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ON $\zeta-$FACTORS AND COMPUTING STRUCTURES IN CYCLIC $n-$ROOTS

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Abstract. In this paper, we introduce a new concept in number theory called $\zeta-$factors associated with a positive integer n. Applications of $\zeta-$factors are in the arrangement of the defining polynomials in cyclic $n-$roots algebraic system and are thoroughly investigated. More precisely, $\zeta-$factors arise in the proofs of vanishing theorems in regard to associated prime factors of the system. Exact computations through concrete examples of positive dimensions for $n = 16, 18$ support the results.

1. Introduction

This research addresses a solid relationship between a novel concept in number theory and a structural arrangement of the indices in the defining polynomials of cyclic $n-$roots algebraic system. Let n be a positive integer and $j = 0, 1, \cdots, n-1$ and $[j]$ be the set of positive integers whose remainder on division by n is j. For $i \in \mathbb{N}$, an important integer-valued function is defined by

$$\varphi_n(i) := \begin{cases} j & \text{if} \qquad i \in [j], \quad \exists j = 1, \cdots, n-1, \\ n & \text{otherwise.} \end{cases} \tag{1}$$

Let $R = \mathbb{C}[X_1, \cdots, X_n]$ be the ring of polynomials of n variables $X_1, \cdots, X_n$ with complex coefficients. Generally, for $f_1, \cdots, f_k \in R$, $I = I(f_1, \cdots, f_k)$ denotes the ideal generated by f_i's in R and $R/I = R_I$ is the associated residue class ring. R_I has a structure of vector space over $\mathbb{C}$. To get a precise definition of the members of R_I, we need to fix a monomial order $>$ on R, say lexicographic order (i.e.; lex-order) with $X_1 > X_2 > \cdots > X_n$. The set of all monomials in the representation of a polynomial $f \in R$ is called the support of f. The lex-order largest monomial in the support of f is called the initial of f or $\mathsf{in}(f)$. For an ideal $I \subset R$, the initial ideal is $\mathsf{in}(I) := I(\{\mathsf{in}(f) : f \in I\})$. See [1] on page 32. If $G(I)$ is the reduced Gröbner basis of I, then every polynomial $f \in R$ has a unique remainder $\bar{f}$ on division by $G(I)$. The reader is referred to [1] for an excellent introduction about Gröbner bases. For $f, g \in R$, let's define an equivalence relation $\sim$ on R by $f \sim g$ if and only if $\overline{f - g} = 0$ (equivalently $f - g \in I$). Now we define R_I by $R_I = \{[f] : f \in R\}$ where $[f]$ is the

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equivalence class corresponding to f. Since $\mathsf{in}(I) = \mathsf{in}(G(I))$, then evidently all the monomials that are not in $\mathsf{in}(G(I))$ form a basis for R_I. Now suppose $n \geqslant 3$. We define n polynomials $h_i \in R$ $(1 \leqslant i \leqslant n)$ by

$$h_i = \sum_{j=1}^{n} \prod_{k=j}^{j+i-1} X_{\varphi_n(k)}, \quad (X_1, \cdots, X_n) \in \mathbb{C}^n.$$

Cyclic $n-$roots is the solution set of the system of polynomial equations $h_1 = 0, \cdots, h_{n-1} = 0, h_n = n$. A simplified notation as $H_i = h_i$, for $i = 1, \cdots, n-1$ with $H_n = \frac{1}{n}h_n - 1$ will be used. Let $IC_n = I(H_1, \cdots, H_n)$ be the ideal generated by the defining polynomials $H_1, \cdots, H_n$ of cyclic $n-$roots. For a detailed history about cyclic $n-$roots and applications see [3,4] and references therein. In section 2, we introduce $\zeta-$factors associated with a given positive integer n. In reality, the size of our algebraic system n is decomposed as a product of two of its divisors λ and μ. Intuitively speaking, a $\zeta-$factor can be referred to a triplet (λ, μ, ζ), where ζ is another positive integer such that $\lambda < \zeta < n$, $\mu < \zeta < n$, $\lambda \mid \zeta$ and $\mu \mid \zeta$. We study some conditions under which the existence of a $\zeta-$factor associated with a given positive integer n is guaranteed. Certain basic definitions and notions in number theory can be found in [2]. In particular, we refer to the fundamental theorem of arithmetic and an evident notation $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$. Section three is devoted to our main computational results about the structures of prime ideals of $\sqrt{IC_n}$. Of particular importance, the most effective result that targets the core structure of the system is in lemma 3.6. The proof of lemma 3.6 is in the appendix. For the sake of clarity, in this paper, some of the equations are given via expanded notations.

2. $\zeta-$Factors

Suppose n is a positive integer and $\mathsf{Div}(n) := \{k : k \mid n, k \neq 1; k < n\}$ denotes the set of all divisors of n except 1 and n. Also, let $\mathfrak{p}(n)$ be the set of prime factors of n. A pair $\lambda, \mu \in \mathsf{Div}(n)$ with $n = \lambda\mu$ (in which we discard 1 and n as divisors of n) is called a *split of* n and is denoted by $(\lambda, \mu)_n$. A split is *pure* if we further have $\lambda \leqslant \mu$ and we denote it by $(\lambda \leqslant \mu)_n$. For a pure split $(\lambda \leqslant \mu)_n$ of n, if there exists a positive integer ζ such that $\lambda < \zeta < n$ and $\mu < \zeta < n$ where $\lambda \mid \zeta$ and $\mu \mid \zeta$, we say that the triplet of integers $(\lambda \leqslant \mu, \zeta)_n$ is a *$\zeta-$factor* of n.

Example 2.1. Let $n = 1000$. The triplet $(\lambda \leqslant \mu, \zeta)_n = (4 \leqslant 250, 500)_{1000}$ is a $500-$factor for 1000 while for $\lambda = 8$ and $\mu = 125$, no ζ exists. As the reader observes, λ does not have to divide μ as in $(4 \leqslant 250, 500)_{1000}$. As another example, consider $(4 \leqslant 14, 28)_{56}$.

If for every pure split $(\lambda \leqslant \mu)_n$ of n, no such ζ exists, then we say n does not admit any $\zeta-$factor. $n = 20$ is an example. The problem of determining whether a given positive integer n admits a $\zeta-$factor seems to be a far-reaching one.

Example 2.2. The following table shows all possible integers $n \leqslant 45$ with their corresponding $\zeta-$factors. Those integers $n \leqslant 45$ that are not in the table do not admit any $\zeta-$factor.

n	$(\lambda \leqslant \mu, \zeta)_n$	n	$(\lambda \leqslant \mu, \zeta)_n$	n	$(\lambda \leqslant \mu, \zeta)_n$
18	$(3 \leqslant 6, 12)_{18}$	24	$(4 \leqslant 6, 12)_{24}$	27	$(3 \leqslant 9, 18)_{27}$
32	$(4 \leqslant 8, 16)_{32}$	32	$(4 \leqslant 8, 24)_{32}$	36	$(3 \leqslant 12, 24)_{36}$
40	$(4 \leqslant 10, 20)_{40}$	45	$(3 \leqslant 15, 30)_{45}$		

Lemma 2.3. *For a given positive integer n, let $(\lambda \leqslant \mu)_n$ be a pure split for n. $(\lambda \leqslant \mu)_n$ does not admit any ζ−factor if and only if $\mathfrak{p}(n) = A \cup B$ where the union is disjoint, $\mathfrak{p}(\lambda) = A$ and $\mathfrak{p}(\mu) = B$.*

Proof. Let $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$. Set $A = \mathfrak{p}(\lambda)$ and $B = \mathfrak{p}(n) \backslash A$. Suppose $(\lambda \leqslant \mu)_n$ does not admit any ζ−factor. We shall show that $B = \mathfrak{p}(\mu)$. For $q \in B$, since $q \notin A$ we must have $q \in \mathfrak{p}(\mu)$ which implies $B \subset \mathfrak{p}(\mu)$. With no loss of generality we may assume that $\varnothing \neq F = \mathfrak{p}(\mu) \backslash B = \{f\}$. The general case where F might have more than one element can be discussed similarly. Let $1 \leqslant j_0 \leqslant k$ be the integer with $f = p_{j_0}$. We may consider $\mathfrak{p}(\lambda) = \{p_1, \cdots, p_{j_0-1}, f\}$ and $\mathfrak{p}(\mu) = \{f, p_{j_0+1}, \cdots, p_k\}$. Since $n = \lambda\mu = p_1^{r_1} \cdots p_{j_0-1}^{r_{j_0-1}} \cdot f^{\alpha_{j_0}} \cdot f^{\beta_{j_0}} \cdot p_{j_0+1}^{r_{j_0+1}} \cdots p_k^{r_k}$ for some integers $\alpha_{j_0}, \beta_{j_0} \geqslant 1$, as a result, $r_{j_0} = \alpha_{j_0} + \beta_{j_0} \geqslant 2$. With $\gamma_{j_0} = \max\{\alpha_{j_0}, \beta_{j_0}\}$, now we define $\zeta = p_1^{r_1} \cdots p_{j_0-1}^{r_{j_0-1}} \cdot f^{\gamma_{j_0}} \cdot p_{j_0+1}^{r_{j_0+1}} \cdots p_k^{r_k} < n$ and evidently $\lambda \mid \zeta$ and $\mu \mid \zeta$. So, $(\lambda \leqslant \mu, \zeta)_n$ is a ζ−factor for n. From these facts, a detailed proof can be set up. □

Corollary 2.4. *A positive integer in the form $n = p_1 \cdots p_k$ does not admit any ζ−factor.*

Example 2.5. To illustrate the construction in the proof of lemma 2.3, let $n = p_1 p_2 p_3^2$ with $p_1 < p_2 < p_3$. Then $(p_1 p_3 \leqslant p_2 p_3, p_1 p_2 p_3)_{p_1 p_2 p_3^2}$ is a ζ−factor. In this situation, $\lambda = p_1 p_3$ and $\mu = p_2 p_3$ where apparently $A = \{p_1, p_3\}$ and $B = \{p_2, p_3\}$ and they are not disjoint.

The following is another applicable corollary of lemma 2.3.

Corollary 2.6. *Let $(\lambda \leqslant \mu)_n$ be a pure split of n. If $(\lambda \leqslant \mu, \zeta)_n$ is a ζ−factor of n, then $\mathfrak{p}(\lambda) \cap \mathfrak{p}(\mu) \neq \varnothing$.*

Remark 2.7. I planned to continue presenting more results in conjunction with the combinatorics of the problem of determination of ζ−factors. Since the main theme of this research is on certain structural properties of cyclic n−roots, they will be postponed to another time.

3. Computing structures in cyclic n−roots

For fixed $n \geqslant 3$, in various situations we will have combinations of a series of assumptions among which: (a) $(\lambda \leqslant \mu)_n$ is a pure split for n (b) $\lambda \mid \mu$ and let $\overline{\kappa} = n - \lambda$ and so. Primitive μ^{th} roots of unity (generally denoted by ω_μ) play important role in the formation of the generators of the prime decomposition of $\sqrt{IC_n}$. In this regard, extensive computation with rigorous proofs was presented in [3,4]. In those proofs, the given forms of the linear generators of a prime helped me to determine the dimension of the prime. In this paper, we only deal with some structural computations in regard to the positive-dimensional prime ideals of $\sqrt{IC_n}$. Consider the following linear binomial polynomials

$$\delta_j = \omega_\mu X_j - X_{\lambda+j}, \;\; j = 1, \cdots, \overline{\kappa}. \tag{2}$$

The generators of the prime ideals (of the minimal prime decomposition of $\sqrt{IC_n}$) in two main examples in this paper are given in different forms than (2). For these examples, we will show that the form (2) can be derived. General arguments in this regard are given in lemmas 3.11 and 3.13 where we will show that the equations (2) constitute linear binomial generators in a prime ideal of higher dimension in the minimal prime decomposition of $\sqrt{IC_n}$. The notations $\mathcal{B} = \{\delta_1, \cdots, \delta_{\overline{\kappa}}\}$ and $\mathcal{B} = I(\delta_1, \cdots, \delta_{\overline{\kappa}})$ will be used interchangeably. Lemma 3.1 below shows the advantage of the form (2), where we may simply convert it into a more convenient form (3) (i.e.; lex-order Gröbner basis). The form (3) plays a central role in the rational parametrization of the system as in lemma 2 in [3].

LEMMA 3.1. *For a given integer $n \geqslant 3$, let $(\lambda \leqslant \mu)_n$ be a pure split with $\overline{\kappa} = n - \lambda$. Then*

$$\widetilde{\mathcal{B}} = \left\{ \ \tilde{\delta}_{t\lambda+j} = X_{t\lambda+j} - (\omega_\mu)^{t+1} X_{\overline{\kappa}+j} : \ j = 1, \cdots, \lambda, \ \ t = 0, 1, \cdots, \mu - 2 \ \right\} \tag{3}$$

is the reduced Gröbner basis of $\mathcal{B} = I(\delta_1, \cdots, \delta_{\overline{\kappa}})$, where δ_r's and X_s's are related via (2).

Proof. By considering the row-reduced echelon form of the coefficient matrix in the equations (2), the result follows. □

EXAMPLE 3.2. Let $n = 16 = 4 \times 4$, $\lambda = 4$, $\mu = 4$ and $\bar{k} = 16 - 4 = 12$. In this case, with the form we introduced in [4], let $\omega_4 = \boldsymbol{i}$ as a primitive 4^{th} root of unity, a set of generators of the ideal $\mathbf{I}^{\mathbf{C_{16}}} = I(\rho_1, \cdots, \rho_{13})$ is given by

$$\begin{array}{ll} \rho_i = X_i + \boldsymbol{i} X_{4+i}, \ \ i = 1, \cdots, 4; & \rho_i = X_{i-4} + X_{4+i}, \ \ i = 5, \cdots, 8; \\ \rho_i = X_{i-8} - \boldsymbol{i} X_{4+i}, \ \ i = 9, \cdots, 12; & \rho_{13} = X_1 X_2 X_3 X_4 - 1. \end{array}$$

We may easily write

$$\begin{array}{rcl} \mathbf{I}^{\mathbf{C_{16}}} & = & I \left(\begin{array}{l} \boldsymbol{i}\rho_1, \boldsymbol{i}\rho_2, \boldsymbol{i}\rho_3, \boldsymbol{i}\rho_4, \rho_1 - \rho_5, \rho_2 - \rho_6, \rho_3 - \rho_7, \rho_4 - \rho_8, \\ \boldsymbol{i}(\rho_5 - \rho_9), \boldsymbol{i}(\rho_6 - \rho_{10}), \boldsymbol{i}(\rho_7 - \rho_{11}), \boldsymbol{i}(\rho_8 - \rho_{12}), \rho_{13} \end{array} \right) \\ & = & I(\delta_1, \cdots, \delta_{12}, X_{13} X_{14} X_{15} X_{16} - 1) \end{array}$$

where ρ_{13} can be written as

$$\begin{array}{rcl} \rho_{13} & = & (\rho_9 + \omega_4 X_{13})(\rho_{10} + \omega_4 X_{14})(\rho_{11} + \omega_4 X_{15})(\rho_{12} + \omega_4 X_{16}) - 1 \\ & = & X_{13} X_{14} X_{15} X_{16} - 1 + \Sigma \end{array}$$

and Σ is a combination of ρ_i's. The reader should notice that δ_i's in $\mathcal{B} = I(\delta_1, \cdots, \delta_{12})$ are in the form (2).

EXAMPLE 3.3. For $n = 18 = 3 \times 6$ where $\lambda = 3$ and $\mu = 6$, the only two primitive 6^{th} roots of unity are $\omega_6^{(\pm)} = \frac{1}{2} \pm \boldsymbol{i}\frac{\sqrt{3}}{2}$. Let $\omega = \frac{1}{2} - \boldsymbol{i}\frac{\sqrt{3}}{2}$. The following generators for $\mathbf{I}^{\mathbf{C_{18}}} = I(\tau_1, \cdots, \tau_{16})$ can also be found in [4].

$$\begin{array}{ll} \tau_i = X_i - \omega X_{i+3}, \ \ i = 1, 2, 3; & \tau_i = X_{i-3} + \overline{\omega} X_{i+3}, \ \ i = 4, 5, 6; \\ \tau_i = X_{i-6} + X_{i+3}, \ \ i = 7, 8, 9; & \tau_i = X_{i-9} + \omega X_{i+3}, \ \ i = 10, 11, 12; \\ \tau_i = X_{i-12} - \overline{\omega} X_{i+3}, \ \ i = 13, 14, 15; & \tau_{16} = X_1 X_2 X_3 - \omega \end{array} \tag{4}$$

where the bars denote complex conjugates. Thus, we may write the following

$$\begin{array}{rcl} \mathbf{I}^{\mathbf{C_{18}}} & = & I\left(\begin{array}{c} \overline{\omega}\tau_1, \overline{\omega}\tau_2, \overline{\omega}\tau_3, \omega(\tau_1-\tau_4), \omega(\tau_2-\tau_5), \\ \omega(\tau_3-\tau_6), \tau_4-\tau_7, \tau_5-\tau_8, \tau_6-\tau_9, \\ \overline{\omega}(\tau_7-\tau_{10}), \overline{\omega}(\tau_8-\tau_{11}), \overline{\omega}(\tau_9-\tau_{12}), \\ -\omega(\tau_{10}-\tau_{13}), -\omega(\tau_{11}-\tau_{14}), -\omega(\tau_{12}-\tau_{15}), \tau_{16} \end{array}\right) \\ & = & I(\delta_1, \cdots, \delta_{15}, X_{16}X_{17}X_{18}+\omega) \end{array}$$

where δ_i's are in the form of (2). From (4) we get the equations for τ_{13}, τ_{14} and τ_{15}. After substitution of X_1, X_2 and X_3 in τ_{16}, it can be written as

$$\begin{array}{rcl} \tau_{16} & = & (\tau_{13}+\overline{\omega}X_{16})(\tau_{14}+\overline{\omega}X_{17})(\tau_{15}+\overline{\omega}X_{18})-\omega \\ & = & -(X_{16}X_{17}X_{18}+\omega)+\Sigma \end{array}$$

and Σ is a combination of τ_i's.

Remark 3.4. By lemma 3.1, based on a well-known property of Gröbner bases, $\mathsf{in}(\widetilde{\mathcal{B}}) = \mathsf{in}(\mathcal{B})$. In particular, using lemma 3.1, $\mathsf{in}(\widetilde{\mathcal{B}}) = I(X_1, \cdots, X_{\overline{\kappa}})$. In order to get a basis for $R_{\widetilde{\mathcal{B}}}$, it is inevitable to consider an identification of the form $X_i \to [X_i]$ where $[\cdot]$ is the notation for the equivalence classes in $R_{\widetilde{\mathcal{B}}}$. Using defining polynomials of $\widetilde{\mathcal{B}}$ in (3) consider

$$[X_{t\lambda+j}] = (\omega_\mu)^{t+1}[X_{\overline{\kappa}+j}], \quad j=1,\cdots,\lambda, \quad t=0,1,\cdots,\mu-2. \tag{5}$$

Example 3.5. In (5), let $n = 16 = 4\times 4$ with $\lambda = 4$ and $\mu = 4$. Thus, $\overline{\kappa} = n-\lambda = 12$. As the reader observes in [4], the linear part of $\mathbf{I}^{\mathbf{C_{16}}}$ can be considered as

$$\widetilde{\mathcal{B}} = \left\{\begin{array}{l} X_1-\boldsymbol{i}X_{13},\, X_2-\boldsymbol{i}X_{14},\, X_3-\boldsymbol{i}X_{15},\, X_4-\boldsymbol{i}X_{16}, X_5+X_{13}, \\ X_6+X_{14},\, X_7+X_{15},\, X_8+X_{16},\, X_9+\boldsymbol{i}X_{13},\, X_{10}+\boldsymbol{i}X_{14}, \\ X_{11}+\boldsymbol{i}X_{15},\, X_{12}+\boldsymbol{i}X_{16} \end{array}\right\}.$$

The above set $\widetilde{\mathcal{B}}$ is the lex-order reduced Gröbner basis. In this example, the set of equations in (5) turn into $[X_{4t+j}] = \boldsymbol{i}^{t+1}[X_{12+j}], \quad j=1,2,3,4; \; t=0,1,2.$

The long proof of the following is given in the appendix.

Lemma 3.6. *For a given positive integer $n \geqslant 3$, let $(\lambda \leqslant \mu)_n$ be a pure split. For $0 \leqslant t \leqslant \mu-1$ and any positive integer j,*

$$\left[X_{\varphi_n(j+t\lambda)}\right] = (\omega_\mu)^t\left[X_{\varphi_n(j)}\right], \tag{6}$$

where X_r's satisfy (2).

From this point to the end of the paper, the reader should notice that, due to the complicated form of the expressions, compact notations for sums and products will only be used in some parts of the context. By (1), for $1 \leqslant i \leqslant n$ and $1 \leqslant j$ consider

$$\mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{j})} = \prod_{l=1}^{i} X_{\varphi_n(j+l-1)} = X_{\varphi_n(j)}X_{\varphi_n(j+1)}\cdots X_{\varphi_n(j+i-1)}. \tag{7}$$

A simple lemma follows. It can be applied in example 3.9.

Lemma 3.7. *For a given integer $n \geqslant 3$ and $1 \leqslant i \leqslant n$, $1 \leqslant j$;* $\mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{j})} = \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{\varphi_n(j)})}$.

Proof. Following the definition (7) we write

$$\mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{\varphi_n(j)})} = \prod_{l=1}^{i} X_{\varphi_n(\varphi_n(j)+l-1)} = X_{\varphi_n(\varphi_n(j))}X_{\varphi_n(\varphi_n(j)+1)}\cdots X_{\varphi_n(\varphi_n(j)+i-1)}. \tag{8}$$

Let $j = kn + u$ where $0 \leqslant u < n$ and k is a non-negative integer. The proof follows by considering two cases. Case 1: $u = 0$. In this case, since $\varphi_n(j) = n$, (8) becomes

$$\begin{aligned} \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{\varphi_n(j)})} &= \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{n})} = \prod_{l=1}^{i} X_{\varphi_n(n+l-1)} = X_{\varphi_n(n)}X_{\varphi_n(n+1)}\cdots X_{\varphi_n(n+i-1)} \\ &= X_{\varphi_n(kn)}X_{\varphi_n(kn+1)}\cdots X_{\varphi_n(kn+i-1)} = \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{j})}. \end{aligned}$$

Case 2: $u > 0$. This implies $\varphi_n(j) = u$ and in turn

$$\mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{\varphi_n(j)})} = \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{u})} = \prod_{l=1}^{i} X_{\varphi_n(u+l-1)} = \prod_{l=1}^{i} X_{\varphi_n(kn+u+l-1)} = \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{j})}.$$

□

Clearly, $\mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{j})}$ is a square-free monomial of total degree j starting with $X_{\varphi_n(j)}$.

Example 3.8. Some instances of $\mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{j})}$'s can be written as

$$\begin{aligned} \mathfrak{m}_{(\mathbf{4},\mathbf{3},\mathbf{3})} &= X_{\varphi_4(3)}X_{\varphi_4(4)}X_{\varphi_4(5)} = X_3X_4X_1 \\ \mathfrak{m}_{(\mathbf{9},\mathbf{7},\mathbf{6})} &= \prod_{i=6}^{12} X_{\varphi_9(i)} = X_6X_7X_8X_9X_1X_2X_3 \\ \mathfrak{m}_{(\mathbf{16},\mathbf{8},\mathbf{8})} &= \prod_{i=8}^{15} X_{\varphi_{16}(i)} = X_8X_9X_{10}X_{11}X_{12}X_{13}X_{14}X_{15} \\ \mathfrak{m}_{(\mathbf{18},\mathbf{8},\mathbf{12})} &= \prod_{l=1}^{8} X_{\varphi_{18}(12+l-1)} = \prod_{u=12}^{19} X_{\varphi_{18}(u)} \\ &= X_{\varphi_{18}(12)}X_{\varphi_{18}(13)}X_{\varphi_{18}(14)}X_{\varphi_{18}(15)}X_{\varphi_{18}(16)} \\ &\quad \times X_{\varphi_{18}(17)}X_{\varphi_{18}(18)}X_{\varphi_{18}(19)} \\ &= X_{12}X_{13}X_{14}X_{15}X_{16}X_{17}X_{18}X_1. \end{aligned}$$

In case where $j > n$ modular calculations must be performed. For $1 \leqslant i < n$, H_i can be written as

$$H_i = \sum_{j=1}^{n} \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{j})} = X_1X_2\cdots X_i + X_2X_3\cdots X_{i+1} + \cdots + X_nX_1\cdots X_{i-1}$$

and $H_n = \frac{1}{n}\sum_{j=1}^{n} \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{n},\boldsymbol{j})} - 1$. Next let $1 \leqslant i < n$, $1 \leqslant j \leqslant \lambda$ and

$$\boldsymbol{\alpha}_{(\boldsymbol{n},\boldsymbol{j},\boldsymbol{i})} = \sum_{l=0}^{\mu-1} \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{j}+\boldsymbol{\lambda l})} = \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{j})} + \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{j}+\boldsymbol{\lambda})} + \cdots + \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{j}+\boldsymbol{\lambda}(\boldsymbol{\mu}-\mathbf{1}))}. \tag{9}$$

Also, let

$$\boldsymbol{\beta}_{(\boldsymbol{n},\boldsymbol{j},\boldsymbol{i})} = \sum_{l=0}^{\mu-1} \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{\mu j}+\boldsymbol{l})} = \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{\mu j})} + \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{\mu j}+\mathbf{1})} + \cdots + \mathfrak{m}_{(\boldsymbol{n},\boldsymbol{i},\boldsymbol{\mu}(\boldsymbol{j}+\mathbf{1})-\mathbf{1})}. \tag{10}$$

Example 3.9. Let $n = 16 = 4 \times 4$, $\lambda = 4$ and $\mu = 4$. To clarify the notations in (9) and (10), the following can be written

$$\begin{aligned}
\boldsymbol{\beta}_{(16,1,3)} &= \mathfrak{m}_{(16,3,4)} + \mathfrak{m}_{(16,3,5)} + \mathfrak{m}_{(16,3,6)} + \mathfrak{m}_{(16,3,7)} \\
&= \prod_{l=1}^{3} X_{\varphi_{16}(3+l)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+4)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+5)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+6)} \\
&= X_4X_5X_6 + X_5X_6X_7 + X_6X_7X_8 + X_7X_8X_9
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\beta}_{(16,2,3)} &= \mathfrak{m}_{(16,3,8)} + \mathfrak{m}_{(16,3,9)} + \mathfrak{m}_{(16,3,10)} + \mathfrak{m}_{(16,3,11)} \\
&= \prod_{l=1}^{3} X_{\varphi_{16}(l+7)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+8)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+9)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+10)} \\
&= X_8X_9X_{10} + X_9X_{10}X_{11} + X_{10}X_{11}X_{12} + X_{11}X_{12}X_{13} \\
\boldsymbol{\beta}_{(16,3,3)} &= \mathfrak{m}_{(16,3,12)} + \mathfrak{m}_{(16,3,13)} + \mathfrak{m}_{(16,3,14)} + \mathfrak{m}_{(16,3,15)} \\
&= \prod_{l=1}^{3} X_{\varphi_{16}(l+11)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+12)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+13)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+14)} \\
&= X_{12}X_{13}X_{14} + X_{13}X_{14}X_{15} + X_{14}X_{15}X_{16} + X_{15}X_{16}X_{17} \\
\boldsymbol{\beta}_{(16,4,3)} &= \mathfrak{m}_{(16,3,16)} + \mathfrak{m}_{(16,3,17)} + \mathfrak{m}_{(16,3,18)} + \mathfrak{m}_{(16,3,19)} \\
&= \prod_{l=1}^{3} X_{\varphi_{16}(l+15)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+16)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+17)} + \prod_{l=1}^{3} X_{\varphi_{16}(l+18)} \\
&= X_{\varphi_{16}(16)}X_{\varphi_{16}(17)}X_{\varphi_{16}(18)} + X_{\varphi_{16}(17)}X_{\varphi_{16}(18)}X_{\varphi_{16}(19)} + \\
&\quad X_{\varphi_{16}(18)}X_{\varphi_{16}(19)}X_{\varphi_{16}(20)} + X_{\varphi_{16}(19)}X_{\varphi_{16}(20)}X_{\varphi_{16}(21)} \\
&= X_{16}X_1X_2 + X_1X_2X_3 + X_2X_3X_4 + X_3X_4X_5.
\end{aligned}$$

Thus, $H_3 = \boldsymbol{\beta}_{(16,1,3)} + \boldsymbol{\beta}_{(16,2,3)} + \boldsymbol{\beta}_{(16,3,3)} + \boldsymbol{\beta}_{(16,4,3)}$. The same can be established for $\boldsymbol{\alpha}_{(16,1,3)}, \boldsymbol{\alpha}_{(16,2,3)}, \boldsymbol{\alpha}_{(16,3,3)}$ and $\boldsymbol{\alpha}_{(16,4,3)}$.

In the calculation of $\boldsymbol{\beta}_{(16,4,3)}$, we see the third indices in the items $\mathfrak{m}_{(16,3,17)}$, $\mathfrak{m}_{(16,3,18)}$ and $\mathfrak{m}_{(16,3,19)}$ are higher than $n = 16$. In such cases, we may efficiently use lemma 3.7.

Lemma 3.10. *For a given integer $n \geqslant 3$, let $(\lambda \leqslant \mu)_n$ be a pure split. Then*

$$H_i = \boldsymbol{\alpha}_{(n,1,i)} + \cdots + \boldsymbol{\alpha}_{(n,\lambda,i)} = \boldsymbol{\beta}_{(n,1,i)} + \cdots + \boldsymbol{\beta}_{(n,\lambda,i)}$$

where $1 \leqslant i < n$.

Proof. For $\boldsymbol{\alpha}$−sum

$$\begin{aligned}
\sum_{j=1}^{\lambda} \boldsymbol{\alpha}_{(n,j,i)} &= \sum_{j=1}^{\lambda}\sum_{l=0}^{\mu-1} \mathfrak{m}_{(n,i,j+\lambda l)} = \sum_{l=0}^{\mu-1}\left(\mathfrak{m}_{(n,i,1+\lambda l)} + \cdots + \mathfrak{m}_{(n,i,\lambda+\lambda l)}\right) \\
&= \left(\mathfrak{m}_{(n,i,1)} + \cdots + \mathfrak{m}_{(n,i,\lambda)}\right) + \left(\mathfrak{m}_{(n,i,\lambda+1)} + \cdots + \mathfrak{m}_{(n,i,2\lambda)}\right) \\
&\quad + \cdots + \left(\mathfrak{m}_{(n,i,1+\lambda(\mu-1))} + \cdots + \mathfrak{m}_{(n,i,\lambda\mu-1)}\right) \\
&= H_i.
\end{aligned}$$

Likewise, for $\boldsymbol{\beta}$−sum

$$\boldsymbol{\beta}_{(n,\ell,i)} = \sum_{t=0}^{\mu-1} \mathfrak{m}_{(n,i,\ell\mu+t)};\ \ \ell = 1, \cdots, \lambda - 1 \tag{11}$$

The last row, after some modular calculations, can be written as

$$\boldsymbol{\beta}_{(n,\lambda,i)} = \mathfrak{m}_{(n,i,n)} + \mathfrak{m}_{(n,i,1)} + \cdots + \mathfrak{m}_{(n,i,\mu-1)}.$$

This modified form with the rest of the equations in (11) gives the conclusion.

□

LEMMA 3.11. *For a given integer $n \geqslant 3$, let $(\lambda \leqslant \mu)_n$ be a pure split. If $1 \leqslant i < n$ and $1 \leqslant j \leqslant \lambda$, then in $R_{\mathcal{B}}$*

(*a*) $\left[\boldsymbol{\alpha}_{(n,j,i)}\right] = \boldsymbol{\Omega}_i^{\mu} \cdot \left[\mathfrak{m}_{(n,i,j)}\right]$ *where* $\boldsymbol{\Omega}_i^{\mu} = \sum_{s=0}^{\mu-1}(\omega_\mu)^{si}$.

(*b*) *For positive integer q, let* $\boldsymbol{\Gamma}_j^{q,\mu} = \sum_{s=j\mu-1}^{\mu(j+1)-2}(\omega_\mu)^{qs}$.
Then $\left[\boldsymbol{\beta}_{(n,j,i)}\right] = \boldsymbol{\Gamma}_j^{q,\mu} \cdot \left[\mathfrak{m}_{(n,i,1)}\right]$ *if and only if* $i = q\lambda$ (*i.e.*; $\lambda \mid i$).

(*c*) $\lambda \mid i$ *if and only if* $\left[\boldsymbol{\beta}_{(n,j,i)}\right] = [0]$ (*i.e.*; $\boldsymbol{\beta}_{(n,j,i)} \in \mathcal{B}$).

Proof. (*a*) By lemma 3.6, for $0 \leqslant t \leqslant \mu - 1$

$$\begin{aligned}\left[\mathfrak{m}_{(n,i,j+t\lambda)}\right] &= \left[X_{\varphi_n(j+t\lambda)}X_{\varphi_n(j+t\lambda+1)}\cdots X_{\varphi_n(j+t\lambda+i-1)}\right]\\ &= \left[X_{\varphi_n(j+t\lambda)}\right]\cdot\left[X_{\varphi_n(j+t\lambda+1)}\right]\cdots\left[X_{\varphi_n(j+t\lambda+i-1)}\right]\\ &= \left((\omega_\mu)^t\left[X_{\varphi_n(j)}\right]\right)\cdot\left((\omega_\mu)^t\left[X_{\varphi_n(j+1)}\right]\right)\cdots\left((\omega_\mu)^t\left[X_{\varphi_n(j+i-1)}\right]\right)\\ &= (\omega_\mu)^{it}\cdot\left[X_{\varphi_n(j)}X_{\varphi_n(j+1)}\cdots X_{\varphi_n(j+i-1)}\right] = (\omega_\mu)^{it}\cdot\left[\mathfrak{m}_{(n,i,j)}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\left[\boldsymbol{\alpha}_{(n,j,i)}\right] &= \left[\mathfrak{m}_{(n,i,j)}\right] + \left[\mathfrak{m}_{(n,i,j+\lambda)}\right] + \cdots + \left[\mathfrak{m}_{(n,i,j+\lambda(\mu-1))}\right]\\ &= \left[\mathfrak{m}_{(n,i,j)}\right] + (\omega_\mu)^i\cdot\left[\mathfrak{m}_{(n,i,j)}\right] + \cdots + (\omega_\mu)^{(\mu-1)i}\cdot\left[\mathfrak{m}_{(n,i,j)}\right]\\ &= (1 + (\omega_\mu)^i + \cdots + (\omega_\mu)^{(\mu-1)i})\cdot\left[\mathfrak{m}_{(n,i,j)}\right] = \boldsymbol{\Omega}_i^{\mu}\cdot\left[\mathfrak{m}_{(n,i,j)}\right].\end{aligned}$$

(*b*) Set $i = \lambda q + t$ where t is a non-negative integer and $t < \lambda$. In this case, $1 \leqslant l < n$, as a backward recurrence relation for $\mathfrak{m}_{(n,i,l+1)}$ and by lemma 3.6 consider

$$\begin{aligned}\left[\mathfrak{m}_{(n,i,l+1)}\right] &= \left[X_{\varphi_n(l+1)}X_{\varphi_n(l+2)}\cdots X_{\varphi_n(l+i-1)}X_{\varphi_n(l+\lambda q+t)}\right]\\ &= \left[X_{\varphi_n(l+1)}X_{\varphi_n(l+2)}\cdots X_{\varphi_n(l+i-1)}\right]\cdot\left[X_{\varphi_n(l+\lambda q+t)}\right]\\ &= \left[X_{\varphi_n(l+1)}X_{\varphi_n(l+2)}\cdots X_{\varphi_n(l+i-1)}\right]\cdot\left((\omega_\mu)^q\left[X_{\varphi_n(l+t)}\right]\right).\end{aligned}$$

The last equality can be written in terms of $\mathfrak{m}_{(n,i,l)}$ if and only if $t = 0$. Thus, $\left[\mathfrak{m}_{(n,i,l+1)}\right] = (\omega_\mu)^q\cdot\left[\mathfrak{m}_{(n,i,l)}\right]$ if and only if $\lambda \mid i$. Then for $1 \leqslant j \leqslant \lambda$, $\left[\boldsymbol{\beta}_{(n,j,i)}\right]$ can be written as

$$\begin{aligned}\left[\boldsymbol{\beta}_{(n,j,i)}\right] &= \left[\mathfrak{m}_{(n,i,j\mu)}\right] + \left[\mathfrak{m}_{(n,i,j\mu+1)}\right] + \cdots + \left[\mathfrak{m}_{(n,i,\mu(j+1)-1)}\right]\\ &= \left((\omega_\mu)^{(j\mu-1)q} + (\omega_\mu)^{j\mu q} + \cdots + (\omega_\mu)^{q(\mu(j+1)-2)}\right)\cdot\left[\mathfrak{m}_{(n,i,1)}\right]\\ &= \boldsymbol{\Gamma}_j^{q,\mu}\cdot\left[\mathfrak{m}_{(n,i,1)}\right]\end{aligned}$$

if and only if $\lambda \mid i$.

(*c*) If $i = q\lambda$, then (*b*) implies that $\left[\boldsymbol{\beta}_{(n,j,i)}\right] = \boldsymbol{\Gamma}_j^{q,\mu}\cdot\left[\mathfrak{m}_{(n,i,1)}\right]$. On the other hand,

$$(\omega_\mu)^q\cdot\boldsymbol{\Gamma}_j^{q,\mu} = \boldsymbol{\Gamma}_j^{q,\mu} - (\omega_\mu)^{q(\mu j-1)} + (\omega_\mu)^{q(\mu(j+1)-1)}. \tag{12}$$

Since in (12),

$$\begin{aligned}-(\omega_\mu)^{q(\mu j-1)} + (\omega_\mu)^{q(\mu(j+1)-1)} &= -(\omega_\mu)^{q\mu j}\cdot(\omega_\mu)^{-q} + (\omega_\mu)^{q\mu(j+1)}\cdot(\omega_\mu)^{-q}\\ &= 0,\end{aligned}$$

therefore $(\omega_\mu)^q\cdot\boldsymbol{\Gamma}_j^{q,\mu} = \boldsymbol{\Gamma}_j^{q,\mu}$. Thus, $\lambda q = i < n = \lambda\mu$ implies $q < \mu$ and in turn $\boldsymbol{\Gamma}_j^{q,\mu} = 0$. Therefore, $\left[\boldsymbol{\beta}_{(n,j,i)}\right] = [0]$. For the other side, suppose $\lambda \nmid i$. By (10), for $1 \leqslant j \leqslant \lambda$, $\boldsymbol{\beta}_{(n,j,i)}$ is a sum of the following square-free monomials of total degree i

$$\mathfrak{m}_{(n,i,\mu j+l)},\quad 0 \leqslant l \leqslant \mu - 1. \tag{13}$$

All the i variables (all of degree one) in each of the monomials in (13), have been arranged with successive indices (mod n based on φ_n in (1)). Now considering $\left[\boldsymbol{\beta}_{(n,j,i)}\right]$

in $R_{\widetilde{\mathcal{B}}}$, where all of its terms are written as a combination of monomials not in $\mathsf{in}(\widetilde{\mathcal{B}})$. For $0 \leqslant l \leqslant \mu - 1$, since $\lambda \nmid i$ each $\left[\mathfrak{m}_{(n,i,\mu j+l)}\right]$ contains different pattern of exponents for each element in the list $[X_{\overline{\kappa}+1}], \cdots, [X_n]$. It means no cancellation occurs. Thus, $\left[\boldsymbol{\beta}_{(n,j,i)}\right] \neq [0]$. □

Example 3.12. In this example, we repeatedly use the form of the generators in (2) and the forms in example 3.2. To simulate parts (*a*) and (*b*) of lemma 3.11, calculations in $R_{\widetilde{\mathcal{B}}}$ show

$$\begin{aligned}
\left[\boldsymbol{\beta}_{(16,1,4)}\right] &= \left[\sum_{l=0}^{3} \mathfrak{m}_{(16,4,4+l)}\right] \\
&= \left[\mathfrak{m}_{(16,4,4)} + \mathfrak{m}_{(16,4,5)} + \mathfrak{m}_{(16,4,6)} + \mathfrak{m}_{(16,4,7)}\right] \\
&= [X_4][X_5][X_6][X_7] + [X_5][X_6][X_7][X_8] \\
&\quad + [X_6][X_7][X_8][X_9] + [X_7][X_8][X_9][X_{10}] \\
&= (\boldsymbol{i}^3 \cdot [X_4][X_1][X_2][X_3]) + (\boldsymbol{i}^4 \cdot [X_1][X_2][X_3][X_4]) + \\
&\quad (\boldsymbol{i}^5 \cdot [X_2][X_3][X_4][X_1]) + (\boldsymbol{i}^6 \cdot [X_1][X_2][X_3][X_4]) \\
&= (\boldsymbol{i}^3 + \boldsymbol{i}^4 + \boldsymbol{i}^5 + \boldsymbol{i}^6) \cdot [X_1][X_2][X_3][X_4] \\
&= \boldsymbol{\Gamma}_1^{1,4} \cdot \left[\mathfrak{m}_{(16,4,1)}\right] = [0]
\end{aligned}$$

where $\boldsymbol{i} = \omega_4$.

To see the functionality of the proof of lemma 3.11 (in case where $\lambda \nmid i$), with almost the same strategy of calculations as above, it is not hard to see that

$$\begin{aligned}
\left[\boldsymbol{\beta}_{(16,1,3)}\right] &= \left[\sum_{l=0}^{3} \mathfrak{m}_{(16,3,4+l)}\right] \\
&= [X_4][X_5][X_6] + [X_5][X_6][X_7] + [X_6][X_7][X_8] \\
&\quad + [X_7][X_8][X_9] \\
&= \bar{\boldsymbol{i}}^7 [X_{13}][X_{14}][X_{16}] + \bar{\boldsymbol{i}}^6 [X_{13}][X_{14}][X_{15}] \\
&\quad + \bar{\boldsymbol{i}}^6 [X_{14}][X_{15}][X_{16}] + \bar{\boldsymbol{i}}^5 [X_{13}][X_{15}][X_{16}] \\
&= \boldsymbol{i}\left([X_{13}][X_{14}][X_{16}] - [X_{13}][X_{15}][X_{16}]\right) \\
&\quad - \left([X_{13}][X_{14}][X_{15}] + [X_{14}][X_{15}][X_{16}]\right)
\end{aligned}$$

where the reader can easily see that no cancellation occurs. For various examples of $\boldsymbol{\alpha}_{(16,\cdot,\cdot)}$'s consider the following table.

i	j	$\left[\boldsymbol{\alpha}_{(16,j,i)}\right]$	$\boldsymbol{\Omega}_i^4$
2	1	$\boldsymbol{\Omega}_2^4 \cdot \left[\mathfrak{m}_{(16,2,1)}\right] = [0]$	$1 + \boldsymbol{i}^2 + \boldsymbol{i}^4 + \boldsymbol{i}^6$
4	1	$\boldsymbol{\Omega}_4^4 \cdot \left[\mathfrak{m}_{(16,4,1)}\right] = 4\left[\mathfrak{m}_{(16,4,1)}\right]$	$1 + \boldsymbol{i}^4 + \boldsymbol{i}^8 + \boldsymbol{i}^{12}$
5	1	$\boldsymbol{\Omega}_5^4 \cdot \left[\mathfrak{m}_{(16,5,1)}\right] = [0]$	$1 + \boldsymbol{i}^5 + \boldsymbol{i}^{10} + \boldsymbol{i}^{15}$
2	2	$\boldsymbol{\Omega}_2^4 \cdot \left[\mathfrak{m}_{(16,2,2)}\right] = [0]$	$1 + \boldsymbol{i}^2 + \boldsymbol{i}^4 + \boldsymbol{i}^6$
4	2	$\boldsymbol{\Omega}_4^4 \cdot \left[\mathfrak{m}_{(16,4,2)}\right] = 4\left[\mathfrak{m}_{(16,4,2)}\right]$	$1 + \boldsymbol{i}^4 + \boldsymbol{i}^8 + \boldsymbol{i}^{12}$

Lemma 3.13. *For a given integer $n \geqslant 3$, let $(\lambda \leqslant \mu)_n$ be a pure split. If $1 < i < n$ and $1 \leqslant j \leqslant \lambda$, then in $R_{\mathcal{B}}$*
(*a*) *$\mu \nmid i$ if and only if $\left[\boldsymbol{\alpha}_{(n,j,i)}\right] = [0]$.*
(*b*) *If $\mu \mid i$ and $\lambda \mid \mu$, then $\left[\boldsymbol{\beta}_{(n,j,i)}\right] = [0]$. In other words, if $(\lambda \leqslant \mu, i)_n$ is a ζ−factor associated with n, then $\left[\boldsymbol{\beta}_{(n,j,i)}\right] = [0]$.*
(*c*) *If $\mu \mid i$ and $\left[\boldsymbol{\beta}_{(n,j,i)}\right] = [0]$, then $\mathfrak{p}(\lambda) \cap \mathfrak{p}(\mu) \neq \varnothing$.*

Proof. (*a*) By lemma 3.11 (*a*), $(\omega_\mu)^i \cdot \boldsymbol{\Omega}_i^\mu = (\omega_\mu)^i + \cdots + (\omega_\mu)^{\mu i} = \boldsymbol{\Omega}_i^\mu - 1 + (\omega_\mu)^{\mu i} = \boldsymbol{\Omega}_i^\mu$ since ω_μ is a primitive μ^{th} root of unity. And this implies $\boldsymbol{\Omega}_i^\mu \cdot (1 - (\omega_\mu)^i) = 0$. Now $1 - (\omega_\mu)^i \neq 0$ if and only if $\mu \nmid i$. In other words, $\boldsymbol{\Omega}_i^\mu = 0$ if and only if $\mu \nmid i$. By lemma 3.11 (*a*) this is equivalent to say $\left[\boldsymbol{\alpha}_{(n,j,i)}\right] = [0]$ if and only if $\mu \nmid i$.
(*b*) Suppose $\mu \mid i$ and $\lambda \mid \mu$. Then $\lambda \mid i$ and lemma 3.11 (*c*) give the result.
(*c*) With $\mu \mid i$, $\left[\boldsymbol{\beta}_{(n,j,i)}\right] = [0]$ and lemma 3.11 (*c*) at hand, consider corollary 2.6. □

Remark 3.14. Again we assume that n is an integer with $n \geqslant 3$ and let $(\lambda \leqslant \mu)_n$ be a pure split for n where $\lambda \mid \mu$. With the aid of lemma 3.1, lemma 3.10 and lemma 3.13 (*a*), (*b*), $[H_i] = 0$ for $i = 1, \cdots, n-1$, can be concluded. Here, $\overline{\kappa} = n - \lambda$ and the factors of $[H_n]$ (in $\widetilde{\mathcal{B}}$) can be considered as follow

$$
\begin{aligned}
[H_n] &= \left[\prod_{i=1}^{n} X_i - 1\right] = \left[\prod_{j=0}^{\mu-2} (X_{\lambda j+1} \cdots X_{\lambda j+\lambda})\right] \cdot [X_{\overline{\kappa}+1} \cdots X_n] - 1 \\
&= \left[\prod_{j=0}^{\mu-2} \left((\omega_\mu)^{\lambda(j+1)} \cdot X_{\overline{\kappa}+1} \cdots X_n\right)\right] \cdot [X_{\overline{\kappa}+1} \cdots X_n] - 1 \\
&= \prod_{j=0}^{\mu-2} \left((\omega_\mu)^{\lambda(j+1)}\right) \cdot [X_{\overline{\kappa}+1} \cdots X_n]^\mu - 1 \\
&= (\omega_\mu)^{\mu \frac{\lambda(\mu-1)}{2}} \cdot [X_{\overline{\kappa}+1} \cdots X_n]^\mu - 1 \\
&= ([X_{\overline{\kappa}+1}] \cdots [X_n])^\mu - 1 = [F_1] \cdots [F_\mu]
\end{aligned}
$$

where $F_k = X_{\overline{\kappa}+1} \cdots X_n - \omega_\mu^{k-1}$, $k = 1, \cdots, \mu$. Let's consider one of these factors say $F_1 = X_{\overline{\kappa}+1} \cdots X_n - 1$. By defining an ideal $J = I(\widetilde{\mathcal{B}}, F_1)$, the above discussion shows that $IC_n \subset J$. As in lemma 3 in [3], one may prove that J is a prime ideal. In that theorem, a rational parametrization of the underlying ideal has been used. In this case, the parametrization shows that $\dim(J) = \lambda - 1$. Thus, it is an associated prime of $\sqrt{IC_n}$ of positive dimension.

4. Conclusions

In this article, we mainly verified that the binomial polynomials in (2) satisfy the equations of the system if some conditions on n are satisfied. This is our so-called vanishing process. In remark 3.14 we considered the case where $\lambda \mid \mu$ and we presented the proof of $[H_n] = 0$, and this completes the proofs of the vanishing process for all defining polynomials of the system. Besides remark 3.14, the other parts of the proofs go through the following path. For a given $1 < i < n$, lemma 3.13 (*a*) gives the result in the case where $\mu \nmid i$ (even if $\lambda \nmid \mu$). Lemma 3.13 (*b*) gives the result in the case where $\mu \mid i$ and $\lambda \mid \mu$ (in which $(\lambda \leqslant \mu, i)_n$ is a $\zeta-$factor associated with n). If $\mu \mid i$ but $\lambda \nmid \mu$, then the proofs of the same results are still unknown to me. We anticipate that the concept of $\zeta-$factors plays an important role in the future of this research.

5. Appendix

Proof. (*Lemma* 3.6) Set $j = \xi n + r$ where $0 \leqslant r < n$ and ξ is a positive integer. Also, let $r = \eta\lambda + s$, where $0 \leqslant s < \lambda$ and η is another positive integer.

Claim 1: $0 \leqslant \eta \leqslant \mu - 1$. Suppose not and $\eta \geqslant \mu$. In this case

$$n > r = s + \eta\lambda \geqslant \lambda\mu + s > n + s,$$

a contradiction. Also, $j + t\lambda = \xi n + (\eta + t)\lambda + s$ which evidently implies

$$\varphi_n(j + t\lambda) = \varphi_n(\xi n + (\eta + t)\lambda + s) = \varphi_n((\eta + t)\lambda + s). \tag{14}$$

(6) is trivially hold for $t = 0$. Consider $t > 0$.

Case $(\boldsymbol{i})$ **:** $(r = 0)$

In this case, $j + t\lambda = \xi n + t\lambda$

$$\begin{aligned}
\left[X_{\varphi_n(j+t\lambda)}\right] &= [X_{t\lambda}] = \left[X_{(t-1)\lambda+\lambda}\right] \\
&= (\omega_\mu)^t \left[X_{\overline{\kappa}+\lambda}\right] (\text{ use } (5) \text{ with } \eta = t-1 \text{ and } j = \lambda) \\
&= (\omega_\mu)^t [X_n] (\text{ since } \overline{\kappa} + \lambda = n) \\
&= (\omega_\mu)^t \left[X_{\varphi_n(j)}\right]
\end{aligned}$$

where with $r = 0$ and $j = \xi n$, $\varphi_n(j) = n$.

Case $(\boldsymbol{ii})$ **:** $(r > 0)$ In this case, since by *Claim 1* , $0 \leqslant \eta \leqslant \mu - 1$, then

$$s < t\lambda + s \leqslant (\eta + t)\lambda + s \leqslant (\mu - 1 + t)\lambda + s = n + (t - 1)\lambda + s.$$

On the other hand, by the condition given in the statement of the lemma, $0 \leqslant t-1 \leqslant \mu - 2$ then $(t-1)\lambda + s \leqslant \lambda(\mu - 2) + s = \lambda\mu - 2\lambda + s < n - \lambda = \overline{\kappa} < n$ (since $n = \lambda\mu$).

$(\boldsymbol{ii} - \mathbf{1})$ **:** $(\eta + t)\lambda + s = n + (t - 1)\lambda + s$

After simplifying the above condition one gets $(\eta + 1)\lambda = n$ and in turn by (14)

$$\begin{aligned}
\varphi_n(j + t\lambda) &= \varphi_n((\eta + t)\lambda + s) = \varphi_n((\eta + 1)\lambda + (t - 1)\lambda + s) \\
&= \varphi_n(n + (t - 1)\lambda + s) = \varphi_n((t - 1)\lambda + s).
\end{aligned}$$

Thus, $\left[X_{\varphi_n(j+t\lambda)}\right] = \left[X_{(t-1)\lambda+s}\right] = (\omega_\mu)^{(t-1)+1}\left[X_{\overline{\kappa}+s}\right]$. Since $(\eta + t)\lambda = n + (t - 1)\lambda$, then $\eta\lambda = n - \lambda = \overline{\kappa}$. Hence,

$$\left[X_{\varphi_n(j+t\lambda)}\right] = (\omega_\mu)^t \left[X_{\overline{\kappa}+s}\right] = (\omega_\mu)^t \left[X_{\eta\lambda+s}\right] = (\omega_\mu)^t [X_r] = (\omega_\mu)^t \left[X_{\varphi_n(j)}\right].$$

$(\boldsymbol{ii} - \mathbf{2})$ **:** $(\eta + t)\lambda + s = n$

The condition implies $(\mu - \eta - t)\lambda = s < \lambda$ which means $s = 0$ or $\mu = \eta + t$. Now

$$\begin{aligned}
\left[X_{\varphi_n(j+t\lambda)}\right] &= \left[X_{(\eta+t)\lambda+s}\right] = [X_n] = \left[X_{\overline{\kappa}+\lambda}\right] \\
&= (\omega_\mu)^{-\eta}\left[X_{(\eta-1)\lambda+\lambda}\right] = (\omega_\mu)^{t-\mu}[X_{\eta\lambda}] = (\omega_\mu)^{t-\mu}[X_r] \\
&= (\omega_\mu)^t \left[X_{\varphi_n(j)}\right]
\end{aligned}$$

since $(\omega_\mu)^{-\mu} = ((\omega_\mu)^\mu)^{-1} = 1$.

$(\boldsymbol{ii} - \mathbf{3})$ **:** $(\eta + t)\lambda + s < n$

In this case, since $(\eta + t)\lambda + s < \lambda\mu$, then $\eta + t \leqslant \mu - 1$.

(a) If $\eta + t = \mu - 1$, then

$$\begin{aligned}
\left[X_{\varphi_n(j+t\lambda)}\right] &= \left[X_{(\eta+t)\lambda+s}\right] = \left[X_{(\mu-1)\lambda+s}\right] = \left[X_{\overline{\kappa}+s}\right] = (\omega_\mu)^{-\eta-1}\left[X_{\eta\lambda+s}\right] \\
&= (\omega_\mu)^{t-\mu}[X_r] = (\omega_\mu)^t \left[X_{\varphi_n(j)}\right].
\end{aligned}$$

(b) If $\eta + t \leqslant \mu - 2$, then

$$\begin{aligned}
\left[X_{\varphi_n(j+t\lambda)}\right] &= \left[X_{(\eta+t)\lambda+s}\right] = (\omega_\mu)^{\eta+t+1}\left[X_{\overline{\kappa}+s}\right] = (\omega_\mu)^t \cdot (\omega_\mu)^{\eta+1}\left[X_{\overline{\kappa}+s}\right] \\
&= (\omega_\mu)^t \cdot \left[X_{\eta\lambda+s}\right] = (\omega_\mu)^t \left[X_{\varphi_n(j)}\right].
\end{aligned}$$

$(\boldsymbol{ii} - \mathbf{4})$ **:** $n < (\eta + t)\lambda + s < n + (t - 1)\lambda + s$

This case also will be treated similarly. There is an s_1 with, $0 \leqslant s_1 < \lambda$, a positive integer ψ such that $(\eta + t)\lambda + s = \lambda\mu + \psi\lambda + s_1$. Notice the above implies $s = s_1$ and $\eta + t = \psi + \mu$. Now

$$\begin{aligned} \left[X_{\varphi_n(j+t\lambda)}\right] &= \left[X_{\varphi_n((\eta+t)\lambda+s)}\right] = [X_{\psi\lambda+s}] = (\omega_\mu)^{\psi+1}[X_{\overline{\kappa}+s}] \\ &= (\omega_\mu)^t \cdot (\omega_\mu)^{\eta+1}[X_{\overline{\kappa}+s}] = (\omega_\mu)^t [X_{\eta\lambda+s}] = (\omega_\mu)^t [X_r] \\ &= (\omega_\mu)^t \left[X_{\varphi_n(j)}\right]. \end{aligned}$$

□

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