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GENERALIZING HARDY TYPE INEQUALITIES VIA k-RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL OPERATORS INVOLVING TWO ORDERS

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Abstract. In this study, We have applied the right operator k-Riemann-Liouville is involving two orders α and β with a positive parameter $p > 0$, further, the left operator k-Riemann-Liouville is used with the negative parameter $p < 0$ to introduce a new version related to Hardy-type inequalities. These inequalities are given and reversed for the cases $0 < p < 1$ and $p < 0$. We then improved and generalized various consequences in the framework of Hardy-type fractional integral inequalities.

1. Introduction

In 2012, W.T. Sulaiman presented the following Hardy type inequality [10](Theorem 3.1).

Let f be positive function defined on $[a, b] \subseteq (0, +\infty)$, $F(x) = \int_a^x f(t)dt$. Then

1. For $p \geq 1$,

$$p \int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq (b-a)^p \int_a^b \left(\frac{f(x)}{x}\right)^p dx - \int_a^b (1-\frac{a}{x})^p f^p(x)dx. \tag{1}$$

2. For $0 < p < 1$,

$$p \int_a^b \left(\frac{F(x)}{x}\right)^p dx \geq (1-\frac{a}{b})^p \int_a^b f(x)^p dx - \frac{1}{b^p} \int_a^b (x-a)^p f^p(x). \tag{2}$$

These inequalities has evoked the interest of many researchers, some generalizations variants and extensions have appeared in the literature, including Sroysang [9] has generalized some integral inequalities similar to Hardy's inequality, in [2], Benaissa gave a further generalization to this inequality, moreover, Benaissa and Benguessoum [3] presented Hardy-type inequalities via Jensen integral inequality , afterward Benaissa in [4] publicized a new version of the reverse Hardy's inequality with two parameters has presented on

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time scales and presented a new result with a negative parameter $p < 0$. On another side, in the fractional calculus, Z. Dahmani et all [6] given the following inequality:

$$\begin{aligned} \mathrm{J}^{\beta}_{a^+}\left(\frac{\left[\mathrm{J}^{\alpha}_{a^+}f(b)\right]^p}{g^q(b)}\right) &\leq \frac{\Gamma(\beta+\alpha-1)\Gamma^{1-p}(\alpha+1)}{\Gamma(\beta)\Gamma(\alpha)(\alpha(p-1)+1)} \\ &\times\left[(b-a)^{\alpha(p-1)+1}\mathrm{J}^{\beta+\alpha-1}_{a^+}\left(\tfrac{f^p(b)}{g^q(b)}\right)-\mathrm{J}^{\beta+\alpha-1}_{a^+}\left(\tfrac{f^p(b)}{g^q(b)}(b-a)^{\alpha(p-1)+1}\right)\right]. \end{aligned} \tag{3}$$

The right-sided k-Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[a, b]$ is defined as

$$\mathrm{J}^{\alpha,k}_{a^+}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x(x-t)^{\frac{\alpha}{k}-1}f^p(t)dt,\quad a<x\leq b. \tag{4}$$

The left-sided k-Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on [a, b] defined as

$$\mathrm{J}^{\alpha,k}_{b^-}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b(t-x)^{\frac{\alpha}{k}-1}f^p(t)dt,\quad a\leq x<b,\ k>0. \tag{5}$$

For more details, some applications on the k-Riemann-Liouville fractional integral are demonstrated in [7].
Motivated by the above literature, in this work we introduce the left and the right k-Riemann-Liouville fractional integral with two orders to generalize inequality (3) for $p > 1$ and we present a new results related to Hardy type inequality for the cases $0 < p < 1$ and $p < 0$.

2. Main results

In this section we present our principal results.

Theorem 2.1. *Let $f \geq 0$ and $g > 0$ on $[a, b] \subseteq [0, \infty[$ such that g is non decreasing. Then, for all $p > 1$, $\frac{\beta}{k} \geq 1$, $\frac{\alpha}{k} \geq 1$, we have*

$$\begin{aligned} \mathrm{J}^{\beta,k}_{a^+}\left(\frac{\left[\mathrm{J}^{\alpha,k}_{a^+}f(b)\right]^p}{g(b)}\right) &\leq c \\ &\times\left[(b-a)^{\frac{\alpha}{k}(p-1)+1}\mathrm{J}^{\beta+\alpha-k,\,k}_{a^+}\left(\tfrac{f^p(b)}{g(b)}\right)-\mathrm{J}^{\beta+\alpha-k,\,k}_{a^+}\left(\tfrac{f^p(b)}{g(b)}(b-a)^{\frac{\alpha}{k}(p-1)+1}\right)\right], \end{aligned} \tag{6}$$

where $c=\dfrac{\Gamma_k(\beta+\alpha-k)\Gamma_k^{1-p}(\alpha+k)}{\Gamma_k(\beta)\Gamma_k(\alpha)(\alpha(p-1)+k)}$.

Proof. For $p>1$, using Hölder inequality for $\frac{1}{p}+\frac{1}{p'}=1$, we have

$$\begin{aligned}\left[\mathrm{J}_{a^+}^{\alpha,k}f(x)\right]^p &= \left(\int_a^x\left[\frac{(x-t)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\right]^{\frac{1}{p'}}\left[\frac{(x-t)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\right]^{\frac{1}{p}}f(t)dt\right)^p\\ &\leq \frac{\Gamma_k^{1-p}(\alpha+k)}{k\Gamma_k(\alpha)}(x-a)^{\frac{\alpha}{k}(p-1)}\left(\int_a^x(x-t)^{\frac{\alpha}{k}-1}f^p(t)dt\right),\end{aligned}$$

thus

$$\mathrm{J}_{a^+}^{\beta,k}\left(g^{-1}(b)\left[\mathrm{J}_{a^+}^{\alpha,k}f(b)\right]^p\right)=\frac{1}{k\Gamma_k(\beta)}\int_a^b(b-x)^{\frac{\beta}{k}-1}g^{-1}(x)\left[\mathrm{J}_{a^+}^{\alpha,k}f(x)\right]^p dx$$

$$\leq\frac{\Gamma_k^{1-p}(\alpha+k)}{k^2\Gamma_k(\beta)\Gamma_k(\alpha)}\int_a^b(b-x)^{\frac{\beta}{k}-1}g^{-1}(x)(x-a)^{\frac{\alpha}{k}(p-1)}\left(\int_a^x(x-t)^{\frac{\alpha}{k}-1}f^p(t)dt\right)dx,$$

since g is non-decreasing function on $[t,b]$ and by using Fubini Theorem, we obtain

$$\mathrm{J}_{a^+}^{\beta,k}\left(g^{-1}(b)\left[\mathrm{J}_{a^+}^{\alpha,k}f(b)\right]^p\right)\leq\frac{\Gamma_k^{1-p}(\alpha+k)}{k^2\Gamma_k(\beta)\Gamma_k(\alpha)}$$

$$\times\int_a^b(b-t)^{\frac{\beta}{k}-1}g^{-1}(t)(b-t)^{\frac{\alpha}{k}-1}f^p(t)\left(\int_t^b(x-a)^{\frac{\alpha}{k}(p-1)}dx\right)dt,$$

$$=\frac{\Gamma_k^{1-p}(\alpha+k)}{k^2\Gamma_k(\beta)\Gamma_k(\alpha)(\frac{\alpha}{k}(p-1)+1)}$$

$$\times\int_a^b(b-t)^{\frac{\beta+\alpha}{k}-2}\frac{f^p(t)}{g(t)}\left((b-a)^{\frac{\alpha}{k}(p-1)+1}-(t-a)^{\frac{\alpha}{k}(p-1)+1}\right)dt.$$

Therefore

$$\int_a^b\frac{\left[\mathrm{J}_{a^+}^{\alpha}f(x)\right]^p}{g(x)}dx$$

$$\leq\frac{\Gamma_k(\beta+\alpha-k)\Gamma_k^{1-p}(\alpha+k)}{\Gamma_k(\beta)\Gamma_k(\alpha)(\alpha(p-1)+k)}\left[\frac{(b-a)^{\frac{\alpha}{k}(p-1)+1}}{k\Gamma_k(\beta+\alpha-k)}\int_a^b(b-t)^{\frac{\beta+\alpha-k}{k}-1}\frac{f^p(t)}{g(t)}dt\right.$$

$$\left.-\frac{1}{k\Gamma_k(\beta+\alpha-k)}\int_a^b(b-t)^{\frac{\beta+\alpha-k}{k}-1}\frac{f^p(t)}{g(t)}(t-a)^{\frac{\alpha}{k}(p-1)+1}dt\right]$$

$$=\frac{\Gamma_k(\beta+\alpha-k)\Gamma_k^{1-p}(\alpha+k)}{\Gamma_k(\beta)\Gamma_k(\alpha)(\alpha(p-1)+k)}$$

$$\left[(b-a)^{\frac{\alpha}{k}(p-1)+1}\mathrm{J}_{a^+}^{\beta+\alpha-k}\left(\frac{f^p(b)}{g(b)}\right)-\mathrm{J}_{a^+}^{\beta+\alpha-k}\left(\frac{f^p(b)}{g(b)}(b-a)^{\frac{\alpha}{k}(p-1)+1}\right)\right].$$

This ends the proof. □

Theorem 2.2. *Let $f \geq 0$ and $g > 0$ on $[a,b] \subseteq [0,\infty[$ such that g is non decreasing. Then, for all $0 < p < 1$, $0 < \frac{\alpha}{k} < \frac{1}{1-p}$ and $\beta + \alpha > k$, we have*

(7)
$$\mathrm{J}_{a^+}^{\beta,k}\left(\frac{\left[\mathrm{J}_{a^+}^{\alpha,k} f(b)\right]^p}{g(b)}\right) \geq c_1$$

$$\times\left[(b-a)^{\frac{\alpha}{k}(p-1)+1}\mathrm{J}_{a^+}^{\beta+\alpha-k,\,k}\left(\tfrac{f^p(b)}{g(b)}\right) - \mathrm{J}_{a^+}^{\beta+\alpha-k,\,k}\left(\tfrac{f^p(b)}{g(b)}(b-a)^{\frac{\alpha}{k}(p-1)+1}\right)\right],$$

where $c_1 = \dfrac{\Gamma_k(\beta+\alpha-k)\Gamma_k^{1-p}(\alpha+k)}{\Gamma_k(\beta)\Gamma_k(\alpha)(k-\alpha(1-p))g(b)}$.

Proof. For $0 < p < 1$, use the reverse Hölder inequality for $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, we have

$$\left[\mathrm{J}_{a^+}^{\alpha,k} f(x)\right]^p \geq \frac{\Gamma_k^{1-p}(\alpha+k)}{k\Gamma_k(\alpha)}(x-a)^{\frac{\alpha}{k}(p-1)}\left(\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t)dt\right),$$

hence

$$\mathrm{J}_{a^+}^{\beta,k}\left(g^{-1}(b)\left[\mathrm{J}_{a^+}^{\alpha,k} f(b)\right]^p\right) = \frac{1}{k\Gamma_k(\beta)}\int_a^b (b-x)^{\frac{\beta}{k}-1} g^{-1}(x)\left[\mathrm{J}_{a^+}^{\alpha,k} f(x)\right]^p dx$$

$$\geq \frac{\Gamma_k^{1-p}(\alpha+k)}{k^2\Gamma_k(\beta)\Gamma_k(\alpha)}\int_a^b (b-x)^{\frac{\beta}{k}-1} g^{-1}(x)(x-a)^{\frac{\alpha}{k}(p-1)}\left(\int_a^x (x-t)^{\frac{\alpha}{k}-1} f^p(t)dt\right)dx,$$

since g is non-decreasing function on $[t,b]$ and by using Fubini Theorem, we obtain

$$\mathrm{J}_{a^+}^{\beta,k}\left(g^{-1}(b)\left[\mathrm{J}_{a^+}^{\alpha,k} f(b)\right]^p\right)$$

$$\geq \frac{\Gamma_k^{1-p}(\alpha+k)}{k^2\Gamma_k(\beta)\Gamma_k(\alpha)}$$

$$\times\int_a^b (b-t)^{\frac{\beta}{k}-1} g^{-1}(b)(b-t)^{\frac{\alpha}{k}-1} f^p(t)\left(\int_t^b (x-a)^{\frac{\alpha}{k}(p-1)}dx\right)dt,$$

$$= \frac{\Gamma_k^{1-p}(\alpha+k)}{k^2\Gamma_k(\beta)\Gamma_k(\alpha)(\frac{\alpha}{k}(p-1)+1)g(b)}$$

$$\times\int_a^b (b-t)^{\frac{\beta+\alpha}{k}-2} f^p(t)\left((b-a)^{\frac{\alpha}{k}(p-1)+1} - (t-a)^{\frac{\alpha}{k}(p-1)+1}\right)dt.$$

Then

$$\int_a^b \frac{\left[\mathrm{J}_{a^+}^{\alpha} f(x)\right]^p}{g(x)} dx$$

$$\geq \frac{\Gamma_k(\beta+\alpha-k)\Gamma_k^{1-p}(\alpha+k)}{\Gamma_k(\beta)\Gamma_k(\alpha)(\alpha(p-1)+k)g(b)} \left[\frac{(b-a)^{\frac{\alpha}{k}(p-1)+1}}{k\Gamma_k(\beta+\alpha-k)} \int_a^b (b-t)^{\frac{\beta+\alpha-k}{k}-1} f^p(t)dt\right.$$

$$\left. -\frac{1}{k\Gamma_k(\beta+\alpha-k)} \int_a^b (b-t)^{\frac{\beta+\alpha-k}{k}-1} f^p(t)(t-a)^{\frac{\alpha}{k}(p-1)+1} dt\right]$$

$$= \frac{\Gamma_k(\beta+\alpha-k)\Gamma_k^{1-p}(\alpha+k)}{\Gamma_k(\beta)\Gamma_k(\alpha)(\alpha(p-1)+k)g(b)}$$

$$\left[(b-a)^{\frac{\alpha}{k}(p-1)+1} \mathrm{J}_{a^+}^{\beta+\alpha-k,\,k}\left(f^p(b)\right) - \mathrm{J}_{a^+}^{\beta+\alpha-k,\,k}\left(f^p(b)(b-a)^{\frac{\alpha}{k}(p-1)+1}\right)\right].$$

This gives us the desired result. □

Now we present a new result according the left-sided k-Riemann-Liouville fractional integral operator.

Theorem 2.3. *Let $f \geq 0$ and $g > 0$ on $[a,b] \subseteq [0,\infty[$ such that g is non increasing. Then, for all $p < 0$, $\frac{1}{1-p} < \frac{\alpha}{k}$ and $\alpha+\beta > k$, we have*

(8)
$$\mathrm{J}_{b^-}^{\beta,k}\left(\frac{\left[\mathrm{J}_{b^-}^{\alpha,k} f(a)\right]^p}{g(a)}\right) \leq c_2$$

$$\times \left[\mathrm{J}_{b^-}^{\beta+\alpha-k,\,k}\left(f^p(a)(b-a)^{\frac{\alpha}{k}(p-1)+1}\right) - (b-a)^{\frac{\alpha}{k}(p-1)+1} \mathrm{J}_{b^-}^{\beta+\alpha-k,\,k}\left(f^p(a)\right)\right],$$

where $c_2 = \dfrac{\Gamma_k(\beta+\alpha-k)\Gamma_k^{1-p}(\alpha+k)}{\Gamma_k(\beta)\Gamma_k(\alpha)(\alpha(1-p)-k)g(a)}$.

Proof. For $p < 0$, using the reverse Hölder inequality [5] for $\dfrac{1}{p}+\dfrac{1}{p'}=1$, we have

$$\begin{aligned}
\mathrm{J}_{b^-}^{\alpha,k} f(x) &= \int_x^b \left[\frac{(t-x)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\right]^{\frac{1}{p'}} \left[\frac{(t-x)^{\frac{\alpha}{k}-1}}{k\Gamma_k(\alpha)}\right]^{\frac{1}{p}} f(t)dt \\
&\geq \left(\frac{1}{\Gamma_k(\alpha+k)}(b-x)^{\frac{\alpha}{k}}\right)^{\frac{1}{p'}} \left(\frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f^p(t), dt\right)^{\frac{1}{p}}
\end{aligned}$$

since $p<0$, we get

$$\mathrm{J}_{b^-}^{\beta,k}\left(\frac{\left[\mathrm{J}_{b^-}^{\alpha,k}f(a)\right]^p}{g(a)}\right)=\frac{1}{k\Gamma_k(\beta)}\int_a^b (x-a)^{\frac{\beta}{k}-1}g^{-1}(x)\left[\mathrm{J}_{b^-}^{\alpha,k}f(x)\right]^p dx$$

$$\leq \frac{\Gamma_k^{1-p}(\alpha+k)}{k^2\Gamma_k(\beta)\Gamma_k(\alpha)}\int_a^b (x-a)^{\frac{\beta}{k}-1}g^{-1}(x)(b-x)^{\frac{\alpha}{k}(p-1)}\left(\int_x^b (t-x)^{\frac{\alpha}{k}-1}f^p(t)dt\right)dx.$$

Since g is a non-increasing function on $[a,t]$ and by using Fubini Theorem, we get

$$\mathrm{J}_{b^-}^{\beta,k}\left(\frac{\left[\mathrm{J}_{b^-}^{\alpha,k}f(a)\right]^p}{g(a)}\right)$$

$$\leq \frac{\Gamma_k^{1-p}(\alpha+k)}{k^2\Gamma_k(\beta)\Gamma_k(\alpha)}\int_a^b (t-a)^{\frac{\beta}{k}-1}g^{-1}(a)(t-a)^{\frac{\alpha}{k}-1}f^p(t)\left(\int_a^t (b-x)^{\frac{\alpha}{k}(p-1)}dx\right)dt$$

$$=\frac{g^{-1}(a)\Gamma_k^{1-p}(\alpha+k)}{k\Gamma_k(\beta)\Gamma_k(\alpha)(\alpha(1-p)-k)}$$

$$\times\int_a^b f^p(t)(t-a)^{\frac{\alpha+\beta}{k}-2}\left((b-t)^{\frac{\alpha}{k}(p-1)+1}-(b-a)^{\frac{\alpha}{k}(p-1)+1}\right)dt,$$

this gives us that

(9)

$$\mathrm{J}_{b^-}^{\beta,k}\left(\frac{\left[\mathrm{J}_{b^-}^{\alpha,k}f(a)\right]^p}{g(a)}\right)$$

$$\leq \frac{\Gamma_k(\beta+\alpha-k)\Gamma_k^{1-p}(\alpha+k)}{\Gamma_k(\beta)\Gamma_k(\alpha)(\alpha(1-p)-k)g(a)}\left[\int_a^b \frac{(t-a)^{\frac{\beta+\alpha-k}{k}-1}}{k\Gamma_k(\beta+\alpha-k)}f^p(t)(b-t)^{\frac{\alpha}{k}(p-1)+1}dt\right.$$

$$\left.-\frac{(b-a)^{\frac{\alpha}{k}(p-1)+1}}{\Gamma_k(\beta+\alpha-k)}\int_a^b (t-a)^{\frac{\beta+\alpha-k}{k}-1}f^p(t)dt\right]$$

$$=\frac{\Gamma_k(\beta+\alpha-k)\Gamma_k^{1-p}(\alpha+k)}{\Gamma_k(\beta)\Gamma_k(\alpha)(\alpha(1-p)-k)g(a)}$$

$$\times\left[\mathrm{J}_{b^-}^{\beta+\alpha-k,\,k}\left(f^p(a)(b-a)^{\frac{\alpha}{k}(p-1)+1}\right)-(b-a)^{\frac{\alpha}{k}(p-1)+1}\mathrm{J}_{b^-}^{\beta+\alpha-k,\,k}\left(f^p(a)\right)\right].$$

That completes our proof. □

3. Applications

Taking $k = 1$ in the above Theorem 2.1, Theorem 2.2 and Theorem 2.3, we deduce the following Corollaries related to Riemann-Liouville inequalities with two orders α and β.

Corollary 3.1. *Let $f \geq 0$ and $g > 0$ on $[a, b] \subseteq [0, \infty[$ such that g is non decreasing. Then, for all $p > 1$, $\beta \geq 1$, $\alpha \geq 1$, we have*

(10)
$$\mathrm{J}^{\beta}_{a^+}\left(\frac{\left[\mathrm{J}^{\alpha}_{a^+} f(b)\right]^p}{g(b)}\right) \leq c$$
$$\times \left[(b-a)^{\alpha(p-1)+1}\mathrm{J}^{\beta+\alpha-1}_{a^+}\left(\tfrac{f^p(b)}{g(b)}\right) - \mathrm{J}^{\beta+\alpha-1}_{a^+}\left(\tfrac{f^p(b)}{g(b)}(b-a)^{\alpha(p-1)+1}\right)\right],$$

where $c = \dfrac{\Gamma(\beta+\alpha-1)\Gamma^{1-p}(\alpha+1)}{\Gamma(\beta)\Gamma(\alpha)(\alpha(p-1)+1)}$.

Remark 3.2. *If we replace g by g^q where $q > 0$, we get Theorem 3 in* [6].

Corollary 3.3. *Let $f \geq 0$ and $g > 0$ on $[a, b] \subseteq [0, \infty[$ such that g is non decreasing. Then, for all $0 < p < 1$, $0 < \alpha < \frac{1}{1-p}$ and $\beta + \alpha > 1$, we have*

(11)
$$\mathrm{J}^{\beta}_{a^+}\left(\frac{\left[\mathrm{J}^{\alpha}_{a^+} f(b)\right]^p}{g(b)}\right) \geq c_1$$
$$\times \left[(b-a)^{\alpha(p-1)+1}\mathrm{J}^{\beta+\alpha-1}_{a^+}\left(\tfrac{f^p(b)}{g(b)}\right) - \mathrm{J}^{\beta+\alpha-1}_{a^+}\left(\tfrac{f^p(b)}{g(b)}(b-a)^{alpha(p-1)+1}\right)\right],$$

where $c_1 = \dfrac{\Gamma(\beta+\alpha-1)\Gamma^{1-p}(\alpha+1)}{\Gamma(\beta)\Gamma(\alpha)(1-\alpha(1-p))g(b)}$.

Corollary 3.4. *Let $f \geq 0$ and $g > 0$ on $[a, b] \subseteq [0, \infty[$ such that g is non increasing. Then, for all $p < 0$, $\frac{1}{1-p} < \alpha$ and $\alpha + \beta > 1$, we have*

(12)
$$\mathrm{J}^{\beta}_{b^-}\left(\frac{\left[\mathrm{J}^{\alpha}_{b^-} f(a)\right]^p}{g(a)}\right) \leq c_2$$
$$\times \left[\mathrm{J}^{\beta+\alpha-1}_{b^-}\left(f^p(a)(b-a)^{\alpha(p-1)+1}\right) - (b-a)^{\alpha(p-1)+1}\mathrm{J}^{\beta+\alpha-1}_{b^-}\left(f^p(a)\right)\right],$$

where $c_2 = \dfrac{\Gamma(\beta+\alpha-1)\Gamma^{1-p}(\alpha+1)}{\Gamma(\beta)\Gamma(\alpha)(\alpha(1-p)-1)g(a)}$.

Now, setting $\beta = k$ in the above Theorem 2.1, Theorem 2.2 and Theorem 2.3, we deduce the following Corollaries related to k-Riemann-Liouville inequalities with the order α.

Corollary 3.5. *Let $f \geq 0$ and $g > 0$ on $[a, b] \subseteq [0, \infty[$ such that g is non decreasing. Then, for all $p > 1$, $\frac{\alpha}{k} \geq 1$, we have*

$$\int_a^b \left(\frac{\left[\mathrm{J}_{a^+}^{\alpha,k} f(x) \right]^p}{g(x)} dx \right) \leq c \tag{13}$$

$$\times \left[(b-a)^{\frac{\alpha}{k}(p-1)+1} \mathrm{J}_{a^+}^{\alpha,\,k} \left(\frac{f^p(b)}{g(b)} \right) - \mathrm{J}_{a^+}^{\alpha,\,k} \left(\frac{f^p(b)}{g(b)} (b-a)^{\frac{\alpha}{k}(p-1)+1} \right) \right],$$

where $c = \dfrac{k\,\Gamma_k^{1-p}(\alpha+k)}{\alpha(p-1)+k}$.

Remark 3.6. *If we replace g by g^q where $q > 0$, we get Theorem 3.2 in* [8].

Corollary 3.7. *Let $f \geq 0$ and $g > 0$ on $[a, b] \subseteq [0, \infty[$ such that g is non decreasing. Then, for all $0 < p < 1$, $0 < \frac{\alpha}{k} < \frac{1}{1-p}$, we have*

$$\int_a^b \left(\frac{\left[\mathrm{J}_{a^+}^{\alpha,k} f(x) \right]^p}{g(x)} dx \right) \geq c_1 \tag{14}$$

$$\times \left[(b-a)^{\frac{\alpha}{k}(p-1)+1} \mathrm{J}_{a^+}^{\alpha,\,k} \left(\frac{f^p(b)}{g(b)} \right) - \mathrm{J}_{a^+}^{\alpha,\,k} \left(\frac{f^p(b)}{g(b)} (b-a)^{\frac{\alpha}{k}(p-1)+1} \right) \right],$$

where $c_1 = \dfrac{k\,\Gamma_k^{1-p}(\alpha+k)}{(k-\alpha(1-p))g(b)}$.

Corollary 3.8. *Let $f \geq 0$ and $g > 0$ on $[a, b] \subseteq [0, \infty[$ such that g is non increasing. Then, for all $p < 0$, $\frac{1}{1-p} < \frac{\alpha}{k}$, we have*

$$\int_a^b \left(\frac{\left[\mathrm{J}_{b^-}^{\alpha,k} f(x) \right]^p}{g(x)} \right) \leq c_2 \tag{15}$$

$$\times \left[\mathrm{J}_{b^-}^{\alpha,\,k} \left(f^p(a)(b-a)^{\frac{\alpha}{k}(p-1)+1} \right) - (b-a)^{\frac{\alpha}{k}(p-1)+1} \mathrm{J}_{b^-}^{\alpha,\,k} \left(f^p(a) \right) \right],$$

where $c_2 = \dfrac{k\,\Gamma_k^{1-p}(\alpha+k)}{(\alpha(1-p)-k)g(a)}$.

Remark 3.9.
1. *Taking $k = 1$ in the Corollary 3.5, we get Theorem 3.2 in* [6].
2. *Taking $k = 1$ in the Corollary 3.7, then for $0 < p < 1$ and $0 < \alpha < \frac{1}{1-p}$, we get the following inequality.*

$$\int_a^b \frac{\left[\mathrm{J}_{a^+}^{\alpha} f(x) \right]^p}{g^q(x)} dx \geq \frac{\Gamma^{1-p}(\alpha+1)}{(\alpha(p-1)+1)g^q(b)} \tag{16}$$

$$\times \left[(b-a)^{\alpha(p-1)+1} \mathrm{J}_{a^+}^{\alpha} \left(f^p(b) \right) - \mathrm{J}_{a^+}^{\alpha} \left(f^p(b)(b-a)^{\alpha(p-1)+1} \right) \right].$$

3. *Setting $k = 1$ in the Corollary 3.8, hence for $p < 0$ and $\frac{1}{1-p} < \alpha$, we obtain the following inequality.*

$$\int_a^b \frac{\left[\mathrm{J}_{b-}^{\alpha} f(x)\right]^p}{g(x)} dx \geq \frac{\Gamma^{1-p}(\alpha+1)}{(\alpha(1-p)-1)g(a)} \tag{17}$$

$$\times \left[\mathrm{J}_{b-}^{\alpha}\left(f^p(a)(b-a)^{\alpha(p-1)+1}\right) - (b-a)^{\alpha(p-1)+1}\mathrm{J}_{b-}^{\alpha}\left(f^p(a)\right)\right].$$

Inequalities (16) and (17) are a new version of Riemann-Liouville integral inequalities. Putting now $\alpha = k = 1$ in the Corollary 3.8, we obtain the following Corollary.

Corollary 3.10. *Let $f \geq 0$, $g > 0$ on $[a, b] \subseteq [0, \infty[$ such that g is non increasing and $F(x) = \int_x^b f(t)dt$. Then, for all $p < 0$, we have*

$$-p\int_a^b \frac{F^p(x)}{g(x)} dx \geq \frac{1}{g(a)}\left[\int_a^b f^p(x)(b-x)^p dx - (b-a)^p \int_a^b f^p(x)dx\right]. \tag{18}$$

Remark 3.11. *The inequality (18) coincide with inequality (4.26) in* [4].

4. Conclusion

We have presented some new reverse Hardy type inequalities introduced via fractional integral operators k-Riemann-Liouville involving two orders *alpha* and *beta* by using the Holder's inequality, moreover new results are obtained with the parameters $0 < p < 1$ and $p < 0$. We then improved and generalized various consequences in the framework of fractional Hardy-type integral inequalities, we also presented new results related to Riemann-Liouville fractional integral operators with two orders.

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