and the unit spacelike tangent vector of $b-lines$ is defined by $\overrightarrow{b} = \frac{\partial \sigma}{\partial b}$, then we get

$$\frac{\partial \Psi}{\partial s} = \frac{\partial \sigma}{\partial s} = \overrightarrow{t} \text{ and } \frac{\partial \Psi}{\partial b} = \frac{\partial \sigma}{\partial b} = \overrightarrow{b}.$$

We obtain

$$\frac{\partial \Psi}{\partial s} \times_L \frac{\partial \Psi}{\partial b} = \overrightarrow{t} \times_L \overrightarrow{b} = \overrightarrow{n}.$$

It follows that $\overrightarrow{n}$ is perpendicular to surface. This means that $\overrightarrow{n}$ parallel to the normal vector field $\overrightarrow{N}$ of the surfaces Ψ, i.e. $\overrightarrow{N} = \overrightarrow{n}$. Thus, the one-parameter family of surfaces Ψ, which contain the $s-lines$ and $b-lines$, are timelike surfaces. We know that

$$\frac{\partial^2 \Psi}{\partial b^2} = \frac{\partial \overrightarrow{b}}{\partial b} = \xi_{bs}\overrightarrow{t} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{n}$$

by Equation 2. Then we obtain the geodesic curvatures of $b-lines$ as follows:

$$k_{g_b} = \left\langle \xi_{bs}\overrightarrow{t} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{n}, \overrightarrow{n} \times_L \overrightarrow{b} \right\rangle_L = \left\langle \xi_{bs}\overrightarrow{t} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{n}, -\overrightarrow{t} \right\rangle_L$$
$$= \xi_{bs}.$$

In similar manner, we get

$$\frac{\partial^2 \Psi}{\partial s^2} = \frac{\partial \overrightarrow{t}}{\partial s} = \kappa\overrightarrow{n}$$

by Equation 1. Thus, we get the geodesic curvatures of $s-lines$ as follows:

$$k_{g_s} = \left\langle \kappa\overrightarrow{n}, \overrightarrow{n} \times_L \overrightarrow{t} \right\rangle = \left\langle \kappa\overrightarrow{n}, -\overrightarrow{b} \right\rangle = 0.$$

This implies that $s-lines$ are geodesics on the surfaces Ψ. Again by using the equation

$$\frac{\partial^2 \Psi}{\partial b^2} = \xi_{bs}\overrightarrow{t} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{n},$$

we obtain the normal curvatures of $b-lines$ as follows:

$$k_{n_b} = \left\langle \xi_{bs}\overrightarrow{t} - (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{n}, \overrightarrow{n} \right\rangle_L = -(\operatorname{div}\overrightarrow{n} + \kappa).$$

On the other hand, the normal curvatures of $s-lines$ are obtained as follows:

$$k_{n_s} = \langle \kappa\overrightarrow{n}, \overrightarrow{n} \rangle_L = \kappa.$$

By

$$k_g^2 + k_n^2 = \kappa^2,$$

we get

$$\xi_{bs}^2 + (\operatorname{div}\overrightarrow{n} + \kappa)^2 = \kappa_b^2$$

where κ_b is the curvature function of the $b-lines$ of one-parameter family of surfaces Ψ. We obtain the geodesic torsion of $b-lines$ as follows:

$$\tau_{g_b} = \left\langle \tau\overrightarrow{t} + (\operatorname{div}\overrightarrow{n} + \kappa)\overrightarrow{b}, -\overrightarrow{t} \right\rangle_L = \tau$$

by Equation 2. Similarly, the geodesic torsion of $s-lines$ as follows:

$$\tau_{g_s} = \left\langle \kappa\overrightarrow{t} + \tau\overrightarrow{b}, -\overrightarrow{b} \right\rangle_L = -\tau$$

by Equation 1. We found the coefficients of first and second fundamental forms of one-parameter family of surfaces Ψ

$$I = \langle d\Psi, d\Psi \rangle_L = \left\langle \overrightarrow{t}\,ds + \overrightarrow{b}\,db, \overrightarrow{t}\,ds + \overrightarrow{b}\,db \right\rangle_L = -ds^2 + db^2.$$

We have found that $g_{11} = -1$, $g_{12} = 0$ and $g_{22} = 1$. Since normal vector field of one-parameter family of surfaces Ψ is equal to $\overrightarrow{n}$, we find the second fundamental form as follows:

$$\begin{aligned} II &= \langle d\Psi, d\overrightarrow{n} \rangle_L \\ &= \left\langle \overrightarrow{t}\, ds + \overrightarrow{b}\, db, (\kappa \overrightarrow{t} + \tau \overrightarrow{b})ds + (\tau \overrightarrow{t} + (\operatorname{div} \overrightarrow{n} + \kappa \overrightarrow{b})db \right\rangle_L \\ &= -\kappa ds^2 + (\operatorname{div} \overrightarrow{n} + \kappa) db^2. \end{aligned}$$

We get $l_{11} = -\kappa$, $l_{12} = 0$ and $l_{22} = \operatorname{div} \overrightarrow{n} + \kappa$. Thus, the Gaussian and mean of the surfaces Ψ are

$$\begin{aligned} K &= \frac{-\kappa(\operatorname{div} \overrightarrow{n} + \kappa)}{-1} = \kappa(\operatorname{div} \overrightarrow{n} + \kappa), \\ H &= \frac{-(\operatorname{div} \overrightarrow{n} + \kappa) - \kappa}{-2} = \frac{\operatorname{div} \overrightarrow{n} + 2\kappa}{2}. \end{aligned}$$

If the following equality is satisfied

$$\kappa(\operatorname{div} n + \kappa) = 0,$$

then one-parameter family of surfaces Ψ is developable. We know that the Gaussian curvature one-parameter family of surfaces Ψ is found as

$$K = \kappa(\operatorname{div} \overrightarrow{n} + \kappa).$$

From following equation

$$\frac{\partial \xi_{bs}}{\partial s} = \xi_{bs}^2 - \tau^2 + \kappa(\operatorname{div} \overrightarrow{n} + \kappa),$$

we get that

$$K = \frac{\partial \xi_{bs}}{\partial s} - \xi_{bs}^2 + \tau^2.$$

If $b - lines$ are geodesics and $s - lines$ are plane curves, then one-parameter family of surfaces Ψ is developable. One-parameter family of surfaces Ψ is minimal if and only if

$$\operatorname{div} \overrightarrow{n} = -2\kappa.$$

It is easily seen that the equality

$$\frac{\partial \Psi}{\partial s} \times_L \frac{\partial^2 \Psi}{\partial s^2} = \frac{\partial \Psi}{\partial b}.$$

One-parameter family of surfaces Ψ is a NLS surface if and only if $\kappa = 1$.

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