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GEOMETRIC CHARACTERISTICS OF GENERIC LIGHTLIKE SUBMANIFOLDS

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Abstract. In the present study, we investigate generic lightlike submanifolds of indefinite nearly Kaehler manifolds. After proving the existence of generic lightlike submanifolds in an indefinite generalized complex space form, a non-trivial example of this class of submanifolds is discussed. Then, we find a characterization theorem enabling the induced connection on a generic lightlike submanifold to be a metric connection. We also derive some conditions for the integrability of distributions defined on generic lightlike submanifolds. Further, we discuss the non-existence of mixed geodesic generic lightlike submanifolds in a generalized complex space form. Finally, we investigate totally umbilical generic lightlike submanifolds and minimal generic lightlike submanifolds of an indefinite nearly Kaehler manifold.

1. Introduction

The concept of the CR-submanifolds of a Kaehler manifold was firstly developed by Bejancu in 1978 ([2]). He studied about totally real and complex submanifolds as sub cases and further the detailed discussion and investigation was done by the many researchers ([3]–[7]). As a hypersurface of a complex manifold, the CR-structure on a five-dimensional manifold has outstanding applications in differential geometry and the general theory of relativity. In this context, Duggal studied the interaction of CR-structures with Lorentzian geometry which was needed for the general theory of relativity ([9]). Furthermore, they introduced the interaction of CR-submanifolds with the theory of relativity and developed new results of geometric and physical importance ([10]). Deshmukh et al. in [8] initiated the study of CR-submanifolds of nearly Kaehler manifolds. Husain et al. in [17] extended this study and obtained their fundamental properties and observed that constant holomorphic sectional curvature in nearly Kaehler manifolds does not admit complex hypersurfaces.

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Furthermore, Duggal et al. in [11] established a new class with exclusion of invariant and totally real cases called CR-lightlike submanifolds of indefinite Kaehler manifolds. Thereafter, Duggal et al. in [14] introduced another class called SCR-lightlike submanifolds of indefinite Kaehler manifolds containing invariant and screen real subcases. They concluded that $SCR(CR)$-lightlike submanifolds are entirely different from each other as CR-lightlike submanifolds are always non-trivial. To achieve the desired connections, Duggal et al. in [15] established another class called GCR-lightlike submanifolds of indefinite Kaehler manifolds. It acts as an umbrella for all above mentioned lightlike submanifolds. Subsequently, Kumar et al. in [19] introduced a new class of submanifolds called GCR-lightlike submanifolds of indefinite nearly Kaehler manifolds taking constant holomorphic sectional curvature c and of constant type α.

On the other hand, Zhu et al. in [22] provided the concept about parallel canonical structure using Generic submanifolds of nearly Kaehler manifolds. The theory of lightlike submanifolds is not having applications in the interpretation of mathematical geometry; rather, it is having excellent applications in mathematical physics too in terms of study of four-dimensional electromagnetic space times, Einstein Field Equations, different types of horizons (Cauchy's horizons, event horizons and Kruskal's horizons). The deep analysis of lightlike submanifolds and their enormous applications in mathematical physics motivated the present authors to work on generic lightlike submanifolds of indefinite nearly Kaehler manifolds.

In the present paper, we introduce the study of generic lightlike submanifolds of indefinite nearly Kaehler manifolds. We obtain the existence of this class and the non-existence of mixed geodesic generic lightlike submanifolds of a generalized complex space form. We also study totally umbilical generic lightlike submanifolds and give some characterization theorems on minimal generic lightlike submanifolds.

2. Preliminaries

2.1. Geometry of lightlike submanifolds

Let $(\bar{K}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$, (K, g) an m-dimensional submanifold of $\bar{K}$, and g the induced metric of $\bar{g}$ on K. If $\bar{g}$ is degenerate on the tangent bundle TK of K, then K is called a lightlike submanifold of $\bar{K}$. For a degenerate metric g on K, one has the equation

$$TK^{\perp} = \cup\{u \in T_x\bar{K} : \bar{g}(u, v) = 0, \forall v \in T_xK, x \in K\}, \tag{1}$$

is a degenerate n-dimensional subspace of $T_x\bar{K}$. Thus, both T_xK and $T_xK^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(T_xK) = T_xK \cap T_xK^{\perp}$, which is known as the radical (null) subspace. If the mapping

$$Rad(TK) : x \in K \longrightarrow Rad(T_xK) \tag{2}$$

defines a smooth distribution on K of rank $r > 0$, then the submanifold K of $\bar{K}$ is called an r-lightlike submanifold and $Rad(TK)$ is called the radical distribution on K. The screen distribution $S(TK)$ is a semi-Riemannian complementary distribution of $Rad(TK)$ in TK, that is,

$$TK = Rad(TK) \perp S(TK) \tag{3}$$

and $S(TK^{\perp})$ is a complementary vector subbundle to $Rad(TK)$ in $TK^{\perp}$.

Theorem 2.1. [11] *For an r-lightlike submanifold* $(K, g, S(TK), S(TK^{\perp}))$ *of a semi-Riemannian manifold* $(\bar{K}, \bar{g})$*, there exists a complementary vector bundle* $ltr(TK)$ *of* $Rad(TK)$ *in* $S(TK^{\perp})^{\perp}$ *and a basis of* $\Gamma(ltr(TK) \mid_{\mathrm{u}})$ *consisting of smooth section* $\{N_i\}$ *of* $S(TK^{\perp})^{\perp} \mid_{\mathrm{u}}$*, where* u *is a coordinate neighborhood of* K *satisfying*

$$\bar{g}(N_i, N_j) = 0, \quad \bar{g}(N_i, \xi_j) = \delta_{ij}, \ \text{for} \quad i, j \in \{1, 2, .., r\}, \tag{4}$$

where $\{\xi_1, ..., \xi_r\}$ *is the lightlike basis of* $\Gamma(Rad(TK))$.

Let $tr(TK)$ and $ltr(TK)$ be complementary (but not orthogonal) vector bundles to TK in $T\bar{K} \mid_K$ and to $Rad(TK)$ in $S(TK^{\perp})^{\perp}$, respectively. Then we have

$$tr(TK) = ltr(TK) \perp S(TK^{\perp}). \tag{5}$$

$$T\bar{K} \mid_K = TK \oplus tr(TK) = (Rad(TK) \oplus ltr(TK)) \perp S(TK) \perp S(TK^{\perp}). \tag{6}$$

If we consider the Levi-Civita connection $\bar{\nabla}$ on $\bar{K}$, then in view of decomposition (6) the Gauss and Weingarten formulae are

$$\bar{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h(Y_1, Y_2), \quad \forall \ Y_1, Y_2 \in \Gamma(TK), \tag{7}$$

$$\bar{\nabla}_{Y_1} U = -A_U Y_1 + \nabla^{\perp}_{Y_1} U, \quad \forall \ Y_1 \in \Gamma(TK), \ U \in \Gamma(tr(TK)), \tag{8}$$

where $\{\nabla_{Y_1} Y_2, A_U Y_1\}$ and $\{h(Y_1, Y_2), \nabla^{\perp}_{Y_1} U\}$ belongs to $\Gamma(TK)$ and $\Gamma(tr(TK))$, respectively. One may note that ∇ is a torsion-free linear connection on K, h is a symmetric bilinear form on $\Gamma(TK)$, which is known as second fundamental form and A_U is a linear operator on K, which is known as the shape operator.

In view of the equation (5) and considering the projection morphisms L and S of $tr(TK)$ on $ltr(TK)$ and $S(TK^{\perp})$, respectively, from the equations (7) and (8), we attain

$$\bar{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h^l(Y_1, Y_2) + h^s(Y_1, Y_2), \tag{9}$$

$$\bar{\nabla}_{Y_1} U = -A_U Y_1 + D^l_{Y_1} U + D^s_{Y_1} U, \tag{10}$$

where $h^l(Y_1,Y_2) = L(h(Y_1,Y_2)), h^s(Y_1,Y_2) = S(h(Y_1,Y_2)), D^l_{Y_1}U = L(\nabla^{\perp}_{Y_1}U)$, $D^s_{Y_1}U = S(\nabla^{\perp}_{Y_1}U)$. As h^l and h^s are $\Gamma(ltr(TK))$-valued and $\Gamma(S(TK^{\perp}))$-valued, respectively, they are called the lightlike second fundamental form and the screen second fundamental form on K. In particular,

$$\bar{\nabla}_{Y_1}N = -A_NY_1 + \nabla^l_{Y_1}N + D^s(Y_1,N), \tag{11}$$

$$\bar{\nabla}_{Y_1}V = -A_VY_1 + \nabla^s_{Y_1}V + D^l(Y_1,V), \tag{12}$$

where $Y_1 \in \Gamma(TK), N \in \Gamma(ltr(TK))$ and $V \in \Gamma(S(TK^{\perp}))$. Employing the equations (9)–(12), we obtain

$$\bar{g}(h^s(Y_1,Y_2),V) + \bar{g}(Y_2, D^l(Y_1,V)) = g(A_VY_1,Y_2), \tag{13}$$

$$\bar{g}(A_NY_1,N') + \bar{g}(N, A_{N'}Y_1) = 0, \tag{14}$$

for $\xi \in \Gamma(Rad(TK))$, $V \in \Gamma(S(TK^{\perp}))$ and $N, N' \in \Gamma(ltr(TK))$.

If we consider $\bar{P}$ and the projection morphism of TK on $S(TK)$, then by employing the equation (3) we obtain

$$\nabla_{Y_1}\bar{P}Y_2 = \nabla^*_{Y_1}\bar{P}Y_2 + h^*(Y_1,Y_2), \tag{15}$$

$$\nabla_{Y_1}\xi = -A^*_{\xi}Y_1 + \nabla^{*t}_{Y_1}\xi, \tag{16}$$

for $Y_1, Y_2 \in \Gamma(TK)$ and $\xi \in \Gamma(Rad(TK))$, where $\{\nabla^*_{Y_1}PY_2, A^*_{\xi}Y_1\}$ and

$$\{h^*(Y_1,Y_2), \nabla^{*t}_{Y_1}\xi\}$$

belong to $\Gamma(S(TK))$ and $\Gamma(Rad(TK))$, respectively. Here, ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TK)$ and $Rad(TK)$, respectively. Moreover, h^* and A^* are $Rad(TK)$-valued and $S(TK)$-valued bilinear forms and called as second fundamental forms of distributions $S(TK)$ and $Rad(TK)$, respectively.

Further, employing the equations (9), (10), (15) and (16), we obtain

$$\bar{g}(h^l(Y_1,PY_2),\xi) = g(A^*_{\xi}Y_1,PY_2), \tag{17}$$

$$\bar{g}(h^*(Y_1,PY_2),N) = \bar{g}(A_NY_1,PY_2), \tag{18}$$

for $Y_1, Y_2 \in \Gamma(TK), \xi \in \Gamma(Rad(TK))$ and $N \in \Gamma(ltr(TK))$.

By considering $\bar{\nabla}$ be a metric connection on $\bar{K}$, we have

$$(\nabla_{Y_1}g)(Y_2,Y_3) = \bar{g}(h^l(Y_1,Y_2),Y_3) + \bar{g}(h^l(Y_1,Y_3),Y_2), \tag{19}$$

for $Y_1, Y_2, Y_3 \in \Gamma(TK)$. Then the equation of Codazzi is given by

$$\begin{aligned}(\bar{R}(Y_1,Y_2)Y_3)^{\perp} &= (\nabla_{Y_1}h^l)(Y_2,Y_3) - (\nabla_{Y_2}h^l)(Y_1,Y_3) + D^l(Y_1,h^s(Y_2,Y_3))\\ &\quad - D^l(Y_2,h^s(Y_1,Y_3)) + (\nabla_{Y_1}h^s)(Y_2,Y_3) - (\nabla_{Y_2}h^s)(Y_1,Y_3)\\ &\quad + D^s(Y_1,h^l(Y_2,Y_3)) - D^s(Y_2,h^l(Y_1,Y_3)),\end{aligned} \tag{20}$$

where

$$(\nabla_{Y_1}h^l)(Y_2,Y_3) = \nabla^l_{Y_1}(h^l(Y_2,Y_3)) - h^l(\nabla_{Y_1}Y_2,Y_3) - h^l(Y_2,\nabla_{Y_1}Y_3), \tag{21}$$

$$(22)\qquad (\nabla_{Y_1}h^s)(Y_2,Y_3)=\nabla^s_{Y_1}(h^l(Y_2,Y_3))-h^s(\nabla_{Y_1}Y_2,Y_3)-h^s(Y_2,\nabla_{Y_1}Y_3).$$

2.2. Indefinite nearly Kaehler manifolds

An indefinite almost Hermitian manifold together with an almost complex structure $\bar{J}$ and almost Hermitian metric $\bar{g}$ is said to be an indefinite nearly Kaehler manifold if

$$(23)\qquad \bar{J}^2=-I,\ \bar{g}(\bar{J}Y_1,\bar{J}Y_2)=\bar{g}(Y_1,Y_2),\ (\bar{\nabla}_{Y_1}\bar{J})Y_2+(\bar{\nabla}_{Y_2}\bar{J})Y_1=0,$$

for $Y_1,Y_2\in\Gamma(T\bar{K})$, where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{K}$ (see [16]).

On the other hand, an indefinite almost Hermitian manifold $(\bar{K},\bar{g},\bar{\nabla},\bar{J})$ is said to be an indefinite RK-manifold if

$$\bar{R}(\bar{J}Y_1,\bar{J}Y_2,\bar{J}Y_3,\bar{J}Y_4)=\bar{R}(Y_1,Y_2,Y_3,Y_4),$$

for $Y_1,Y_2,Y_3,Y_4\in\Gamma(T\bar{K})$. Further, an indefinite RK-manifold of constant holomorphic sectional curvature c and of constant type α is known as an indefinite generalized complex space form and it is denoted by $\bar{K}(c,\alpha)$. For an indefinite generalized complex space form $\bar{K}(c,\alpha)$, the curvature tensor $\bar{R}$ is defined as follows:

$$\begin{aligned}\bar{R}(Y_1,Y_2)Y_3=&\frac{c+3\alpha}{4}\{\bar{g}(Y_2,Y_3)Y_1-\bar{g}(Y_1,Y_3)Y_2\}\\ (24)\qquad &+\frac{c-\alpha}{4}\{\bar{g}(Y_1,\bar{J}Y_3)\bar{J}Y_2-\bar{g}(Y_2,\bar{J}Y_3)\bar{J}Y_1+2\bar{g}(Y_1,\bar{J}Y_2)\bar{J}Y_3\},\end{aligned}$$

where $Y_1,Y_2,Y_3\in\Gamma(T\bar{K})$.

3. Generic lightlike submanifolds

Firstly, we define generic lightlike submanifolds of indefinite nearly Kaehler manifolds following the paper [18].

Definition 3.1. *Let K be a real r-lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{K}$. Then, K is said to be a generic lightlike submanifold if the screen distribution $S(TK)$ of K is expressed as*

$$\begin{aligned}S(TK)=&\bar{J}(S(TK)^{\perp})\oplus_{orth}D_0\\ (25)\qquad =&\bar{J}(Rad(TK))\oplus\bar{J}(ltr(TK))\oplus_{orth}\bar{J}(S(TK^{\perp}))\oplus_{orth}D_0,\end{aligned}$$

where D_0 is a non-degenerate almost complex distribution on K with respect to $\bar{J}$, i.e., $\bar{J}(D_0)=D_0$ and D' is an r-lightlike distribution on $S(TK)$ such that $\bar{J}(D')\subset tr(TK)$, where $D'=\bar{J}(ltr(TK))\oplus_{orth}\bar{J}(S(TK^{\perp}))$.

Therefore, by using the equation (25) the general decompositions of the equations (3) and (6) become

$$TK=D\oplus D',\quad T\bar{K}=D\oplus D'\oplus tr(TK),$$

where D is a $2r$-lightlike almost complex distribution on K such that $D = Rad(TK) \oplus_{orth} \bar{J}(Rad(TK)) \oplus_{orth} D_0$.

Consider the projections Q, P_1, and P_2 from TK to D, $\bar{J}ltr(TK)$, and $\bar{J}S(TK^{\perp})$, respectively. Then for $Y \in \Gamma(TK)$ we have

$$Y = QY + P_1Y + P_2Y. \tag{26}$$

Thus by applying $\bar{J}$ to the equation (26), we obtain

$$\bar{J}Y = TY + wP_1Y + wP_2Y, \tag{27}$$

and we can write the equation (27) as

$$\bar{J}Y = TY + wY, \tag{28}$$

where TY and wY denote the tangential and transversal components of $\bar{J}Y$, respectively. Similarly, for $V \in \Gamma(tr(TK))$

$$\bar{J}V = EV, \tag{29}$$

where EV is the section of TK.

Differentiating the equation (27) and using the equations (9), (11), (12) and (29), we derive

$$\begin{aligned}(\nabla_{Y_1}T)Y_2 + (\nabla_{Y_2}T)Y_1 =& A_{wP_1Y_2}Y_1 + A_{wP_1Y_1}Y_2 + A_{wP_2Y_2}Y_1 + A_{wP_2Y_1}Y_2 \\ &+ 2Eh(Y_1, Y_2),\end{aligned} \tag{30}$$

$$\begin{aligned}D^s(Y_1, wP_1Y_2) + D^s(Y_2, wP_1Y_1) =& -\nabla^s_{Y_1}wP_2Y_2 - \nabla^s_{Y_2}wP_2Y_1 + wP_2\nabla_{Y_1}Y_2 \\ &+ wP_2\nabla_{Y_2}Y_1 - h^s(Y_1, TY_2) - h^s(Y_2, TY_1),\end{aligned} \tag{31}$$

and

$$\begin{aligned}D^l(Y_1, wP_2Y_2) + D^l(Y_2, wP_2Y_1) =& -\nabla^l_{Y_1}wP_1Y_2 - \nabla^l_{Y_2}wP_1Y_1 + wP_1\nabla_{Y_1}Y_2 \\ &+ wP_1\nabla_{Y_2}Y_1 - h^l(Y_1, TY_2) - h^l(Y_2, TY_1).\end{aligned} \tag{32}$$

Example 3.2. *Consider a submanifold K of $(R^8_2, \bar{g})$ with signature*

$$(+, +, -, +, +, -, +, +)$$

given by the equations $u_3 = u_8$ and $u_5 = \sqrt{1 - u_6^2}$ with respect to the basis

$$(\partial u_1, \partial u_2, \partial u_3, \partial u_4, \partial u_5, \partial u_6, \partial u_7, \partial u_8).$$

The tangent bundle of K is given by

$$U_1 = \partial u_1, \quad U_2 = \partial u_2, \quad U_3 = \partial u_3 + \partial u_8, \quad U_4 = \partial u_4,$$

$$U_5 = -u_6\partial u_5 + u_5\partial u_6, \quad U_6 = \partial u_7.$$

It is easy to see that K is a 1-lightlike submanifold with $Rad(TK) = Span\{U_3\}$ and $\bar{J}U_3 = U_4 - U_6 \in \Gamma(S(TK))$. Moreover, $\bar{J}U_1 = U_2$ and $\bar{J}U_2 = -U_1$ and therefore $D_0 = Span\{U_1, U_2\}$. By direct calculations, we get $S(TK^{\perp}) = Span\{V = x_5\partial x_5 + x_6\partial x_6\}$. Thus, $\bar{J}V = U_5$ and thus $\bar{J}S(TK^{\perp}) \subset S(TK)$. On the other hand, $ltr(TK)$ is spanned by $N = \frac{1}{2}(-\partial x_3 + \partial x_8)$. Then $\bar{J}N =$

$-\frac{1}{2}(\partial x_4 + \partial x_7) = -\frac{1}{2}(U_4 + U_6)$ *and* $D' = \{\bar{J}N, \bar{J}V\}$. *Thus,* K *is a proper 6-dimensional generic lightlike submanifold of* $(R^8_2, \bar{g})$.

Note: In the forthcoming part of the paper, we shall write **gc.l.s.** for a generic lightlike submanifold, $\bar{K}$ for an indefinite nearly Kaehler manifold and $\bar{K}(c, \alpha)$ for an indefinite generalized complex space form, unless otherwise indicated.

Theorem 3.3. *A lightlike submanifold* K *of* $\bar{K}(c, \alpha)$ *(provided,* $c = -3\alpha$ *and* $\alpha \neq 0$*) is* **gc.l.s.** *with* $D_0 \neq 0$ *if and only if*

(a) *The maximal complex subspace of* $T_pK, p \in K$ *defines a distribution*

$$D = Rad(TK) \perp \bar{J}(Rad(TK)) \perp D_0,$$

where D_0 *is a non-degenerate complex distribution.*

(b) *There exists a lightlike transversal vector bundle* $ltr(TK)$ *such that*

$$\bar{g}(\bar{R}(\xi, N)\xi, N) = 0,$$

for $\xi \in \Gamma(Rad(TK))$ *and* $N \in \Gamma(ltr(TK))$.

(c) *There exists a non-degenerate vector bundle* $S(TK^{\perp})$ *of* K *such that*

$$\bar{g}(\bar{R}(V, V')V, V') = 0,$$

for $V, V' \in \Gamma(S(TK^{\perp}))$.

Proof. Assume that K is a **gc.l.s.** of $\bar{K}(c, \alpha)$ provided, $c = -3\alpha$ and $\alpha \neq 0$. Then, in view of Definition 3.1, $D = Rad(TK) \perp \bar{J}Rad(TK) \perp D_0$ is a maximal subspace, which proves (a). Now for $\xi \in \Gamma(Rad(TK))$ and $N \in \Gamma(ltr(TK))$, from the equation (24), we obtain

$$\bar{g}(\bar{R}(\xi, N)\xi, N) = 0, \tag{33}$$

which satisfies (b). Similarly, using the equation (24), for $V, V' \in \Gamma(S(TK^{\perp}))$, one has

$$\bar{g}(\bar{R}(V, V')V, V') = 0. \tag{34}$$

Hence, (c) follows.

Conversely, consider (a), (b) and (c) hold. Then, from (a), it follows that $Rad(TK)$ in K satisfies $\bar{J}Rad(TK) \cap Rad(TK) = \{0\}$ and we obtain a non-degenerate distribution D_0 on $S(TK)$. Further, as the distribution $ltr(TK)$ is orthogonal to $S(TK)$, therefore, for $\xi \in \Gamma(Rad(TK))$, we obtain $\bar{g}(\xi, \bar{J}N) = -\bar{g}(\bar{J}\xi, N) = 0$. This yields that $\bar{J}ltr(TK)$ defines a distribution on M. Moreover, employing (c), it is clear that there exists a distribution $\bar{J}(S(TK^{\perp}))$, which is orthogonal to $D \oplus \bar{J}ltr(TK)$, which ensures definition 3.1, thereby we achieve the result. □

Lemma 3.4. *([21]) For a nearly Kaehler manifold* $\bar{K}$*, one has*

$$(\bar{\nabla}_{Y_1}\bar{J})Y_2 + (\bar{\nabla}_{\bar{J}Y_1}\bar{J})\bar{J}Y_2 = 0, \; N(Y_1, Y_2) = -4\bar{J}((\bar{\nabla}_{Y_1}\bar{J})Y_2), \tag{35}$$

for $Y_1, Y_2 \in \Gamma(T\bar{K})$, where Nijenhuis tensor $N(Y_1, Y_2)$ is given by

$$N(Y_1, Y_2) = [\bar{J}Y_1, \bar{J}Y_2] - \bar{J}[Y_1, \bar{J}Y_2] - \bar{J}[\bar{J}Y_1, Y_2] - [Y_1, Y_2]. \tag{36}$$

Theorem 3.5. *Consider a* **gc.l.s.** *K of $\bar{K}$. If D is an integrable distribution on K, then $h(Y_1, \bar{J}Y_2) = h(\bar{J}Y_1, Y_2)$ for $Y_1, Y_2 \in \Gamma(D)$.*

Proof. For $Y_1, Y_2 \in \Gamma(D)$, by employing the equations (9) and (36) we obtain

$$\begin{aligned} \bar{J}N(Y_1, Y_2) &= 2(\nabla_{Y_1}\bar{J}Y_2 - \nabla_{Y_2}\bar{J}Y_1) + 2(h(Y_1, \bar{J}Y_2) \\ &\quad - h(\bar{J}Y_1, Y_2)) - 2\bar{J}[Y_1, Y_2]. \end{aligned} \tag{37}$$

As D is an integrable distribution, therefore $\bar{J}N(Y_1, Y_2) \in \Gamma(TK)$ and

$$\bar{J}[Y_1, Y_2] \in \Gamma(TK).$$

Further, by equating the transversal components we obtain

$$h(Y_1, \bar{J}Y_2) = h(\bar{J}Y_1, Y_2),$$

which proves the assertion. □

Theorem 3.6. *Let K be a* **gc.l.s.** *of $\bar{K}$. If D defines a totally geodesic foliation in K, then K is D-geodesic.*

Proof. Assume that D defines a totally geodesic foliation in $\bar{K}$. Then for $Y_1, Y_2 \in \Gamma(D)$, $\bar{\nabla}_{Y_1}Y_2 \in \Gamma(D)$. Further, using the equation (9), we obtain

$$\bar{g}(\bar{\nabla}_{Y_1}Y_2, \xi) = \bar{g}(h^l(Y_1, Y_2), \xi) = 0 \tag{38}$$

for $\xi \in \Gamma(Rad(TK))$ and

$$\bar{g}(\bar{\nabla}_{Y_1}Y_2, V) = \bar{g}(h^s(Y_1, Y_2), V) = 0 \tag{39}$$

for $V \in \Gamma(S(TK^\perp))$. □

Theorem 3.7. *There does not exist any proper mixed geodesic* **gc.l.s.** *of $\bar{K}(c, \alpha)$ with D_0 to be a totally geodesic foliation such that $c \neq \alpha$.*

Proof. For $Y \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}S(TK^\perp)) \subset \Gamma(D')$, by employing the equation (24) we attain

$$\bar{g}(\bar{R}(Y, \bar{J}Y)Z, \bar{J}Z) = -\frac{c-\alpha}{2}||Y||^2||Z||^2. \tag{40}$$

Then, taking into account, hypothesis with the equation (20), we derive

$$\bar{g}(\bar{R}(Y, \bar{J}Y)Z, \bar{J}Z) = \bar{g}((\nabla_Y h^s)(\bar{J}Y, Z) - (\nabla_{\bar{J}Y} h^s)(Y, Z), \bar{J}Z), \tag{41}$$

for $Y \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}S(TK^\perp))$. Next, from employing the equation (22), we obtain

$$(\nabla_Y h^s)(\bar{J}Y, Z) = -h^s(\nabla_Y \bar{J}Y, Z) - h^s(\bar{J}Y, \nabla_Y Z) \tag{42}$$

and

$$(\nabla_{\bar{J}Y} h^s)(Y, Z) = -h^s(\nabla_{\bar{J}Y} Y, Z) - h^s(Y, \nabla_{\bar{J}Y} Z). \tag{43}$$

From the equations (42) and (43), we have

$$(\nabla_{\bar{J}Y}h^s)(Y,Z)-(\nabla_{\bar{J}Y}h^s)(Y,Z)=h^s([\bar{J}Y,Y],Z)-h^s(\bar{J}Y,\nabla_Y Z)+h^s(Y,\nabla_{\bar{J}Y}Z). \tag{44}$$

As D_0 defines a totally geodesic foliation, thus for $Y\in\Gamma(D_0)$ and $Z\in\Gamma(\bar{J}S(TK^{\perp}))$, we attain

$$g(T\nabla_Y Z,Y)=-\,g(\nabla_Y Z,TY)=-\bar{g}(\bar{\nabla}_Y Z,TY)=\bar{g}(Z,\bar{\nabla}_Y TY)=0. \tag{45}$$

Further, the non-degeneracy of D_0 gives that $\nabla_Y Z\in\Gamma(D')$, hence the equation (44) yields

$$(\nabla_{\bar{J}Y}h^s)(Y,Z)-(\nabla_{\bar{J}Y}h^s)(Y,Z)=0,$$

which further gives $\bar{g}(\bar{R}(Y,\bar{J}Y)Z,\bar{J}Z)=0$. Then, from the equation (40) we have

$$-\frac{c-\alpha}{2}||Y||^2||Z||^2=0.$$

Using the non-degeneracy of D_0 and $\bar{J}S(TK^{\perp})$, we obtain $c=\alpha$. □

4. Totally umbilical generic lightlike submanifolds

Definition 4.1. *([12]). A lightlike submanifold (K,g) of a semi-Riemannian manifold $(\bar{K},\bar{g})$ is said to be totally umbilical in $\bar{K}$ if there exist a smooth transversal vector field $H\in\Gamma(tr(TK))$ on K, called as the transversal curvature vector field of K such that for $Y_1,Y_2\in\Gamma(TK)$*

$$h(Y_1,Y_2)=H\bar{g}(Y_1,Y_2). \tag{46}$$

From the equation (12), K is totally umbilical if and only if, on each coordinate neighborhood u, *there exist smooth vector fields $H^l\in\Gamma(ltr(TK))$ and $H^s\in\Gamma(S(TK^{\perp}))$ satisfying*

$$h^l(Y_1,Y_2)=H^l g(Y_1,Y_2),\quad h^s(Y_1,Y_2)=H^s g(Y_1,Y_2),\quad D^l(Y_1,V)=0, \tag{47}$$

for $Y_1,Y_2\in\Gamma(TK)$ and $V\in\Gamma(S(TK^{\perp}))$.

From the equation (19), it is clear that the induced connection on the submanifold K from the Levi-Civita connection $\bar{\nabla}$ need not be metric connection. Therefore, in the following result we derive the conditions under which the induced connection becomes metric connection.

Theorem 4.2. *For a totally umbilical* **gc.l.s.** *K of $\bar{K}$ such that D_0 defines a totally geodesic foliation, then the induced connection ∇ is always a metric connection.*

Proof. For $Y \in \Gamma(D_0)$, using the equation (32), we have

$$\omega P_1 \nabla_Y Y + \omega P_1 \nabla_Y Y = h^l(Y, \bar{J}Y) + h^l(\bar{J}Y, Y) \tag{48}$$

On taking the inner product of above equation with respect to $\xi \in \Gamma(Rad(TK))$, we derive

$$\bar{g}(h^l(Y,Y),\xi) = -\bar{g}(\bar{J}h^l(Y,\phi Y),\xi) - \bar{g}(\bar{J}D^l(Y,\omega Y),\xi). \tag{49}$$

Then, using the hypothesis of being totally umbilical of K, we obtain

$$\bar{g}(H^l,\xi)g(Y,Y) = -\bar{g}(\bar{J}H^l,\xi)g(Y,\phi Y) = 0. \tag{50}$$

In particular, for $Y \in \Gamma(D_0)$, $\bar{g}(H^l,\xi)g(Y,Y) = 0$. Using the non-degeneracy of the distribution D_0 and Theorem 2.1, we get $H^l = 0$, further, employing the equation (47), we arrive at $h^l = 0$. Hence, from the equation (19), the assertion follows. □

Theorem 4.3. *For a proper totally umbilical* **gc.l.s.** *K of $\bar{K}$, we have $\nabla_Y Y \in \Gamma(D)$ for $Y \in \Gamma(D)$.*

Proof. Since $D' = \bar{J}(ltr(TK)) \perp \bar{J}(S(TK^{\perp}))$, therefore $\nabla_Y Y \in \Gamma(D)$ if and only if

$$g(\nabla_Y Y, \bar{J}\xi) = 0 \quad and \quad \bar{g}(\nabla_Y Y, \bar{J}V) = 0, \tag{51}$$

where $Y \in \Gamma(D)$, $\xi \in \Gamma(Rad(TK))$ and $V \in \Gamma(S(TK^{\perp}))$. Using the hypothesis that K is totally umbilical, we have

$$\begin{aligned} g(\nabla_Y Y, \bar{J}\xi) &= -\,\bar{g}(\bar{\nabla}_Y \bar{J}Y, \xi) = -\bar{g}(h^l(Y,\bar{J}Y),\xi) \\ &= -\,\bar{g}(H^l,\xi)g(Y,\bar{J}Y) \\ &= 0 \end{aligned} \tag{52}$$

and

$$\begin{aligned} \bar{g}(\nabla_Y Y, \bar{J}V) &= -\,\bar{g}(\bar{\nabla}_Y \bar{J}Y, V) = -\bar{g}(h^s(Y,\bar{J}Y),V) \\ &= -\,\bar{g}(H^s,V)g(Y,\bar{J}Y) \\ &= 0. \end{aligned} \tag{53}$$

Hence, the proof follows. □

Theorem 4.4. *For a proper totally umbilical proper* **gc.l.s.** *K of $\bar{K}$, one of the following holds:*

(i) *K is totally geodesic.*
(ii) *$h^s = 0$ or $\dim(\bar{J}S(TK^{\perp})) = 1$ if D_0 does not define a totally geodesic foliation in K.*

Proof. Firstly, assume that D_0 defines a totally geodesic foliation in K. In view of Theorem 4.2, we get $h^l = h^s = 0$, which proves (i). Next, suppose that D_0 does not define a totally geodesic foliation in K, then, employing the equations (9), (11), (12), (27) and (28) and considering the tangential parts, we get

$$-A_{\bar{J}Y_2}Y_1 - A_{\bar{J}Y_1}Y_2 = T\nabla_{Y_1}Y_2 + T\nabla_{Y_2}Y_1 + 2Eh(Y_1, Y_2), \tag{54}$$

for $Y_1, Y_2 \in \Gamma(\bar{J}S(TK^{\perp}))$. Taking the inner product with Y_1 and then, employing the equations (13) and (28), we derive

$$\bar{g}(h^s(Y_1, Y_1), \bar{J}Y_2) = -\bar{g}(h^s(Y_1, Y_2), \bar{J}Y_1), \tag{55}$$

which further yields

$$\bar{g}(H^s, \bar{J}Y_2)g(Y_1, Y_1) = -\bar{g}(H^s, \bar{J}Y_1)g(Y_1, Y_2). \tag{56}$$

Interchanging the roles of Y_1 and Y_2 in the above equation, we derive

$$(H^s, \bar{J}Y_1)g(Y_2, Y_2) = -\bar{g}(H^s, \bar{J}Y_2)g(Y_1, Y_2). \tag{57}$$

From the equations (56) and (57), we obtain

$$\bar{g}(H^s, \bar{J}Y_1) = \frac{g(Y_1, Y_2)^2}{g(Y_1, Y_1)g(Y_2, Y_2)}\bar{g}(H^s, \bar{J}Y_1). \tag{58}$$

In view of the non-degeneracy of $S(TK^{\perp})$ and taking into account non-null vectors Y_1 and Y_2 in the equation (58), we must have $H^s = 0$ or Y_1 and Y_2 are linearly dependent, which proves (ii). □

Theorem 4.5. *There exist no totally umbilical proper* **gc.l.s.** *of $\bar{K}(c, \alpha)$ such that $c \neq \alpha$.*

Proof. Let K be a totally umbilical proper **gc.l.s.** of $\bar{K}(c, \alpha)$. Then, for $Y \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}S(TK^{\perp})) \subset \Gamma(D')$, employing the equation (24), we attain

$$\bar{g}(\bar{R}(Y, \bar{J}Y)Z, \bar{J}Z) = -\frac{c - \alpha}{2}||Y||^2||Z||^2. \tag{59}$$

Further, using the equation (20), we get

$$\bar{g}(\bar{R}(Y, \bar{J}Y)Z, \bar{J}Z) = \bar{g}((\nabla_Y h^s)(\bar{J}Y, Z) - (\nabla_{\bar{J}Y} h^s)(Y, Z), \bar{J}Z), \tag{60}$$

for $Y \in \Gamma(D_0)$ and $Z \in \Gamma(\bar{J}S(TK^{\perp}))$. Further, from the equations (59) and (60), we have

$$-\frac{c - \alpha}{2}||Y||^2||Z||^2 = \bar{g}((\nabla_Y h^s)(\bar{J}Y, Z) - (\nabla_{\bar{J}Y} h^s)(Y, Z), \bar{J}Z). \tag{61}$$

Now, as K is totally umbilical, thus we have

$$\begin{aligned}(\nabla_Y h^s)(\bar{J}Y, Z) =& \nabla_Y h^s(\bar{J}Y, Z) - h^s(\nabla_Y \bar{J}Y, Z) - h^s(\bar{J}Y, \nabla_Y Z)\\ =& -\{\bar{g}(\nabla_Y \bar{J}Y, Z) + \bar{g}(\bar{J}Y, \nabla_Y Z)\}H^s.\end{aligned} \tag{62}$$

Since we have $\bar{g}(\bar{J}Y, Z) = 0$, by differentiating this equation with respect to Y we get $g(\nabla_Y \bar{J}Y, Z) = -g(\bar{J}Y, \nabla_Y Z)$, which further yields

$$(\nabla_Y h^s)(\bar{J}Y, Z) = 0. \tag{63}$$

Similarly, we derive

$$(64) \qquad (\nabla_{\bar{J}Y} h^s)(Y, Z) = 0.$$

Thus, from the equation (61), we obtain $-\frac{c-\alpha}{2}||Y||^2||Z||^2 = 0$ and the non-degeneracy of D_0 and $\bar{J}S(TK^{\perp})$ implies that $c = \alpha$. □

5. Minimal generic lightlike submanifolds

Definition 5.1. [1] *A lightlike submanifold $(K, g, S(TK))$ of semi-Riemannian manifold $(\bar{K}, \bar{g})$ is said to be minimal if*

(i) $h^s(\xi_1, \xi_2) = 0$ *for* $\xi_1, \xi_2 \in \Gamma(Rad(TK))$ *and*
(ii) $\operatorname{trace} h|_{S(TK)} = 0$.

One may note that Definition 5.1 is independent of the choice of $S(TK)$ and $S(TK^{\perp})$ but it depends on $tr(TK)$. The minimal lightlike submanifolds have been dealt in detail by Duggal and Jin in [13] and Kumar in [20].

Example 5.2. *Let $(\bar{K}, \bar{g}) = (R^{10}_2, \bar{g})$ be a semi-Riemannian manifold with signature $(-, -, +, +, +, +, +, +, +, +)$ with respect to the canonical basis*

$$(\partial x_1, \partial x_2, \partial u_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10})$$

and g is the inner product of R^{10}_2. Let K be a submanifold of R^{10}_2 given by

$$x_1 = u_1, \quad x_2 = u_2, \quad x_3 = u_1, \quad x_4 = u_3, \quad x_5 = \cos u_4 \cosh u_5,$$
$$x_6 = \sin u_4 \sinh u_5, \quad x_7 = \cos u_6 \cosh u_7, \quad x_8 = \cos u_6 \sinh u_7,$$
$$x_9 = \sin u_6 \cosh u_7, \quad x_{10} = \sin u_6 \sinh u_7,$$

where $u_4, u_6 \in R - \{\frac{n\pi}{2}, n \in Z\}$. Then TK is spanned by

$$U_1 = \partial x_1 + \partial x_3, \quad U_2 = \partial x_2, \quad U_3 = \partial x_4,$$
$$U_4 = -\sin u_4 \cosh u_5 \partial x_5 + \cos u_4 \sinh u_5 \partial x_6,$$
$$U_5 = \cos u_4 \sinh u_5 \partial x_5 + \sin u_4 \cosh u_5 \partial x_6,$$
$$U_6 = -\sin u_6 \cosh u_7 \partial x_7 - \sin u_6 \sinh u_7 \partial x_8 + \cos u_6 \cosh u_7 \partial x_9 + \cos u_6 \sinh u_7 \partial x_{10},$$
$$U_7 = \cos u_6 \sinh u_7 \partial x_7 + \cos u_6 \cosh u_7 \partial x_8 + \sin u_6 \sinh u_7 \partial x_9 + \sin u_6 \cosh u_7 \partial x_{10}.$$

Clearly, K is a 1-lightlike submanifold with $Rad(TK) = Span\{U_1\}$ and $\bar{J}U_1 = U_2 + U_3 \in \Gamma(S(TK))$. Moreover, $\bar{J}U_4 = U_5$ therefore $D_0 = \{U_4, U_5\}$. Next we see that $\bar{J}U_6$ and $\bar{J}U_7$ are orthogonal to TK and therefore, we have $S(TK^{\perp}) = \{\bar{J}U_6, \bar{J}U_7\}$. Thus we conclude that K a proper **gc.l.s.** *of R^{10}_2. The lightlike transversal bundle $ltr(TK)$ is spanned by*

$$N_1 = \frac{1}{2}\{-\partial x_1 + \partial x_3\}$$

Now $\bar{J}N_1 = -\frac{1}{2}Z_2 - \frac{1}{2}Z_3$. *Hence,* $ltr(TK) = \{N_1\}$. *Now, by direct calculations, using the Gauss and Weingartan formulae, we obtain*

$$h^s(Y, U_1) = h^s(Y, U_2), \quad h^s(Y, U_3) = 0,$$
$$h^s(Y, U_4) = 0, \quad h^s(Y, U_5) = 0, \quad \forall\, Y \in \Gamma(TK),$$
$$h^s(U_6, U_6) = \left(\frac{1}{1 + 2\sinh^2 u_7}\right) \bar{J}U_7,$$
$$h^s(U_7, U_7) = -\left(\frac{1}{1 + 2\sinh^2 u_7}\right) \bar{J}U_7.$$

Thus, the induced connection is a metric connection and K *is not totally geodesic, but it is a proper minimal* **gc.l.s.** *of* R^{10}_2.

Theorem 5.3. *Consider a totally umbilical* **gc.l.s.** K *of* $\bar{K}$. *Then* K *is minimal if and only if*

$$\operatorname{trace} A_{V_p} = 0 \quad \text{and} \quad \operatorname{trace} A^*_{\xi_k} = 0 \quad \text{on} \quad D_0 \perp \bar{J}S(TK^{\perp}),$$

for $V_p \in \Gamma(S(TK^{\perp}))$, *where* $k \in \{1, 2, \ldots, r\}$ *and* $p \in \{1, 2, \ldots, n-r\}$.

Proof. Taking into account the hypothesis and the equation (46), we get $h^s(X, Y) = 0$ for $X, Y \in \Gamma(Rad(TK))$. Now from the definition of **gc.l.s.**, we have

$$\operatorname{trace} h|_{S(TK)} = \sum_{i=1}^{2p} h(Y_i, Y_i) + \sum_{j=1}^{r} h(\bar{J}\xi_j, \bar{J}\xi_j) + \sum_{j=1}^{r} h(\bar{J}N_j, N_j) + \sum_{l=1}^{n-r} h(\bar{J}V_l, \bar{J}V_l),$$

where $2p = \dim(D_0)$, $r = \dim(Rad(TK))$ and $n - r = \dim(S(TK^{\perp}))$. Further, employing the equation (46), we derive $h(\bar{J}\xi_j, \bar{J}\xi_j) = h(\bar{J}N_j, N_j) = 0$. Thus, the above equation becomes

$$\begin{aligned} \operatorname{trace} h|_{S(TK)} &= \sum_{i=1}^{2p} h(Y_i, Y_i) + \sum_{l=1}^{n-r} h(\bar{J}V_l, \bar{J}V_l) \\ &= \sum_{i=1}^{2p} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(h^l(Y_i, Y_i), \xi_k) N_k \\ &\quad + \sum_{i=1}^{2p} \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(h^s(Y_i, Y_i), V_p) V_p \\ &\quad + \sum_{l=1}^{n-r} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(h^l(\bar{J}V_l, \bar{J}V_l), \xi_k) N_k \\ &\quad + \sum_{l=1}^{n-r} \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(h^s(\bar{J}V_l, \bar{J}V_l), V_p) V_p \end{aligned} \tag{65}$$

where $\{V_1, V_2, \ldots, V_{n-r}\}$ is an orthonormal basis of $S(TK^\perp)$. Then employing the equations (13) and (17) in the equation (65), we attain

$$\begin{aligned}\operatorname{trace} h|_{S(TK)} =& \sum_{i=1}^{2p} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(A^*_{\xi_k} Y_i, Y_i) N_k + \sum_{i=1}^{2p} \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(A_{V_p} Y_i, Y_i) V_p \\ &+ \sum_{l=1}^{n-r} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(A^*_{\xi_k} \bar{J} V_l, \bar{J} V_l) N_k \\ &+ \sum_{l=1}^{n-r} \frac{1}{n-r} \sum_{p=1}^{n-r} \bar{g}(A_{V_p} \bar{J} V_l, \bar{J} V_l) V_p \qquad (66)\end{aligned}$$

Thus, $\operatorname{trace} h|_{S(TK)} = 0$ if and only if $\operatorname{trace} A_{V_p} = 0$ and $\operatorname{trace} A_{\xi^*_k} = 0$ on $D_0 \perp \bar{J} S(TK^\perp)$, which proves the theorem. □

Definition 5.4. [11] *A lightlike submanifold K of a semi-Riemannian manifold $\bar{K}$ is said to be irrotational if and only if $\bar{\nabla}_Y \xi \in \Gamma(TK)$ for $Y \in \Gamma(TK)$ and $\xi \in \Gamma(Rad(TK))$.*

Theorem 5.5. *Let K be an irrotational* **gc.l.s.** *of $\bar{K}$. If D is integrable distribution, then K is minimal if and only if*

$$\operatorname{trace} A^*_\xi|_{\bar{J}Rad(TK) \oplus D'} = 0 \quad \text{and} \quad \operatorname{trace} A_V|_{\bar{J}Rad(TK) \oplus D'} = 0.$$

Proof. As K is irrotational, thus we have $h^s(Y, \xi) = 0$, for $Y \in \Gamma(TK)$ and $\xi \in \Gamma(Rad(TK))$ and hence $h^s = 0$ on $Rad(TK)$. Then, the integrability of D gives that $h(Y_1, \bar{J}Y_2) = h(\bar{J}Y_1, Y_2)$ for $Y_1, Y_2 \in \Gamma(D)$, which further yields $h(\bar{J}Y_1, \bar{J}Y_2) = -h(Y_1, Y_2)$. Next, choose an orthonormal basis $\{e_1, e_2, ..., e_p, \bar{J}e_1, \bar{J}e_2, \ldots, \bar{J}e_p\}$ of D_0, thus we have

$$\operatorname{trace} h|_{D_0} = \sum_{i=1}^{2p} \epsilon_i h(e_i, e_i) = \sum_{i=1}^{2p} \epsilon_i (h(e_i, e_i) + h(\bar{J}e_i, \bar{J}e_i)) = 0.$$

Hence K is minimal if and only if

$$\sum_{j=1}^{r} h(\bar{J}\xi_j, \bar{J}\xi_j) = \sum_{j=1}^{r} h(\bar{J}N_j, N_j) = \sum_{l=1}^{n-r} h(\bar{J}V_l, \bar{J}V_l) = 0, \qquad (67)$$

where $r = \dim(Rad(TK))$ and $n - r = \dim(S(TK^\perp))$. Then, employing the equations (13) and (17) in the equation (67), the result follows. □

Theorem 5.6. *Assume that K is a* **gc.l.s.** *of $\bar{K}$. Then the distribution D_0 is minimal if and only if*

$$A_N \bar{J} Y + \bar{J} A_N Y \text{ and } A_{N'} Y - \bar{J} A_{N'} \bar{J} Y \text{ have no components in } D_0$$

for $Y \in \Gamma(D_0)$ and $N, N' \in \Gamma(ltr(TK))$.

Proof. For $Y \in \Gamma(D_0)$ and $\xi \in \Gamma(Rad(TK))$, using the equations (9), (16), and (23), we obtain

$$g(\nabla_Y Y + \nabla_{\bar{J}Y} \bar{J}Y, \bar{J}\xi) = -g(\bar{J}Y, A^*_\xi Y) + g(A^*_\xi \bar{J}Y, Y). \tag{68}$$

As the shape operator of $S(TK)$ is self-adjoint, thus from the equation (68), we derive

$$g(\nabla_Y Y + \nabla_{\bar{J}Y} \bar{J}Y, \bar{J}\xi) = 0, \quad for\ Y \in \Gamma(D_0)\ and\ \xi \in \Gamma(Rad(TK)). \tag{69}$$

Similarly, employing the equations (10) and (23), we have

$$\bar{g}(\nabla_Y Y + \nabla_{\bar{J}Y} \bar{J}Y, \bar{J}V) = 0, \quad for\ Y \in \Gamma(D_0)\ and\ V \in \Gamma(S(TK^\perp). \tag{70}$$

Further, from the equations (9), (12) and (23), we derive

$$g(\nabla_Y Y + \nabla_{\bar{J}Y} \bar{J}Y, \bar{J}N) = \bar{g}(Y, A_N \bar{J}Y + \bar{J}A_N Y) \tag{71}$$

for $Y \in \Gamma(D_0)$ and $N \in \Gamma(ltr(TK))$. Similarly, we have

$$g(\nabla_Y Y + \nabla_{\bar{J}Y} \bar{J}Y, N') = \bar{g}(Y, A_{N'} Y - \bar{J}A_{N'} \bar{J}Y) \tag{72}$$

for $Y \in \Gamma(D_0)$ and $N' \in \Gamma(ltr(TK))$. Hence, the proof follows from the equations (69)–(72). □

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