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BI-ROTATIONAL HYPERSURFACE SATISFYING $\Delta^{III}\mathbf{x} = \mathcal{A}\mathbf{x}$ IN 4-SPACE

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Abstract. We examine the bi-rotational hypersurface $\mathbf{x} = \mathbf{x}(u, v, w)$ with the third Laplace-Beltrami operator in the four dimensional Euclidean space $\mathbb{E}^4$. Giving the i-th curvatures of the hypersurface $\mathbf{x}$, we obtain the third Laplace-Beltrami operator of the bi-rotational hypersurface satisfying $\Delta^{III}\mathbf{x} = \mathcal{A}\mathbf{x}$ for some 4×4 matrix $\mathcal{A}$.

1. Introduction

Chen ([13, 14, 15, 16]) examined the submanifolds of finite type whose immersion into $\mathbb{E}^m$ (or $\mathbb{E}^m_\nu$) via a finite number of eigenfunctions of their Laplacian.

Takahashi ([46]) introduced that a connected Euclidean submanifold is of 1-type if and only if it is either minimal in $\mathbb{E}^m$ or minimal in some hypersphere of $\mathbb{E}^m$. 2-type spherical closed submanifolds were studied by [9, 10, 14]. Garay ([28]) worked an extension of Takahashi's theorem in $\mathbb{E}^m$. Cheng and Yau gave the hypersurfaces with the constant scalar curvature. Chen and Piccinni ([17]) considered the submanifolds with the finite type Gauss map in $\mathbb{E}^m$. Dursun ([23]) focused on the hypersurfaces with the pointwise 1-type Gauss map in $\mathbb{E}^{n+1}$.

In $\mathbb{E}^3$, Takahashi ([46]) gave that the minimal surfaces and spheres are the only surfaces satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$. Ferrandez et al. ([25]) classified that the surfaces satisfying $\Delta H = AH$, $A \in Mat(3,3)$ are either minimal, or an open piece of sphere or of a right circular cylinder. Choi and Kim ([20]) found the minimal helicoid in terms of the pointwise 1-type Gauss map of the first kind. Garay ([27]) worked on the certain class of the finite type surfaces of revolution. Dillen et al. ([21]) focused that the only surfaces satisfying $\Delta r = Ar + B$, $A \in Mat(3,3)$, $B \in Mat(3,1)$ are the minimal surfaces, the spheres and the circular cylinders. Stamatakis and Zoubi ([45]) obtained the surfaces of revolution satisfying $\Delta^{III}x = Ax$. Senoussi and Bekkar ([44])

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introduced the helicoidal surfaces M^2 which are of finite type with respect to the fundamental forms I, II and III. Kim et al. ([37]) gave the Cheng-Yau's operator and the Gauss map of the surfaces of revolution.

In $\mathbb{E}^4$, Moore ([41, 42]) gave the general rotational surfaces. Hasanis and Vlachos ([34]) studied the hypersurfaces with harmonic mean curvature vector field. Cheng and Wan ([18]) considered complete hypersurfaces with CMC. Kim and Turgay ([38]) studied surfaces with L_1-pointwise 1-type Gauss map. Arslan et al. ([3]) introduced the Vranceanu surface with the pointwise 1-type Gauss map. Arslan et al. ([4]) introduced the generalized rotational surfaces. Arslan et al. ([5]) obtained the tensor product surfaces with the pointwise 1-type Gauss map. Kahraman Aksoyak and Yaylı ([35]) considered the rotational surfaces with the pointwise 1-type Gauss map. Güler et al. ([32]) worked on the helicoidal hypersurfaces. Güler et al. ([31]) introduced the Gauss map and the third Laplace-Beltrami operator of the rotational hypersurface. Güler and Turgay ([33]) obtained the Cheng$-$Yau's operator and the Gauss map of the rotational hypersurfaces. Güler ([30]) worked on the rotational hypersurfaces satisfying $\Delta^I R = AR$, where $A \in Mat(4,4)$. He ([29]) also examined the fundamental form IV and curvature formulas of the hypersphere. Arslan et al. ([7]) introduced the rotational λ-hypersurfaces in Euclidean spaces.

In Minkowski 4-space $\mathbb{E}^4_1$, Ganchev and Milousheva ([26]) indicated the analogue of the surfaces of the Moore ([41, 42]). Arvanitoyeorgos et al. ([8]) studied that if the mean curvature vector field of M^3_1 satisfies the equation $\Delta H = \alpha H$ (α a constant), then M^3_1 has CMC; Arslan and Milousheva considered the meridian surfaces of elliptic or hyperbolic type with the pointwise 1-type Gauss map. Turgay introduced some classifications of a Lorentzian surfaces with the finite type Gauss map. Dursun and Turgay gave the space-like surfaces with the pointwise 1-type Gauss map. Kahraman Aksoyak and Yaylı ([36]) worked on the general rotational surfaces with the pointwise 1-type Gauss map in $\mathbb{E}^4_2$. Bektaş et al. ([11]) considered the surfaces in a pseudo-sphere with the 2-type pseudo-spherical Gauss map in $\mathbb{E}^5_2$. They ([12]) also gave the pseudo-spherical submanifolds with the 1-type pseudo-spherical Gauss map.

We give the bi-rotational hypersurface with the third Laplace-Beltrami operator in the four dimensional Euclidean space $\mathbb{E}^4$. In Section 2, we show the fundamental notions of the four dimensional Euclidean geometry. We find the curvature formulas of a hypersurface in $\mathbb{E}^4$ in Section 3. We give the bi-rotational hypersurface in Section 4. Moreover, we study the bi-rotational hypersurface satisfying $\Delta^{III}\mathbf{x} = \mathcal{A}\mathbf{x}$ for some 4×4 matrix $\mathcal{A}$ in $\mathbb{E}^4$ in Section 5. We serve a conclusion in the last section.

2. Preliminaries

In this section, giving some of basic facts and definitions, we describe the notations used in the paper.

Let $\mathbb{E}^m$ denote a Euclidean m-space with the canonical Euclidean metric tensor given by $\widetilde{g} = \langle \ , \ \rangle = \sum_{i=1}^{m} dx_i^2$, where $(x_1, x_2, \ldots, x_m)$ is a rectangular coordinate system in $\mathbb{E}^m$. Consider an m-dimensional Riemannian submanifold of the space $\mathbb{E}^m$. We denote the Levi-Civita connections of $\mathbb{E}^m$ and M by $\widetilde{\nabla}$ and ∇, respectively. We shall use the letters X, Y, Z, W (resp., ξ, η) to denote the vectors fields tangent (resp., normal) to M. The Gauss and Weingarten formulas are given, respectively, by

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \widetilde{\nabla}_X \xi &= -A_\xi(X) + D_X \xi, \end{aligned}$$

where h, D and A are the second fundamental form, the normal connection and the shape operator of M, respectively.

For each $\xi \in T_p^\perp M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_p M$ at $p \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle .$$

The Gauss and Codazzi equations are given, respectively, by

$$\begin{aligned} \langle R(X, Y,)Z, W \rangle &= \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle, \\ (\bar{\nabla}_X h)(Y, Z) &= (\bar{\nabla}_Y h)(X, Z), \end{aligned}$$

where R, R^D are the curvature tensors associated with connections ∇ and D, respectively, and $\bar{\nabla} h$ is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

2.1. Hypersurfaces of Euclidean space

Now, let M be an oriented hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $\mathbf{S}$ its shape operator (i.e. Weingarten map) and x its position vector. We consider a local orthonormal frame field $\{e_1, e_2, \ldots, e_n\}$ consisting of principal directions of M corresponding from the principal curvature k_i for $i = 1, 2, \ldots, n$. Let $\{\theta_1, \theta_2, \ldots, \theta_n\}$ be its dual basis. Then the first structural equation of Cartan is

$$d\theta_i = \sum_{i=1}^{n} \theta_j \wedge \omega_{ij}, \quad i, j = 1, 2, \ldots, n,$$

where ω_{ij} denotes the connection forms corresponding to the chosen frame field. Then, from the Codazzi equation (1), we have

$$\begin{aligned} e_i(k_j) &= \omega_{ij}(e_j)(k_i - k_j), \\ \omega_{ij}(e_l)(k_i - k_j) &= \omega_{il}(e_j)(k_i - k_l) \end{aligned}$$

for distinct $i, j, l = 1, 2, \ldots, n$.

We put $s_j = \sigma_j(k_1, k_2, \ldots, k_n)$, where σ_j is the j-th elementary symmetric function given by

$$\sigma_j(a_1, a_2, \ldots, a_n) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} a_{i_1} a_{i_2} \ldots a_{i_j}.$$

We use the following notation

$$r_i^j = \sigma_j(k_1, k_2, \ldots, k_{i-1}, k_{i+1}, k_{i+2}, \ldots, k_n).$$

By the definition, we have $r_i^0 = 1$ and $s_{n+1} = s_{n+2} = \cdots = 0$. We call the function s_k as the k-th mean curvature of M. We would like to note that the functions $H = \frac{1}{n} s_1$ and $K = s_n$ are called the mean curvature and the Gauss-Kronecker curvature of M, respectively. In particular, M is said to be j-minimal if $s_j \equiv 0$ on M.

In $\mathbb{E}^{n+1}$, to find the i-th curvature formulas $\mathfrak{C}_i$ (sometimes the curvature formulas are represented as the mean curvature H_i, and sometimes as the Gaussian curvature K_i by the different writers, such as [1] and [39]. We will call it just the i-th curvature $\mathfrak{C}_i$ in this paper.), where $i = 0, \ldots, n$, firstly, we use the characteristic polynomial of **S:**

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda \mathcal{I}_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k},$$

where $i = 0, \ldots, n$, $\mathcal{I}_n$ denotes the identity matrix of order n. Then, we get the curvature formulas $\binom{n}{i} \mathfrak{C}_i = s_i$. That is, $\binom{n}{0} \mathfrak{C}_0 = s_0 = 1$ (by definition), $\binom{n}{1} \mathfrak{C}_1 = s_1, \ldots, \binom{n}{n} \mathfrak{C}_n = s_n = K$.

For a Euclidean submanifold x: $M \longrightarrow \mathbb{E}^m$, the immersion (M, x) is called *finite type,* if x can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of (M, x), i.e. $x = x_0 + \sum_{i=1}^{k} x_i$, where x_0 is a constant map, $x_1, \ldots, x_k$ non-constant maps, and $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. If λ_i are different, M is called *k-type.* See [14] for details.

Let $\mathbf{x} = \mathbf{x}(u, v, w)$ be an isometric immersion from $M^3 \subset \mathbb{E}^3$ to $\mathbb{E}^4$. The product of $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$, $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ of $\mathbb{E}^4$ is defined by

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

For a hypersurface $\mathbf{x}$ in 4-space, we consider the symmetric matrices $(g_{ij})_{3\times3}$, $(h_{ij})_{3\times3}$, and $(t_{ij})_{3\times3}$, where (g_{ij}), (h_{ij}), and (t_{ij}) are the first, second, and the third fundamental form matrices, respectively. The coefficients of the symmetric matrices are given by $g_{11} = \mathbf{x}_u \cdot \mathbf{x}_u$, $g_{12} = \mathbf{x}_u \cdot \mathbf{x}_v$, $g_{22} = \mathbf{x}_v \cdot \mathbf{x}_v$, $g_{13} = \mathbf{x}_u \cdot \mathbf{x}_v$, $g_{23} = \mathbf{x}_v \cdot \mathbf{x}_w$, $g_{33} = \mathbf{x}_w \cdot \mathbf{x}_w$, $h_{11} = \mathbf{x}_{uu} \cdot e$, $h_{12} = \mathbf{x}_{uv} \cdot e$, $h_{22} = \mathbf{x}_{vv} \cdot e$, $h_{13} = \mathbf{x}_{uw} \cdot e$, $h_{23} = \mathbf{x}_{vw} \cdot e$, $h_{33} = \mathbf{x}_{ww} \cdot e$, $t_{11} = e_u \cdot e_u$,

$t_{12} = e_u \cdot e_v$, $t_{13} = e_u \cdot e_w$, $t_{22} = e_v \cdot e_v$, $t_{23} = e_v \cdot e_w$, $t_{33} = e_w \cdot e_w$, respectively. Here,

$$e = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w}{\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w\|} \tag{1}$$

is the unit normal (i.e. the Gauss map) of the hypersurface $\mathbf{x}$.

The product matrices $(g_{ij})^{-1} \cdot (h_{ij})$ gives the matrix of the shape operator $\mathbf{S}$ of hypersurface $\mathbf{x}$ in 4-space. See [31, 32, 33] for details. Also, the product matrices $(h_{ij}) \cdot \mathbf{S}$ gives the matrix of the third fundamental form matrix of hypersurface $\mathbf{x}$ in 4-space.

3. i-th Curvatures

In $\mathbb{E}^4$, we use the following characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$ to compute the i-th mean curvature formula $\mathfrak{C}_i$, where $i = 0, 1, 2, 3$,

$$P_{\mathbf{S}}(\lambda) = \det(\mathbf{S} - \lambda \mathcal{I}_3) = 0.$$

Then, we obtain $\mathfrak{C}_0 = 1$ (by definition), $\binom{3}{1}\mathfrak{C}_1 = \binom{3}{1}H = -\frac{b}{a}$, $\binom{3}{2}\mathfrak{C}_2 = \frac{c}{a}$, $\binom{3}{3}\mathfrak{C}_3 = K = -\frac{d}{a}$.

Therefore, we find the i-th curvature formulas depends on the coefficients of the first and second fundamental forms in 4-space.

Theorem 1. *Any hypersurface* $\mathbf{x}$ *in* $\mathbb{E}^4$ *has the following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{\left\{\begin{array}{c} (g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12})g_{33} \\ +(g_{11}g_{22} - g_{12}^2)h_{33} - g_{23}^2 h_{11} - g_{13}^2 h_{22} \\ -2(g_{13}h_{13}g_{22} - g_{23}h_{13}g_{12} - g_{13}h_{23}g_{12} \\ +g_{11}g_{23}h_{23} - g_{13}g_{23}h_{12}) \end{array}\right\}}{3\left[(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2\right]}, \tag{2}$$

$$\mathfrak{C}_2 = \frac{\left\{\begin{array}{c} (g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12})h_{33} \\ +(h_{11}h_{22} - g_{12}^2)g_{33} - g_{11}h_{23}^2 - g_{22}h_{13}^2 \\ -2(g_{13}h_{13}h_{22} - g_{23}h_{13}h_{12} - g_{13}h_{23}h_{12} \\ +g_{23}h_{23}h_{11} - h_{13}h_{23}g_{12}) \end{array}\right\}}{3\left[(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2\right]}, \tag{3}$$

$$\mathfrak{C}_3 = \frac{\left(h_{11}h_{22} - h_{12}^2\right)h_{33} - h_{11}h_{23}^2 + 2h_{12}h_{13}h_{23} - h_{22}h_{13}^2}{(g_{11}g_{22} - g_{12}^2)g_{33} - g_{11}g_{23}^2 + 2g_{12}g_{13}g_{23} - g_{22}g_{13}^2}. \tag{4}$$

See [29] for details.

4. Bi-Rotational Hypersurface

In this section, we define the rotational hypersurface, then find its curvatures in Euclidean 4-space $\mathbb{E}^4$. We would like to note that the definition of rotational hypersurfaces in Riemannian space forms were defined in [22]. A rotational hypersurface $M \subset \mathbb{E}^{n+1}$ generated by a curve γ around an axis γ that does not meet γ is obtained by taking the orbit of γ under those orthogonal transformations of $\mathbb{E}^{n+1}$ that leaves $\mathfrak{r}$ pointwise fixed (See [22, Remark 2.3]).

We use the curve γ as $(\mathbf{f}(u), 0, \mathbf{g}(u), 0)$ with following rotation:

$$\begin{pmatrix} \cos v & -\sin v & 0 & 0 \\ \sin v & \cos v & 0 & 0 \\ 0 & 0 & \cos w & -\sin w \\ 0 & 0 & \sin w & \cos w \end{pmatrix} \begin{pmatrix} \mathbf{f}(u) \\ 0 \\ \mathbf{g}(u) \\ 0 \end{pmatrix},$$

and then give the following definition:

Definition 1. *A bi-rotational hypersurface* in $\mathbb{E}^4$ is *defined by*

$$\mathbf{x}(u, v, w) = (\mathbf{f}(u)\cos v, \mathbf{f}(u)\sin v, \mathbf{g}(u)\cos w, \mathbf{g}(u)\sin w), \tag{5}$$

where $\mathbf{f}, \mathbf{g}$ *are differentiable functions, and* $0 \leq v, w \leq 2\pi$.

Remark 1. *While* $\mathbf{f}(u) = \mathbf{g}(u) = 1$ *in* (5), *we obtain the Clifford torus in* $\mathbb{E}^4$. *See* [2, 48] *for details. Moreover, when* $v = w$ *in* (5), *we get the tensor product surface in* $\mathbb{E}^4$. *See* [5, 43] *for details.*

Considering the first derivatives of (5) with respect to u, v, w, respectively,

$$\mathbf{x}_u = \begin{pmatrix} \mathbf{f}'\cos v \\ \mathbf{f}'\sin v \\ \mathbf{g}'\cos w \\ \mathbf{g}'\sin w \end{pmatrix}, \ \mathbf{x}_v = \begin{pmatrix} -\mathbf{f}\sin v \\ \mathbf{f}\cos v \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{x}_w = \begin{pmatrix} 0 \\ 0 \\ -\mathbf{g}\sin w \\ \mathbf{g}\cos w \end{pmatrix},$$

we find the first quantities of (5):

$$(g_{ij}) = diag\left(\mathbf{f}'^2 + \mathbf{g}'^2, \mathbf{f}^2, \mathbf{g}^2\right), \tag{6}$$

where $\mathbf{f}'$ and $\mathbf{g}'$ denote the first order derivative of $\mathbf{f}$ and $\mathbf{g}$ respect to u, respectively. Here,

$$g = \det(g_{ij}) = \mathbf{f}^2\mathbf{g}^2\left(\mathbf{f}'^2 + \mathbf{g}'^2\right).$$

By using (1), we get the Gauss map of the bi-rotational hypersurface (5):

$$e = \frac{1}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}\left(-\mathbf{g}'\cos v, -\mathbf{g}'\sin v, \mathbf{f}'\cos w, \mathbf{f}'\sin w\right). \tag{7}$$

With the help of the second derivatives of (5) with respect to u, v, w, and the Gauss map (7) of the bi-rotational hypersurface (5), we have the second quantities:

$$(h_{ij}) = diag\left(\frac{\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}''}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}, \frac{\mathbf{f}\mathbf{g}'}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}, -\frac{\mathbf{g}\mathbf{f}'}{\left(\mathbf{f}'^2 + \mathbf{g}'^2\right)^{1/2}}\right), \tag{8}$$

where $\mathbf{f}''$ and $\mathbf{g}''$ denote the second order derivative of $\mathbf{f}$ and $\mathbf{g}$ respect to u, respectively. Then, we get

$$h = \det(h_{ij}) = -\frac{\mathbf{f}\mathbf{g}\mathbf{f}'\mathbf{g}'(\mathbf{f}'\mathbf{g}'' - \mathbf{f}''\mathbf{g}')}{(\mathbf{f}'^2 + \mathbf{g}'^2)^{3/2}}.$$

By using (6) and (8), we calculate the following shape operator matrix of the bi-rotational hypersurface (5):

$$\mathbf{S} = diag\left(\frac{\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}''}{(\mathbf{f}'^2 + \mathbf{g}'^2)^{3/2}}, \frac{\mathbf{g}'}{\mathbf{f}(\mathbf{f}'^2 + \mathbf{g}'^2)^{1/2}}, -\frac{\mathbf{f}'}{\mathbf{g}(\mathbf{f}'^2 + \mathbf{g}'^2)^{1/2}}\right).$$

Finally, using (2), (3), and (4) , with (6), (8), respectively, we find the curvatures of the bi-rotational hypersurface (5):

Corollary 1. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (5). *The* $\mathbf{x}$ *has the following (mean) 1-curvature*

$$\mathfrak{C}_1 = \frac{(\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}'')\,\mathbf{f}\mathbf{g} - \left(\mathbf{f}'^2 + \mathbf{g}'^2\right)\left(\mathbf{f}\mathbf{f}' - \mathbf{g}\mathbf{g}'\right)}{3\mathbf{f}\mathbf{g}(\mathbf{f}'^2 + \mathbf{g}'^2)^{3/2}}.$$

Corollary 2. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (5). *The* $\mathbf{x}$ *has the following 2-curvature*

$$\mathfrak{C}_2 = \frac{\left(\mathbf{f}\mathbf{f}' - \mathbf{g}\mathbf{g}'\right)(\mathbf{g}'\mathbf{f}'' - \mathbf{f}'\mathbf{g}'') - \left(\mathbf{f}'^2 + \mathbf{g}'^2\right)\mathbf{f}'\mathbf{g}'}{3\mathbf{f}\mathbf{g}(\mathbf{f}'^2 + \mathbf{g}'^2)^2}.$$

Corollary 3. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (5). *The* $\mathbf{x}$ *has the following (Gaussian) 3-curvature*

$$\mathfrak{C}_3 = -\frac{\mathbf{f}'\mathbf{g}'(\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}'')}{\mathbf{f}\mathbf{g}(\mathbf{f}'^2 + \mathbf{g}'^2)^{5/2}}.$$

Example 1. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (5). *When the curve* γ *of* $\mathbf{x}$ *is parametrized by the arc length, and* $\mathbf{f}(u) = \cos u$, $\mathbf{g}(u) = \sin u$, *the bi-rotational hypersurface has the following curvatures*

$$\begin{aligned} \mathfrak{C}_1 &= 1, \text{ i.e. } \mathbf{x} \text{ has positive CMC,} \\ \mathfrak{C}_2 &= 1, \text{ i.e. } \mathbf{x} \text{ has positive constant 2-curvature,} \\ \mathfrak{C}_3 &= 1, \text{ i.e. } \mathbf{x} \text{ has positive CGC.} \end{aligned}$$

Example 2. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (5). *When the curve* γ *of* $\mathbf{x}$ *is parametrized by the arc length, and* $\mathbf{f}(u) = \mathbf{g}(u) = \frac{u}{\sqrt{2}}$, *the bi-rotational hypersurface has the following curvatures*

$$\begin{aligned} \mathfrak{C}_1 &= 0, \text{ i.e. } \mathbf{x} \text{ is 1-minimal,} \\ \mathfrak{C}_2 &= -\frac{1}{3u^2}, \text{ i.e. } \mathbf{x} \text{ has negative 2-curvature,} \\ \mathfrak{C}_3 &= 0, \text{ i.e. } \mathbf{x} \text{ is 3-minimal.} \end{aligned}$$

Next, with the help of the Gauss map (7) , and of the third fundamental form matrix $III = (t_{ij})$, we obtain the following coefficients of the third fundamental form matrix of (5):

$$(t_{ij}) = diag\left(\frac{(\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}'')^2}{(\mathbf{f}'^2 + \mathbf{g}'^2)^2}, \frac{\mathbf{g}'^2}{\mathbf{f}'^2 + \mathbf{g}'^2}, \frac{\mathbf{f}'^2}{\mathbf{f}'^2 + \mathbf{g}'^2}\right). \tag{9}$$

5. Bi-Rotational Hypersurface Satisfying $\Delta^{III}\mathbf{x} = \mathcal{A}\mathbf{x}$

In this section, we give the third Laplace−Beltrami operator of a smooth function. Then we calculate the third Laplace−Beltrami operator of the bi-rotational hypersurface.

The inverse of the matrix III, i.e. (t_{ij}) is given by

$$\frac{1}{t}\begin{pmatrix} t_{22}t_{33} - t_{23}t_{32} & -(t_{12}t_{33} - t_{13}t_{32}) & t_{12}t_{23} - t_{13}t_{22} \\ -(t_{21}t_{33} - t_{31}t_{23}) & t_{11}t_{33} - t_{13}t_{31} & -(t_{11}t_{23} - t_{21}t_{13}) \\ t_{21}t_{32} - t_{22}t_{31} & -(t_{11}t_{32} - t_{12}t_{31}) & t_{11}t_{22} - t_{12}t_{21} \end{pmatrix},$$

where

$$\begin{aligned} t &= \det(t_{ij}) \\ &= t_{11}t_{22}t_{33} - t_{11}t_{23}t_{32} + t_{12}t_{31}t_{23} - t_{12}t_{21}t_{33} + t_{21}t_{13}t_{32} - t_{13}t_{22}t_{31}. \end{aligned}$$

Definition 2. *The third Laplace−Beltrami operator of a smooth function* $\phi = \phi(x^1, x^2, x^3)\mid_{\mathbf{D}}$ $(\mathbf{D} \subset \mathbb{R}^3)$ *of class* C^3 *is defined by*

$$\Delta^{III}\phi = \frac{1}{t^{1/2}}\sum_{i,j=1}^{3}\frac{\partial}{\partial x^i}\left(t^{1/2}t^{ij}\frac{\partial\phi}{\partial x^j}\right), \tag{10}$$

where $\left(t^{ij}\right) = (t_{kl})^{-1}$ *and* $t = \det(t_{ij})$.

We can also write (10), as follows

$$\Delta^{III}\phi = \frac{1}{|t|^{1/2}}\left\{\begin{array}{c} \frac{\partial}{\partial x^1}\left(|t|^{1/2}t^{11}\frac{\partial\phi}{\partial x^1}\right) - \frac{\partial}{\partial x^1}\left(|t|^{1/2}t^{12}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^1}\left(|t|^{1/2}t^{13}\frac{\partial\phi}{\partial x^3}\right) \\ -\frac{\partial}{\partial x^2}\left(|t|^{1/2}t^{21}\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(|t|^{1/2}t^{22}\frac{\partial\phi}{\partial x^2}\right) - \frac{\partial}{\partial x^2}\left(|t|^{1/2}t^{23}\frac{\partial\phi}{\partial x^3}\right) \\ +\frac{\partial}{\partial x^3}\left(|t|^{1/2}t^{31}\frac{\partial\phi}{\partial x^1}\right) - \frac{\partial}{\partial x^3}\left(|t|^{1/2}t^{32}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(|t|^{1/2}t^{33}\frac{\partial\phi}{\partial x^3}\right) \end{array}\right\}.$$

For any rotational hypersurface $t_{ij} = 0$, when $i \neq j$. Hence, we re-write the third Laplace−Beltrami operator $\Delta^{III}\phi$:

$$\Delta^{III}\phi = \frac{1}{|t|^{1/2}}\left\{\frac{\partial}{\partial x^1}\left(|t|^{1/2}t^{11}\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(|t|^{1/2}t^{22}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(|t|^{1/2}t^{33}\frac{\partial\phi}{\partial x^3}\right)\right\}.$$

Therefore, more clear form of the third Laplace−Beltrami operator of any rotational hypersurface $\mathbf{x}(u,v,w)$ is given by

(11)

$$\Delta^{III}\mathbf{x} = \frac{1}{|t|^{1/2}}\left\{\frac{\partial}{\partial u}\left(\frac{t_{22}t_{33}}{|t|^{1/2}}\mathbf{x}_u\right)+\frac{\partial}{\partial v}\left(\frac{t_{11}t_{33}}{|t|^{1/2}}\mathbf{x}_v\right)+\frac{\partial}{\partial w}\left(\frac{t_{11}t_{22}}{|t|^{1/2}}\mathbf{x}_w\right)\right\}.$$

Considering (9), and differentiating $\frac{t_{22}t_{33}}{|t|^{1/2}}\mathbf{x}_u$, $\frac{t_{11}t_{33}}{|t|^{1/2}}\mathbf{x}_v$, $\frac{t_{11}t_{22}}{|t|^{1/2}}\mathbf{x}_w$, with respect to u, v, w, respectively, and substituting them into (11), we get the following

$$\Delta^{III}\mathbf{x} = \begin{pmatrix}\Delta^{III}\mathbf{x}_1\\ \Delta^{III}\mathbf{x}_2\\ \Delta^{III}\mathbf{x}_3\\ \Delta^{III}\mathbf{x}_4\end{pmatrix} = \begin{pmatrix}\mathbf{F}(u)\cos v\\ \mathbf{F}(u)\sin v\\ \mathbf{G}(u)\cos w\\ \mathbf{G}(u)\sin w\end{pmatrix},$$

where

$$\mathbf{F}(u) = -\frac{\left\{\begin{array}{c} -\mathbf{f}'\mathbf{g}'^3\left(\mathbf{f}'^2+\mathbf{g}'^2\right)\mathbf{f}'''+\mathbf{f}'^2\mathbf{g}'^2\left(\mathbf{f}'^2+\mathbf{g}'^2\right)\mathbf{g}''' \\ +\left(\mathbf{f}'\mathbf{g}''-\mathbf{g}'\mathbf{f}''\right)\left\{\begin{array}{c}\mathbf{f}\mathbf{g}'^2\mathbf{f}''^2\\ -2\left(\mathbf{f}\mathbf{f}'\mathbf{g}'\mathbf{g}''+\mathbf{g}'^2\left(\mathbf{f}'^2+\mathbf{g}'^2\right)\right)\mathbf{f}''\\ +\mathbf{f}'\mathbf{g}''\left(\mathbf{f}\mathbf{f}'\mathbf{g}''-\mathbf{g}'\left(\mathbf{f}'^2+\mathbf{g}'^2\right)\right)\end{array}\right\}\end{array}\right\}\left(\mathbf{f}'^2+\mathbf{g}'^2\right)}{\mathbf{g}'^2\left(\mathbf{f}'\mathbf{g}''-\mathbf{g}'\mathbf{f}''\right)^3},$$

$$\mathbf{G}(u) = -\frac{\left\{\begin{array}{c} -\mathbf{f}'^2\mathbf{g}'^2\left(\mathbf{f}'^2+\mathbf{g}'^2\right)\mathbf{f}'''+\mathbf{f}'^3\mathbf{g}'\left(\mathbf{f}'^2+\mathbf{g}'^2\right)\mathbf{g}''' \\ +\left(\mathbf{f}'\mathbf{g}''-\mathbf{g}'\mathbf{f}''\right)\left\{\begin{array}{c}\mathbf{g}\mathbf{g}'^2\mathbf{f}''^2\\ -\mathbf{f}'\mathbf{g}'\left(2\mathbf{g}\mathbf{g}''+\left(\mathbf{f}'^2+\mathbf{g}'^2\right)\right)\mathbf{f}''\\ +\mathbf{f}'^2\mathbf{g}''\left(\mathbf{g}\mathbf{g}''-2\left(\mathbf{f}'^2+\mathbf{g}'^2\right)\right)\end{array}\right\}\end{array}\right\}\left(\mathbf{f}'^2+\mathbf{g}'^2\right)}{\mathbf{f}'^2\left(\mathbf{f}'\mathbf{g}''-\mathbf{g}'\mathbf{f}''\right)^3},$$

and $\mathbf{f}'''$ and $\mathbf{g}'''$ denote the third order derivative of $\mathbf{f}$ and $\mathbf{g}$ respect to u, respectively.

6. Conclusion

With the help of the findings of the previous section, we have the following results:

Corollary 4. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (5). *Then bi-rotational hypersurface* $\mathbf{x}$ *satisfies* $\Delta^{III}\mathbf{x} = \mathcal{A}\mathbf{x}$, *where*

$$\mathcal{A} = diag\left(\frac{\mathbf{F}}{\mathbf{f}}\mathcal{I}_2, \frac{\mathbf{G}}{\mathbf{g}}\mathcal{I}_2,\right),$$

and $\mathcal{A} \in Mat\,(4,4)$, $\mathcal{I}_2$ *is identity matrix.*

Corollary 5. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (5). *When the curve* γ *of bi-rotational hypersurface* $\mathbf{x}$ *is parametrized by the arc length, the* $\mathbf{x}$ *holds* $\Delta^{III}\mathbf{x} = \mathcal{B}\mathbf{x}$, *where*

$$\mathcal{B} = diag\left(\mathfrak{p}\mathcal{I}_2, \mathfrak{q}\mathcal{I}_2\right),$$

and

$$\mathfrak{p}(u) = \frac{(\mathbf{g}'\mathbf{f}''' - \mathbf{f}'\mathbf{g}''')\mathbf{g}'\mathbf{f}'^2 + (\mathbf{g}'\mathbf{f}'' - \mathbf{f}'\mathbf{g}'')\left\{\begin{array}{c}\mathbf{f}\left(\mathbf{g}'^2\mathbf{f}''^2 + \mathbf{f}'^2 g''^2\right) \\ -\mathbf{g}'\left(2\mathbf{g}'\mathbf{f}'' + \mathbf{f}'\mathbf{g}'' + 2\mathbf{f}\mathbf{f}'\mathbf{f}''\mathbf{g}''\right)\end{array}\right\}}{\mathbf{g}'^2(\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}'')^3},$$

$$\mathfrak{q}(u) = \frac{(\mathbf{g}'\mathbf{f}''' - \mathbf{f}'\mathbf{g}''')\mathbf{g}'\mathbf{f}'^2 + (\mathbf{g}'\mathbf{f}'' - \mathbf{f}'\mathbf{g}'')\left\{\begin{array}{c}\mathbf{g}\left(\mathbf{g}'^2\mathbf{f}''^2 + \mathbf{f}'^2\mathbf{g}''^2\right) \\ -\mathbf{f}'\left(\mathbf{g}'\mathbf{f}'' + 2\mathbf{f}'\mathbf{g}'' + 2\mathbf{g}\mathbf{g}'\mathbf{g}''\mathbf{f}''\right)\end{array}\right\}}{\mathbf{f}'^2(\mathbf{f}'\mathbf{g}'' - \mathbf{g}'\mathbf{f}'')^3},$$

and $\mathcal{B} \in Mat(4,4)$, $\mathcal{I}_2$ *is the identity matrix.*

Corollary 6. *Let* $\mathbf{x} : M^3 \longrightarrow \mathbb{E}^4$ *be an immersion given by* (5). *When the curve* γ *of* $\mathbf{x}$ *is parametrized by the arc length, and* $\mathbf{f}(u) = \cos u$, $\mathbf{g}(u) = \sin u$, *the bi-rotational hypersurface* $\mathbf{x}$ *supplies* $\Delta^{III}\mathbf{x} = \mathcal{C}\mathbf{x}$, *where*

$$\mathcal{C} = diag(\mathfrak{a}\mathcal{I}_2, \mathfrak{b}\mathcal{I}_2),$$

$\mathfrak{a}(u) = -3\cos u$, $\mathfrak{b}(u) = -3\sin u$, *and* $\mathcal{C} \in Mat(4,4)$, $\mathcal{I}_2$ *is the identity matrix.*

Example 3. *The bi-rotational hypersurface*

$$\xi(u,v,w) = \left(\frac{1-u^2}{1+u^2}\cos v, \frac{1-u^2}{1+u^2}\sin v, \frac{2u}{1+u^2}\cos w, \frac{2u}{1+u^2}\sin w\right) \tag{12}$$

satisfies $\xi = e$, *has the following shape operator matrix* $\mathbf{S} = -\mathcal{I}_3$, *where* $\mathcal{I}_3$ *is the identity matrix, and has the following curvatures* $\mathfrak{C}_i = -1$, *where* $i = 1,2,3$. *Therefore, the hypersurface* ξ *supplies*

$$\Delta^{III}\xi = -3\xi.$$

Here, $H\mathfrak{C}_2 = -K$, $H^2 = -\mathfrak{C}_2$, *and* $H^3 = K$, *i.e., the hypersurface* (12) *is the bi-rotational umbilical hypersphere.*

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