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BL-ALGEBRAS DEFINED BY AN OPERATOR

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Abstract. In this paper, Sheffer stroke BL-algebra and its properties are investigated. It is shown that a Cartesian product of two Sheffer stroke BL-algebras is a Sheffer stroke BL-algebra. After describing a filter of Sheffer stroke BL-algebra, a congruence relation on a Sheffer stroke BL-algebra is defined via its filter, and quotient of a Sheffer stroke BL-algebra is constructed via a congruence relation. Also, it is defined a homomorphism between Sheffer stroke BL-algebras and is presented its properties. Thus, it is stated that the class of Sheffer stroke BL-algebras forms a variety.

1. Introduction

The concept of BL-algebra, where the letters BL is an abridgment of Basic Logic, was primarily introduced by Petr Hájek as an algebraic structure of his Basic logic. He presented filters and prime filters on this algebraic structure and gave the completeness proof of Basic Logic by using these prime filters [9]. The interval $[0, 1]$ given with the structure induced by a continuous t-norm is the well-known example of BL-algebra. Cignoli et al. proved that Basic logic is the logic of continuous t-norms as postulated by Hájek [9]. Also, the most fundamental examples of BL-algebras are MV-algebras, product algebras and Gödel algebras [11]. Hence, this algebraic structure have an important position among logical algebras. Particularly, various filters on a BL-algebra are the most popular workspaces. Esko Turunen studied some features of prime filters on BL-algebras ([22], [23]), and he and Sessa analyzed local BL-algebras [25]. Also, Esteva et al. introduced Strict Basic Logic SBL, where the negation definable in Basic Logic BL is strict and searched the extensions of SBL [6]. Georgescu et al. researched right, left and pseudo-BL algebra which is a noncommutative extension of BL-algebra ([5], [7]). In recent years, Liu and Li investigated on fuzzy filters on a BL-algebra ([11], [12]), and Kondo and Dudek analyzed Boolean, positive implicative, maximal, prime, proper filters and implicative deductive systems on BL-algebras [10]. Saeid et al. studied

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on various filters and open problems on fuzzy filters of BL-algebras ([1], [2]). Besides, Meng and Xin studied on fuzzy ideals on BL-algebras [14].

The Sheffer stroke operation, which was first introduced by H. M. Sheffer [21], engages many scientists' attention, because any Boolean function or axiom can be expressed by means of this operation [13]. Since Sheffer stroke operation is a commutative, applying to many logical algebras reveals quite useful results, and it reducts axiom systems of many algebraic structures. Recently, (fuzzy) filters of Sheffer stroke BL-algebras, filters and neutrosophic structures of a strong Sheffer stroke non-associative MV-algebras, Sheffer stroke Hilbert algebras and fuzzy filter, and also Sheffer stroke UP-algebras are studied ([15]-[20]).

It is given basic definitions and notions of Sheffer stroke BL-algebra. By presenting some properties of this algebraic structure, it is proved that a Sheffer stroke BL-algebra is a BL-algebra which $c_1 \odot c_2 := (c_1|c_2)|(c_1|c_2)$ and $c_1 \longrightarrow c_2 := c_1|(c_2|c_2)$, and a Cartesian product of two Sheffer stroke BL-algebras is a Sheffer stroke BL-algebra. By describing a filter on a Sheffer stroke BL-algebra and presenting its features, it is showed that the family of all filters of a Sheffer stroke BL-algebra forms a complete lattice, and that for a subset of a Sheffer stroke BL-algebra there exists the minimal filter containing this subset. Then a congruence relation on a Sheffer stroke BL-algebra determined by its filter and related notions are given. It is showed that a quotient of a Sheffer stroke BL-algebra defined by a congruence relation is a Sheffer stroke BL-algebra. Finally, after defining a homomorphism between Sheffer stroke BL-algebras, it is demonstrated that mentioned notions such as filter and quotient are preserved under this homomorphism.

2. Preliminaries

In this section, we present basic definitions and notions about Sheffer stroke operation and BL-algebra.

Definition 2.1. [3] *Let $\mathcal{C} = \langle C, |\rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:*

$(S1)$ $c_1|c_2 = c_2|c_1$,
$(S2)$ $(c_1|c_1)|(c_1|c_2) = c_1$,
$(S3)$ $c_1|((c_2|c_3)|(c_2|c_3)) = ((c_1|c_2)|(c_1|c_2))|c_3$,
$(S4)$ $(c_1|((c_1|c_1)|(c_2|c_2)))|(c_1|((c_1|c_1)|(c_2|c_2))) = c_1$.

Definition 2.2. [9] *A BL-algebra is an algebra $(C, \vee, \wedge, \odot, \longrightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(C, \vee, \wedge, 0, 1)$ is a bounded lattice, $(C, \odot, 1)$ is a commutative monoid and the following conditions hold for all $c_1, c_2, c_3 \in C$:*

$(A1)$ $c_1 \odot c_2 \leq c_3$ *if and only if* $c_1 \leq c_2 \longrightarrow c_3$,
$(A2)$ $c_1 \wedge c_2 = c_1 \odot (c_1 \longrightarrow c_2)$, *and*
$(A3)$ $(c_1 \longrightarrow c_2) \vee (c_2 \longrightarrow c_1) = 1$.

Proposition 2.3. [11] *In a BL-algebra C, the following properties hold for all $c_1, c_2, c_3 \in C$:*

1. $c_2 \longrightarrow (c_1 \longrightarrow c_3) = c_1 \longrightarrow (c_2 \longrightarrow c_3)$,
2. $1 \longrightarrow c_1 = c_1$,
3. $c_1 \leq c_2$ *if and only if* $c_1 \longrightarrow c_2 = 1$,
4. $c_1 \vee c_2 = ((c_1 \longrightarrow c_2) \longrightarrow c_2) \wedge ((c_2 \longrightarrow c_1) \longrightarrow c_1)$,
5. $c_1 \leq c_2 \Rightarrow c_2 \longrightarrow c_3 \leq c_1 \longrightarrow c_3$,
6. $c_1 \leq c_2 \Rightarrow c_3 \longrightarrow c_1 \leq c_3 \longrightarrow c_2$,
7. $c_1 \longrightarrow c_2 \leq (c_3 \longrightarrow c_1) \longrightarrow (c_3 \longrightarrow c_2)$,
8. $c_1 \longrightarrow c_2 \leq (c_2 \longrightarrow c_3) \longrightarrow (c_1 \longrightarrow c_3)$,
9. $c_1 \leq (c_1 \longrightarrow c_2) \longrightarrow c_2$.

Definition 2.4. [26] *A filter of a BL-algebra C is a nonempty subset P of C such that for all $c_1, c_2 \in C$*

$(F1)$ *if* $c_1, c_2 \in P$, *then* $c_1 \odot c_2 \in P$,
$(F2)$ *if* $c_1 \in P$ *and* $c_1 \leq c_2$, *then* $c_2 \in P$.

Proposition 2.5. [26] *Let P be a nonempty subset of a BL-algebra C. Then P is a filter of C if and only if the following conditions hold*

1. $1 \in P$, *and*
2. $c_1, c_1 \longrightarrow c_2 \in P$ *implies* $c_2 \in P$.

3. Sheffer stroke on a bounded lattice

In this section, we give Sheffer stroke BL-algebra and some properties about it.

Definition 3.1. [15] *A Sheffer stroke BL-algebra is an algebra $(C, \vee, \wedge, |, 0, 1)$ of type $(2, 2, 2, 0, 0)$ satisfying the following conditions*

$(sBL-1)$ $(C, \vee, \wedge, 0, 1)$ *is a bounded lattice,*
$(sBL-2)$ $(C, |)$ *is a groupoid with the Sheffer stroke operation,*
$(sBL-3)$ $c_1 \wedge c_2 = (c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2)))$, *for all* $c_1, c_2 \in C$ *and*
$(sBL-4)$ $(c_1|(c_2|c_2)) \vee (c_2|(c_1|c_1)) = 1$, *for all* $c_1, c_2 \in C$.

$1 = 0|0$ *is the greatest element and* $0 = 1|1$ *is the least element of* C

Example 3.2. [15] *Consider a Sheffer stroke BL-algebra $(C, \vee, \wedge, |, 0, 1)$ with the Hasse diagram in Figure 1, where the set $C = \{0, a, b, c, d, e, f, 1\}$ and the binary operations $|$, $\vee$ and $\wedge$ on C have Cayley tables in Table 1, respectively:*

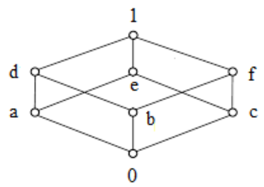

FIGURE 1. Hasse diagram of a Sheffer stroke BL-algebra $(C, \vee, \wedge, |, 0, 1)$

TABLE 1. Cayley tables of Sheffer stroke $|$, $\vee$ and $\wedge$ on C

$\mid$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	0

$\vee$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	d	e	d	e	1	1
b	b	d	b	f	d	1	f	1
c	c	e	f	c	1	e	f	1
d	d	d	d	1	d	1	1	1
e	e	e	1	e	1	e	1	1
f	f	1	f	f	1	1	f	1
1	1	1	1	1	1	1	1	1

$\wedge$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	a	0	a
b	0	0	b	0	b	0	b	b
c	0	0	0	c	0	c	c	c
d	0	a	b	0	d	a	b	d
e	0	a	0	c	a	e	c	e
f	0	0	b	c	b	c	f	f
1	0	a	b	c	d	e	f	1

Example 3.3. *Consider the interval* $[0, 1]$. *Define* $|$ *on* $[0, 1]$ *as below:*

if $c_1 \leq c_2$, $c_1|(c_2|c_2) = 1$ *otherwise* $c_1|(c_2|c_2) = c_2$.

Then $([0,1], \vee, \wedge, |, 0, 1)$ *is a Sheffer stroke BL-algebra.*

Proposition 3.4. [15] *In any Sheffer stroke BL-algebra* C*, the following features hold, for all* $c_1, c_2, c_3 \in C$*:*

1. $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = c_2|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$,
2. $c_1|(c_1|c_1) = 1$,
3. $1|(c_1|c_1) = c_1$,
4. $c_1|(1|1) = 1$,
5. $(c_1|1)|(c_1|1) = c_1$,
6. $(c_1|c_2)|(c_1|c_2) \leq c_3 \Leftrightarrow c_1 \leq c_2|(c_3|c_3)$
7. $c_1 \leq c_2$ *iff* $c_1|(c_2|c_2) = 1$,
8. $c_1 \leq c_2|(c_1|c_1)$,
9. $c_1 \leq (c_1|c_2)|c_2$,
10. (a) $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_1$,
    (b) $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_2$,
11. *If* $c_1 \leq c_2$*, then*
    (i) $c_3|(c_1|c_1) \leq c_3|(c_2|c_2)$,
    (ii) $(c_1|c_3)|(c_1|c_3) \leq (c_2|c_3)|(c_2|c_3)$,
    (iii) $c_2|(c_3|c_3) \leq c_1|(c_3|c_3)$,
12. $c_1|(c_2|c_2) \leq (c_3|(c_1|c_1))|((c_3|(c_2|c_2))|(c_3|(c_2|c_2)))$,
13. $c_1|(c_2|c_2) \leq (c_2|(c_3|c_3))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$.

Theorem 3.5. *Let* $(C, \vee, \wedge, |, 0, 1)$ *be a Sheffer stroke BL-algebra. If we define*

$$c_1 \odot c_2 := (c_1|c_2)|(c_1|c_2) \text{ and } c_1 \longrightarrow c_2 := c_1|(c_2|c_2),$$

then $(C, \vee, \wedge, \odot, \longrightarrow, 0, 1)$ *is a BL-algebra.*

Proof. It is known that $(C, \vee, \wedge, 0, 1)$ is a bounded lattice. Since we have from $(S1)$, $(S3)$ and Proposition 3.4 (5) that

$$\begin{aligned}(c_1 \odot c_2) \odot c_3 &= (((c_1|c_2)|(c_1|c_2))|c_3)|(((c_1|c_2)|(c_1|c_2))|c_3) \\ &= (c_1|((c_2|c_3)|(c_2|c_3)))|(c_1|((c_2|c_3)|(c_2|c_3))) \\ &= c_1 \odot (c_2 \odot c_3),\end{aligned}$$

$$c_1 \odot c_2 = (c_1|c_2)|(c_1|c_2) = (c_2|c_1)|(c_2|c_1) = c_2 \odot c_1$$

and

$$c_1 \odot 1 = (c_1|1)|(c_1|1) = c_1 = (1|c_1)|(1|c_1) = 1 \odot c_1,$$

for all $c_1, c_2, c_3 \in C$, $(C, \odot, 1)$ is a commutative monoid.

$(A1)$: Let c_1, c_2 and c_3 be arbitrary elements in C such that $c_1 \odot c_2 \leq c_3$, i. e., $(c_1|c_2)|(c_1|c_2) \leq c_3$. Then we have from Proposition 3.4 (6) that $c_1 \leq c_2|(c_3|c_3)$, i. e., $c_1 \leq c_2 \longrightarrow c_3$.

Conversely, let c_1, c_2 and c_3 be any elements in C such that $c_1 \leq c_2 \longrightarrow c_3$, i. e., $c_1 \leq c_2|(c_3|c_3)$. Thus, it is obtained from Proposition 3.4 (6) that $(c_1|c_2)|(c_1|c_2) \leq c_3$, i. e., $c_1 \odot c_2 \leq c_3$.

$(A2)$: It follows $c_1 \wedge c_2 = (c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) = c_1 \odot (c_1 \longrightarrow c_2)$, for all $c_1, c_2 \in C$.

$(A3)$: We have $1 = (c_1|(c_2|c_2)) \vee (c_2|(c_1|c_1)) = (c_1 \longrightarrow c_2) \vee (c_2 \longrightarrow c_1)$, for all $c_1, c_2 \in A$. □

Definition 3.6. *Let $(C, \vee_C, \wedge_C, |_C, 0_C, 1_C)$ and $(D, \vee_D, \wedge_D, |_D, 0_D, 1_D)$ be Sheffer stroke BL-algebras. Then the set $C \times D$ is the Cartesian product of C and D, the operations $|_{C\times D}$, $\vee_{C\times D}$ and $\wedge_{C\times D}$ on $C \times D$ are defined by $(c_1, d_1)|_{C\times D}(c_2, d_2) = (c_1|_C c_2, d_1|_D d_2)$, $(c_1, d_1) \vee_{C\times D} (c_2, d_2) = (c_1 \vee_C c_2, d_1 \vee_D d_2)$, $(c_1, d_1) \wedge_{C\times D} (c_2, d_2) = (c_1 \wedge_C c_2, d_1 \wedge_D d_2)$, respectively. Also, $0_{C\times D} = (0_C, 0_D)$ and $1_{C\times D} = (1_C, 1_D)$.*

Theorem 3.7. *Let $(C, \vee_C, \wedge_C, |_C, 0_C, 1_C)$ and $(D, \vee_D, \wedge_D, |_D, 0_D, 1_D)$ be Sheffer stroke BL-algebras. Then $(C \times D, \vee_{C\times D}, \wedge_{C\times D}, |_{C\times D}, 0_{C\times D}, 1_{C\times D})$ is a Sheffer stroke BL-algebra.*

Proof. It is obvious from Definition 3.6. □

4. On Filters of Sheffer stroke BL-algebras

We introduce the notion of filter of a Sheffer stroke BL-algebra in this section. Let $(C, \vee, \wedge, |, 0, 1)$ be a Sheffer stroke BL-algebra, unless otherwise is indicated.

Definition 4.1. [15] *A filter of a Sheffer stroke BL-algebra C is a nonempty subset $P \subseteq C$ satisfying*
$(SF-1)$ if $c_1, c_2 \in P$, then $(c_1|c_2)|(c_1|c_2) \in P$,
$(SF-2)$ if $c_1 \in P$ and $c_1 \leq c_2$, then $c_2 \in P$.

Proposition 4.2. [15] *Let P be a nonempty subset of C. Then P is a filter of C if and only if the following hold*
$(SF-3)$ $1 \in P$
$(SF-4)$ $c_1 \in P$ and $c_1|(c_2|c_2) \in P$ imply $c_2 \in P$.

Theorem 4.3. *The family K_C of all filters of C forms a complete lattice.*

Proof. Let $\{P_i\}_{i\in I}$ be a family of filters of C. Because we know from $(SF-3)$ that $1 \in P_i$ for all $i \in I$, it follows that $1 \in \bigcup_{i\in I} P_i$ and $1 \in \bigcap_{i\in I} P_i$.

- (i) Suppose that $c_1 \in \bigcap_{i\in I} P_i$ and $c_1|(c_2|c_2) \in \bigcap_{i\in I} P_i$ hold for any $c_1, c_2 \in C$, i. e., $c_1 \in P_i$ and $c_1|(c_2|c_2) \in P_i$ hold for all $i \in I$. Then it is obtained from $(SF-4)$ that $c_2 \in P_i$ for all $i \in I$. Thus, we have $c_2 \in \bigcap_{i\in I} P_i$.
- (ii) Let η be the family of all filters of C containing the union $\bigcup_{i\in I} P_i$. Then $\bigcap \eta$ is a filter of C from (i). If $\bigwedge_{i\in I} P_i = \bigcap_{i\in I} P_i$ and $\bigvee_{i\in I} P_i = \bigcap \eta$, then $(K_C, \bigwedge, \bigvee)$ is a complete lattice.

□

Corollary 4.4. *Let D be a subset of a Sheffer stroke BL-algebra C. Then there is the minimal filter $\langle D\rangle$ containing the subset D.*

Proof. Let $\varepsilon = \{P : P$ is a filter of C containing $D \subseteq C\}$. Then $\bigcap \varepsilon$ is the minimal filter of C containing $D \subseteq C$. □

Definition 4.5. *Let P be a filter of C. Define the binary relation β_P on C as follows:*

$$c_1 \beta_P c_2 :\Leftrightarrow c_1|(c_2|c_2) \in P \text{ and } c_2|(c_1|c_1) \in P, \tag{1}$$

for all $c_1, c_2 \in C$.

Example 4.6. *Given the Sheffer stroke BL-algebra C in Example 3.2. For the filter $P = \{a, d, e, 1\}$ of C, $\beta_P = \{(0,0),(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(1,1),(a,d),(d,a,),(a,e),(e,a),(a,1),(1,a),(d,e),(e,d),(d,1),(1,d),(e,1),(1,e),(b,0),(0,b),(b,c),(c,b),(b,f),(f,b),(c,0),(0,c),(c,f),(f,c),(f,0),(0,f)\}$ is a binary relation on C. It can be shown easily that β_P is an equivalence relation on C.*

Definition 4.7. *If $c_1 \xi c_2$ implies $c_1|c_3 \xi c_2|c_3$ and for all $c_1, c_2, c_3 \in C$, then the equivalence relation ξ is called a congruence relation on C.*

Example 4.8. *Consider the Sheffer stroke BL-algebra in Example 3.2. Then the equivalence relation $\xi = \{(0,0),(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(1,1),(e,1),(1,e),(0,b),(b,0),(a,d),(d,a),(c,f),(f,c)\}$ is a congruence relation on C.*

Lemma 4.9. *An equivalence relation ξ is a congruence relation if and only if $c_1 \xi c_2$ and $p \xi r$ imply $c_1|p \xi c_2|r$ and $c_1 \wedge p \xi c_2 \wedge r$.*

Proof. ($\Rightarrow$) Let ξ be a congruence relation on C and c_1, c_2, p, r be arbitrary elements in C such that $c_1 \xi c_2$ and $p \xi r$. Then we have $c_1|p \xi c_2|p$ and $c_2|p \xi c_2|r$ from $(S1)$. So, it follows from transitivity of ξ that $c_1|p \xi c_2|r$. Also, it is obtained from $(sBL-3)$ that $c_1 \wedge p \xi c_2 \wedge r$.

($\Leftarrow$) Assume that $c_1 \xi c_2$ and $p \xi r$ imply $c_1|p \xi c_2|r$ and $c_1 \wedge p \xi c_2 \wedge r$ for arbitrary elements $c_1, c_2, p, r \in C$. Let c_1, c_2, c_3 be any elements in C such that $c_1 \xi c_2$. Because $c_3 \xi c_3$, we obtain $c_1|c_3 \xi c_2|c_3$ from the hypothesis. Then ξ is a congruence relation on C. □

Lemma 4.10. *Let P be a filter of C and the binary relation β_P is defined as the statement (1). Then, β_P is a congruence relation on C.*

Proof. We show that the binary relation β_P is an equivalence relation on C.

• Reflexive: It is obtained from Proposition 3.4 (2) and $(SF-3)$ that $c_1|(c_1|c_1) = 1 \in P$, i.e., $c_1 \beta_P c_1$, for all $c_1 \in C$.

• Symmetric: Let c_1, c_2 be any elements in C such that $c_1 \beta_P c_2$, i.e., $c_1|(c_2|c_2) \in P$ and $c_2|(c_1|c_1) \in P$. Then we have $c_2|(c_1|c_1) \in P$ and $c_1|(c_2|c_2) \in P$, i.e., $c_2 \beta_P c_1$.

• Transitive: Let c_1, c_2, c_3 be any elements in C such that $c_1\beta_P c_2$ and $c_2\beta_P c_3$, i.e., $c_1|(c_2|c_2), c_2|(c_1|c_1) \in P$ and $c_2|(c_3|c_3), c_3|(c_2|c_2) \in P$. Since it is known from Proposition 3.4 (13) that $c_1|(c_2|c_2) \leq (c_2|(c_3|c_3))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$, for all $c_1, c_2, c_3 \in C$. Then it is obtained from $(SF-2)$ that $(c_2|(c_3|c_3))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))) \in P$, and so it follows from $(SF-4)$ that $c_1|(c_3|c_3) \in P$. Similarly, we have $(c_2|(c_1|c_1))|((c_3|(c_1|c_1))|(c_3|(c_1|c_1))) \in P$ since $c_3|(c_2|c_2) \leq (c_2|(c_1|c_1))|((c_3|(c_1|c_1))|(c_3|(c_1|c_1)))$, for all $c_1, c_2, c_3 \in C$. Then we get from $(SF-4)$ that $c_3|(c_1|c_1) \in P$. Thus, $c_1\beta_P c_3$.

Now, we show that the equivalence relation β_P is a congruence relation on C. Let c_1, c_2, p, r be any elements in C such that $c_1\beta_P p$ and $c_2\beta_P r$, i.e., $c_1|(p|p), p|(c_1|c_1) \in P$ and $c_2|(r|r), r|(c_2|c_2) \in P$.

(1) It follows from $(S1)-(S2)$ that $(c_1|c_1)|((p|p)|(p|p)), (p|p)|((c_1|c_1)|(c_1|c_1)) \in P$ and $(c_2|c_2)|((r|r)|(r|r)), (r|r)|((c_2|c_2)|(c_2|c_2)) \in P$. Then it follows from Proposition 3.4 (12) and $(S2)$ that $(p|p)|((c_1|c_1)|(c_1|c_1)) \leq (c_2|((p|p)|(p|p)))|((c_2|((c_1|c_1)|(c_1|c_1)))|(c_2|((c_1|c_1)|(c_1|c_1))) = (c_2|p)|((c_2|c_1)|(c_2|c_1))$, and so, $(c_2|p)|((c_2|c_1)|(c_2|c_1)) \in P$ from $(SF-2)$. Similarly, it is obtained $(c_2|c_1)|((c_2|p)|(c_2|p)) \in P$ by putting, simultaneously, $[c_1 := p]$ and $[p := c_1]$ in above statement. Thus, $c_2|c_1\beta_P c_2|p$.

(2) In a similar way, it follows $p|c_2\beta_P p|r$ by substituting, simultaneously, $[c_2 := p]$, $[c_1 := c_2]$ and $[p := r]$ in (1).

Therefore, it follows from $(S1)$ and the transitivity of β_P that $c_1|c_2\beta_P p|r$. So, $c_1 \wedge c_2 \xi p \wedge r$ by $(sBL-3)$. □

Theorem 4.11. *Let P be a filter of C and β be the congruence relation on C described by P. Then $C/P \equiv C/\beta = \{[c]_\beta : c \in C\}$ is also Sheffer stroke BL-algebra with the Sheffer stroke operation $|_\beta$ determined by $[c_1]_\beta|_\beta[c_2]_\beta = [c_1|c_2]_\beta$, the join operation $\vee_\beta$ determined by $[c_1]_\beta \vee_\beta [c_2]_\beta = [c_1 \vee c_2]_\beta$, the meet operation $\wedge_\beta$ determined by $[c_1]_\beta \wedge_\beta [c_2]_\beta = [c_1 \wedge c_2]_\beta$ for all $c_1, c_2 \in C$ and $P = [1]_\beta$.*

Proof. Suppose that P is a filter of C and β is the congruence relation on C defined by P. Let $C/P \equiv C/\beta = \{[c]_\beta : c \in C\}$ be a structure with the operation $|_\beta$ defined by $[c_1]_\beta|_\beta[c_2]_\beta = [c_1|c_2]_\beta$, the operation $\vee_\beta$ defined by $[c_1]_\beta \vee_\beta [c_2]_\beta = [c_1 \vee c_2]_\beta$ and the operation $\wedge_\beta$ defined by $[c_1]_\beta \wedge_\beta [c_2]_\beta = [c_1 \wedge c_2]_\beta$ for all $c_1, c_2 \in C$.

We show $P = [1]_\beta$. For arbitrary $c_1 \in [1]_\beta$, it is obtained $c_1\beta 1$, i.e., $1 = c_1|(1|1) \in P$ and $c_1 = 1|(c_1|c_1) \in P$ from Proposition 3.4 (3)-(4). Hence, $[1]_\beta \subseteq P$. Because it is known from Proposition 3.4 (3)-(4) and $(SF-3)$ that $1 = c_1|(1|1) \in P$ and $c_1 = 1|(c_1|c_1) \in P$ for all $c_1 \in P$, we get $c_1\beta 1$, i.e., $c_1 \in [1]_\beta$. Then $P \subseteq [1]_\beta$. Also, $[0]_\beta$ is an element in C/P by the definition of C/P.

Now, we demonstrate that the structure $C/P \equiv C/\beta = \{[c]_\beta : c \in C\}$ is a Sheffer stroke BL-algebra. By the definitions of binary operations $|_\beta$, $\vee_\beta$ and $\wedge_\beta$, and the fact that $(C, \vee, \wedge, |, 0, 1)$ is a Sheffer stroke BL-algebra, we have

$(sBL-1)$: • idempotency: $[c_1]_\beta \wedge_\beta [c_1]_\beta = [c_1 \wedge c_1]_\beta = [c_1]_\beta$ and $[c_1]_\beta \vee_\beta [c_1]_\beta = [c_1 \vee x]_\beta = [c_1]_\beta$,

• commutativity: $[c_1]_\beta \wedge_\beta [c_2]_\beta = [c_1 \wedge c_2]_\beta = [c_2 \wedge c_1]_\beta = [c_2]_\beta \wedge_\beta [c_1]_\beta$ and $[c_1]_\beta \vee_\beta [c_2]_\beta = [c_1 \vee c_2]_\beta = [c_2 \vee c_1]_\beta = [c_2]_\beta \vee_\beta [c_1]_\beta$,

• associativity: $[c_1]_\beta \wedge_\beta ([c_2]_\beta \wedge_\beta [c_3]_\beta) = [c_1 \wedge (c_2 \wedge c_3)]_\beta = [(c_1 \wedge c_2) \wedge c_3]_\beta = ([c_1]_\beta \wedge_\beta [c_2]_\beta) \wedge_\beta [c_3]_\beta$, and similarly, $[c_1]_\beta \vee_\beta ([c_2]_\beta \vee_\beta [c_3]_\beta) = ([c_1]_\beta \vee_\beta [c_2]_\beta) \vee_\beta [c_3]_\beta$,

• absorption laws: $([c_1]_\beta \vee_\beta [c_2]_\beta) \wedge_\beta [c_1]_\beta = [(c_1 \vee c_2) \wedge c_1]_\beta = [c_1]_\beta$, and similarly, $([c_1]_\beta \wedge_\beta [c_2]_\beta) \vee_\beta [c_1]_\beta = [c_1]_\beta$,

• $[c_1]_\beta \wedge_\beta [1]_\beta = [c_1]_\beta$, $[c_1]_\beta \vee_\beta [1]_\beta = [1]_\beta$, $[c_1]_\beta \wedge_\beta [0]_\beta = [0]_\beta$, and $[c_1]_\beta \vee_\beta [0]_\beta = [c_1]_\beta$,

$(sBL-2)$: •$[c_1]_\beta|_\beta[c_2]_\beta = [c_1|c_2]_\beta = [c_2|c_1]_\beta = [c_2]_\beta|_\beta[c_1]_\beta$,

• $([c_1]_\beta|_\beta[c_1]_\beta)|_\beta([c_1]_\beta|_\beta[c_2]_\beta) = [(c_1|c_1)|(c_1|c_2)]_\beta = [c_1]_\beta$,

• $[c_1]_\beta|_\beta(([c_2]_\beta|_\beta[c_3]_\beta)|_\beta([c_2]_\beta|_\beta[c_3]_\beta))$
$= [c_1|((c_2|c_3)|(c_2|c_3))]_\beta$
$= [((c_1|c_2)|(c_1|c_2))|c_3]_\beta$
$= (([c_1]_\beta|_\beta[c_2]_\beta)|_\beta([c_1]_\beta|_\beta[c_2]_\beta))|_\beta[c_3]_\beta$,

• $([c_1]_\beta|_\beta(([c_1]_\beta|_\beta[c_1]_\beta)|_\beta([c_2]_\beta|_\beta[c_2]_\beta)))|_\beta$
$([c_1]_\beta|_\beta(([c_1]_\beta|_\beta[c_1]_\beta)|_\beta([c_2]_\beta|_\beta[c_2]_\beta)))$
$= [(c_1|((c_1|c_1)|(c_2|c_2)))|(c_1|((c_1|c_1)|(c_2|c_2)))]_\beta$
$= [c_1]_\beta$,

$(sBL-3)$: $[c_1]_\beta \wedge_\beta [c_2]_\beta = ([c_1]_\beta|_\beta([c_1]_\beta|_\beta([c_2]_\beta|_\beta[c_2]_\beta)))|_\beta([c_1]_\beta|_\beta([c_1]_\beta|_\beta([c_2]_\beta|_\beta[c_2]_\beta)))$ and

$(sBL-4)$: $([c_1]_\beta|_\beta([c_2]_\beta|_\beta[c_2]_\beta)) \vee_\beta ([c_2]_\beta|_\beta([c_1]_\beta|_\beta[c_1]_\beta)) = [1]_\beta$. □

Example 4.12. *Consider the Sheffer stroke BL-algebra in Example 3.2. For the filter* $P = \{d, 1\}$ *of* C, $\beta_P = \{(0,0),(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(1,1),(0,c),(c,0),(1,d),(d,1),(a,e),(e,a),(b,f),(f,b)\}$ *is a congruence on* C *defined by* P. *Then* $(C/P, \vee_{\beta_P}, \wedge_{\beta_P}, |_{\beta_P}, [0]_{\beta_P}, [1]_{\beta_P})$ *is a Sheffer stroke BL-algebra with the Hasse diagram in Figure 2 in which the quotient set* $C/P \equiv C/\beta = \{[0]_{\beta_P}, [a]_{\beta_P}, [b]_{\beta_P}, [1]_{\beta_P}\}$

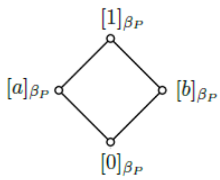

FIGURE 2. Hasse diagram of $(C/P, \vee_{\beta_P}, \wedge_{\beta_P}, |_{\beta_P}, [0]_{\beta_P}, [1]_{\beta_P})$

The binary operations $|_{\beta_P}$, $\vee_{\beta_P}$ *and* $\wedge_{\beta_P}$ *on* C/P *have Cayley tables in Table 2:*

TABLE 2. Cayley tables of the binary operations $|_{\beta_P}$, $\vee_{\beta_P}$ and $\wedge_{\beta_P}$ on C/P

$\vert_{\beta_P}$	$[0]_{\beta_P}$	$[a]_{\beta_P}$	$[b]_{\beta_P}$	$[1]_{\beta_P}$
$[0]_{\beta_P}$	$[1]_{\beta_P}$	$[1]_{\beta_P}$	$[1]_{\beta_P}$	$[1]_{\beta_P}$
$[a]_{\beta_P}$	$[1]_{\beta_P}$	$[b]_{\beta_P}$	$[1]_{\beta_P}$	$[b]_{\beta_P}$
$[b]_{\beta_P}$	$[1]_{\beta_P}$	$[1]_{\beta_P}$	$[a]_{\beta_P}$	$[a]_{\beta_P}$
$[1]_{\beta_P}$	$[1]_{\beta_P}$	$[b]_{\beta_P}$	$[a]_{\beta_P}$	$[0]_{\beta_P}$

$\vee_{\beta_P}$	$[0]_{\beta_P}$	$[a]_{\beta_P}$	$[b]_{\beta_P}$	$[1]_{\beta_P}$
$[0]_{\beta_P}$	$[0]_{\beta_P}$	$[a]_{\beta_P}$	$[b]_{\beta_P}$	$[1]_{\beta_P}$
$[a]_{\beta_P}$	$[a]_{\beta_P}$	$[a]_{\beta_P}$	$[1]_{\beta_P}$	$[1]_{\beta_P}$
$[b]_{\beta_P}$	$[b]_{\beta_P}$	$[1]_{\beta_P}$	$[b]_{\beta_P}$	$[1]_{\beta_P}$
$[1]_{\beta_P}$	$[1]_{\beta_P}$	$[1]_{\beta_P}$	$[1]_{\beta_P}$	$[1]_{\beta_P}$

$\wedge_{\beta_P}$	$[0]_{\beta_P}$	$[a]_{\beta_P}$	$[b]_{\beta_P}$	$[1]_{\beta_P}$
$[0]_{\beta_P}$	$[0]_{\beta_P}$	$[0]_{\beta_P}$	$[0]_{\beta_P}$	$[0]_{\beta_P}$
$[a]_{\beta_P}$	$[0]_{\beta_P}$	$[a]_{\beta_P}$	$[0]_{\beta_P}$	$[a]_{\beta_P}$
$[b]_{\beta_P}$	$[0]_{\beta_P}$	$[0]_{\beta_P}$	$[b]_{\beta_P}$	$[b]_{\beta_P}$
$[1]_{\beta_P}$	$[0]_{\beta_P}$	$[a]_{\beta_P}$	$[b]_{\beta_P}$	$[1]_{\beta_P}$

5. Homomorphisms on Sheffer stroke BL-algebras

In this section, we present some definitions and notions about homomorphism between Sheffer stroke BL-algebras.

Definition 5.1. *Let $(C, \vee_C, \wedge_C, |_C, 0_C, 1_C)$ and $(D, \vee_D, \wedge_D, |_D, 0_D, 1_D)$ be Sheffer stroke BL-algebras. A mapping $f : C \longrightarrow D$ is called homomorphism if*

$$f(c_1|_C c_2) = f(c_1)|_D f(c_2),$$
$$f(c_1 \vee_C c_2) = f(c_1) \vee_D f(c_2)$$

and

$$f(c_1 \wedge_C c_2) = f(c_1) \wedge_D f(c_2),$$

for all $c_1, c_2 \in C$.

Remark 5.2. *The class of Sheffer stroke BL-algebras forms a variety.*

Lemma 5.3. *Let $(C, \vee_C, \wedge_C, |_C, 0_C, 1_C)$ and $(D, \vee_D, \wedge_D, |_D, 0_D, 1_D)$ be Sheffer stroke BL-algebras, and the mapping $f : C \longrightarrow D$ be a homomorphism between these algebras. Then $f(C)$ is a filter of D.*

Proof. By Proposition 3.4 (2) and the fact that f is a a homomorphism, we have $1_D = f(1_C)|_D(f(1_C)|_D f(1_C)) = f(1_C|_C(1_C|_C 1_C)) = f(1_C) \in f(C)$. Let $f(c_1), f(c_1|_C(c_2|_C c_2)) = f(c_1)|_D(f(c_2)|_D f(c_2)) \in f(C)$. Then we get $c_1, c_1|_C(c_2|_C c_2) \in C$. Since C itself is a filter of C, it is obtained $c_2 \in C$, i. e., $f(c_2) \in f(C)$. □

Theorem 5.4. *Let $(C, \vee_C, \wedge_C, |_C, 0_C, 1_C)$ and $(D, \vee_D, \wedge_D, |_D, 0_D, 1_D)$ be Sheffer stroke BL-algebras, and the mapping $f : C \longrightarrow D$ be a homomorphism between these algebras. Then the following features are satisfied:*
(a) If P is a filter of C, then $f(P)$ is a filter of D.
(b) If R is a filter of D, then $f^{-1}(R)$ is a filter of C.

Proof. (a) Suppose that P is a filter of C. Because $1_C \in P$, we get $1_D = f(1_C) \in f(P)$. Let $f(c_1), f(c_1|_C(c_2|_C c_2)) = f(c_1)|_D(f(c_2)|_D f(c_2)) \in f(P)$. Then it is obtained $c_1, c_1|_C(c_2|_C c_2) \in P$. Since P is a filter of C, it follows that $c_2 \in P$, i. e., $f(c_2) \in f(P)$.

(b) Assume that R is a filter of D. Because $f(1_C) = 1_D \in R$, we have $1_C = f^{-1}f(1_C) = f^{-1}(1_D) \in f^{-1}(R)$. Let $c_1 \in f^{-1}(R)$ and $c_1|_C(c_2|_C c_2) \in f^{-1}(R)$, i. e., $f(c_1) \in ff^{-1}(R) \subseteq R$ and $f(c_1)|_D(f(c_2)|_D f(c_2)) = f(c_1|_C(c_2|_C c_2)) \in ff^{-1}(R) \subseteq R$. Since R is a filter of D, we get that $f(c_2) \in R$, i. e., $c_2 \in f^{-1}(R)$.

□

Theorem 5.5. *Let P be a filter of Sheffer stroke BL-algebra $(C, \vee, \wedge, |, 0, 1)$. R is a filter of C such that $P \subseteq R$ if and only if the set $R/P = \{[c]_P \in C/P : c \in R\}$ is a filter of a Sheffer stroke BL-algebra C/P.*

Proof. • Necessary condition: It is known from Theorem 4.11 that $(C/P, \vee_P, \wedge_P, |_P, 0_P, 1_P)$ is a Sheffer stroke BL-algebra. Then we get $P = [1]_P \in R/P$ by the definition of R/P. Let $[c_1]_P, [c_1|(c_2|c_2)]_P = [c_1]_P|_P([c_2]_P|_P[c_2]_P) \in R/P$, i. e., $c_1, c_1|(c_2|c_2) \in R$. Since R is a filter of C, we have $c_2 \in R$, i. e., $[c_2]_P \in R/P$.
• Sufficient condition follows from the definition of R/P. □

Theorem 5.6. *Let P and R be filters of Sheffer stroke BL-algebra $(C, \vee, \wedge, |, 0, 1)$ such that $P \subseteq R$. Then $(C/P)/(R/P) \cong C/R$.*

Proof. It is known from Theorem 5.5 know $R/P = \{[c]_P \in C/P : c \in r\}$ is a filter of Sheffer stroke BL-algebra $C/P = \{[c]_P : c \in C\}$. The factor-set $(C/P)/(R/P) = \{[[c]_P]_{R/P} : [c]_P \in C/P\}$ can be properly described as a Sheffer stroke BL-algebra on the Sheffer stroke BL-algebra C/P by its filter R/P, and also $C/R = \{[c]_R : c \in C\}$ is a Sheffer stroke BL-algebra on the Sheffer stroke BL-algebra C by its filter R from Theorem 4.11. We determine $\lambda : (C/P)/(R/P) \longrightarrow C/R$, $[[c]_P]_{R/P} \longmapsto [c]_R$. Because

$$\begin{aligned}
\lambda([[c_1]_P]_{R/P}|_{R/P}[[c_2]_P]_{R/P}) &= \lambda([[c_1]_P|_P[c_2]_P]_{R/P}) \\
&= \lambda([[c_1|c_2]_P]_{R/P}) \\
&= [c_1|c_2]_R \\
&= [c_1]_R|_R[c_2]_R \\
&= \lambda([[c_1]_P]_{R/P})|_R\lambda([[c_2]_P]_{R/P}),
\end{aligned}$$

similarly

$$\lambda([[c_1]_P]_{R/P} \wedge_{R/P} [[c_2]_P]_{R/P}) = \lambda([[c_1]_P]_{R/P}) \wedge_R \lambda([[c_2]_P]_{R/P}),$$

$$\lambda([[c_1]_P]_{R/P} \vee_{R/P} [[c_2]_P]_{R/P}) = \lambda([[c_1]_P]_{R/P}) \vee_R \lambda([[c_2]_P]_{R/P}),$$

for any elements $[[c_1]_P]_{R/P}, [[c_2]_P]_{R/P} \in (C/P)/(R/P)$,

$$R = [1]_R = \lambda([[1]_P]_{R/P}) \textit{ and } [0]_R = \lambda([[0]_P]_{R/P}),$$

it is obtained that λ is a homomorphism.

• λ is a monomorphism: Let $[c_1]_R, [c_2]_R \in C/R$ such that $\lambda([[c_1]_P]_{R/P}) = [c_1]_R = [c_2]_R = \lambda([[c_2]_P]_{R/P})$, i. e., $c_1\beta_R c_2$. Then we get $c_1|(c_2|c_2) \in R$ and $c_2|(c_1|c_1) \in R$, i. e., $[c_1]_P|_P([c_2]_P|_P[c_2]_P) = [c_1|(c_2|c_2)]_P \in R/P$ and $[c_2]_P|_F([c_1]_P|_P[c_1]_P) = [c_2|(c_1|c_1)]_P \in R/P$. Hence, we obtain $[c_1]_P\beta_{R/P}[c_2]_P$, i. e., $[[c_1]_P]_{R/P} = [[c_2]_P]_{R/P}$.

• λ is a epimorphism: it is obvious from the definition of λ.

So, λ is a isomorphism, i.e., $(C/P)/(R/P) \cong C/R$. □

6. Conclusion

In the present study, we have given a Sheffer stroke BL-algebra, and study Cartesian product, filter, congruence relation, homomorphism between Sheffer stroke BL-algebras, whether Sheffer stroke BL-algebras, filters and related concepts are preserved under this homomorphism, and many features in Sheffer stroke BL-algebras. After giving basic definitions and notions about Sheffer stroke operation and a BL-algebra, we describe a Sheffer stroke BL-algebra and present basic notions about this algebraic structure. So, it is said that the class of Sheffer stroke BL-algebras forms a variety. Then we show that a Sheffer stroke BL-algebra is a BL-algebra if $c_1 \odot c_2 := (c_1|c_2)|(c_1|c_2)$ and $c_1 \longrightarrow c_2 := c_1|(c_2|c_2)$, and that a Cartesian product of two Sheffer stroke BL-algebras is a Sheffer stroke BL-algebra. Besides, by defining a filter on Sheffer stroke BL-algebra, it is proved that the family of all filters of a Sheffer stroke BL-algebra forms a complete lattice, and that for a subset of a Sheffer stroke BL-algebra there exists the minimal filter containing this subset. Also, it is expressed a congruence relation on a Sheffer stroke BL-algebra determined by its filter and related notions, and proved that a quotient of a Sheffer stroke BL-algebra by a congruence relation is a Sheffer stroke BL-algebra. Finally, a homomorphism between two Sheffer stroke BL-algebras is described and it is stated that mentioned notions are preserved under this homomorphism. Particularly, it is studied on filters and quotients of a Sheffer stroke BL-algebra.

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