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BERNSTEIN-TYPE INEQUALITIES PRESERVED BY MODIFIED SMIRNOV OPERATOR

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Abstract. In this paper we consider a modified version of Smirnov operator and obtain some Bernstein-type inequalities preserved by this operator. In particular, we prove some results which in turn provide the compact generalizations of some well-known inequalities for polynomials.

1. Introduction

Let $\mathbb{P}_n$ denote the class of polynomials $f(z) = \sum_{j=0}^{n} a_j z^j$ in $\mathbb{C}$ of degree atmost $n \in \mathbb{N}$. Let $\mathbb{D}$ be the open unit disk $\{z \in \mathbb{C}; |z| < 1\}$, so that $\overline{\mathbb{D}}$ is its closure and $\delta\mathbb{D}$ denotes the boundary. For any polynomial $f \in \mathbb{P}_n$, we have the following result due to Bernstein [3].

Theorem 1.1. *Let $f \in \mathbb{P}_n$, then*

$$\max_{z\in\delta\mathbb{D}} |f^{'}(z)| \leq n \max_{z\in\delta\mathbb{D}} |f(z)|. \tag{1}$$

The result is best possible and equality holds for the polynomials having zeros at the origin.

Aziz and Dawood proved that if $f(z)$ has all its zeros in $\overline{\mathbb{D}}$, then

$$\min_{z\in\delta\mathbb{D}} |f^{'}(z)| \geq n \min_{z\in\delta\mathbb{D}} |f(z)| \tag{2}$$

and for $R \geq 1$

$$\min_{z\in\delta\mathbb{D}} |f(Rz)| \geq R^n \min_{z\in\delta\mathbb{D}} |f(z)|. \tag{3}$$

Inequalities (2) and (3) are sharp and equality holds for the polynomials having all zeros at the origin.

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For the class of polynomials having no zeros in $\mathbb{D}$, inequality (1.1) can be sharpened. In fact , if $f(z) \neq 0$ in $\mathbb{D}$, then

$$\max_{z\in\delta\mathbb{D}} |f^{'}(z)| \leq \frac{n}{2} \max_{z\in\delta\mathbb{D}} |f(z)| \tag{4}$$

and for $R > 1$,

$$\max_{z\in\delta\mathbb{D}} |f(Rz)| \leq \left(\frac{R^n+1}{2}\right) \max_{z\in\delta\mathbb{D}} |f(z)|. \tag{5}$$

Inequality (4) was conjectured by Erdös and later verified by Lax [8], whereas Ankeny and Rivilin [1] used (4) to prove (5). Inequalities (4) and (5) were further improved by Aziz and Dawood [2], where under the same hypothesis, it was shown that

$$\max_{z\in\delta\mathbb{D}} |f^{'}(z)| \leq \frac{n}{2}\left\{ \max_{z\in\delta\mathbb{D}} |f(z)| - \min_{z\in\delta\mathbb{D}} |f(z)|\right\} \tag{6}$$

and for $R > 1$

$$\max_{z\in\delta\mathbb{D}} |f(Rz)| \leq \left(\frac{R^n+1}{2}\right) \max_{z\in\delta\mathbb{D}} |f(z)| - \left(\frac{R^n-1}{2}\right) \min_{z\in\delta\mathbb{D}} |f(z)|. \tag{7}$$

Equality in (4)-(7) holds for the polynomials of the form $f(z) = \alpha z^n + \beta$, with $|\alpha| = |\beta|$. In 1930 Bernstein [4] also proved the following result:

Theorem 1.2. *Let $F(z)$ be a polynomial in $\mathbb{P}_n$ having all zeros in $\overline{\mathbb{D}}$ and $f(z)$ be a polynomial of degree not exceeding that of $F(z)$. If $|f(z)| \leq |F(z)|$ on $\delta\mathbb{D}$, then*

$$|f^{'}(z)| \leq |F^{'}(z)| \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{D}.$$

Equality holds only if $f = e^{i\gamma}F, \gamma \in \mathbb{R}$.

For $z \in \mathbb{C} \setminus \mathbb{D}$, denoting by $\Omega_{|z|}$ the image of the disc $\{t \in \mathbb{C}; |t| \leq |z|\}$ under the mapping $\psi(t) = \dfrac{t}{1+t}$, Smirnov [9] as a generalization of Theorem 1.2 proved the following:

Theorem 1.3. *Let f and F be polynomials possessing conditions as in Theorem 1.2. Then for $z \in \mathbb{C} \setminus \mathbb{D}$*

$$|\mathbb{S}_\alpha[f](z)| \leq |\mathbb{S}_\alpha[F](z)| \tag{8}$$

for all $\alpha \in \overline{\Omega_{|z|}}$, with $\mathbb{S}_\alpha[f](z) := zf^{'}(z) - n\alpha f(z)$, where α is a constant.
For $\alpha \in \overline{\Omega_{|z|}}$ in (8) equality holds at a point $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ only if $f = e^{i\gamma}F, \gamma \in \mathbb{R}$.

We note that for fixed $z \in \mathbb{C} \setminus \mathbb{D}$, (8) can be replaced by (see for reference [6])

$$|zf^{'}(z) - n\frac{az}{1+az}f(z)| \leq |zF^{'}(z) - n\frac{az}{1+az}F(z)|,$$

where a is arbitrary number from $\overline{\mathbb{D}}$.
Equivalently for $z \in \mathbb{C} \setminus \mathbb{D}$

$$|\tilde{\mathbb{S}}_a[f](z)| \leq |\tilde{\mathbb{S}}_a[F](z)|$$

where $\tilde{\mathbb{S}}_a[f](z) = (1+az)f^{'}(z) - naf(z)$ is known as modified Smirnov operator.
The modified Smirnov operator $\tilde{\mathbb{S}}_a$ is more preferred in a sense than Smirnov operator $\mathbb{S}_\alpha$, because the parameter a of $\tilde{\mathbb{S}}_a$ does not depend on z unlike parameter α of $\mathbb{S}_\alpha$.

2. Main Results

Before writing our main results, we prove the following lemmas which are required for their proofs.

LEMMA 2.1. *Let $F \in \mathbb{P}_n$, and has all zeros in $\overline{\mathbb{D}}$. Let $a \in \delta\mathbb{D}$ be not the exceptional value for F. Then all zeros of $\tilde{\mathbb{S}}_a[F]$ lie in $\overline{\mathbb{D}}$.*

The above lemma is due to Ganenkova and Starkov [6].

LEMMA 2.2. *If $f \in \mathbb{P}_n$, such that $f(z) \neq 0$ in $\mathbb{D}$, then*

$$|\tilde{\mathbb{S}}_a[f](z)| \leq |\tilde{\mathbb{S}}_a[g](z)| \quad \textit{for} \quad z \in \mathbb{C} \setminus \mathbb{D}, \tag{9}$$

where $g(z) = z^n\overline{f(\frac{1}{\bar{z}})}$.

Proof. Since $g(z) = z^n\overline{f(\frac{1}{\bar{z}})}$, therefore $|g(z)| = |f(z)|$ for $z \in \delta\mathbb{D}$, and hence $\dfrac{g(z)}{f(z)}$ is analytic in $\overline{\mathbb{D}}$. By Maximum Modulus Principle, we have

$$|g(z)| \leq |f(z)| \text{ for } z \in \overline{\mathbb{D}}.$$

Or equivalently,

$$|f(z)| \leq |g(z)| \text{ for } z \in \mathbb{C} \setminus \mathbb{D}.$$

Therefore for every β with $|\beta| > 1$, the polynomial $f(z) - \beta g(z)$ has all zeros in $\mathbb{C} \setminus \mathbb{D}$. By Lemma 2.1, $\tilde{\mathbb{S}}_a[f - \beta g](z)$ has all its zeros in $\overline{\mathbb{D}}$. Since $\tilde{\mathbb{S}}_a$ is linear, therefore $\tilde{\mathbb{S}}_a[f](z) - \beta\tilde{\mathbb{S}}_a[g](z)$ has all its zeros in $\overline{\mathbb{D}}$, which in particular gives

$$|\tilde{\mathbb{S}}_a[f](z)| \leq |\tilde{\mathbb{S}}_a[g](z)| \text{ for } z \in \mathbb{C} \setminus \mathbb{D}.$$

Because, if this is not true, then there exists some z_0 with $z_0 \in \mathbb{C} \setminus \mathbb{D}$, such that

$$|\tilde{\mathbb{S}}_a[f](z_0)| > |\tilde{\mathbb{S}}_a[g](z_0)|.$$

Choosing $\beta = \dfrac{\tilde{\mathbb{S}}_a[f](z_0)}{\tilde{\mathbb{S}}_a[g](z_0)}$, so that $|\beta| > 1$. For this value of β, $\tilde{\mathbb{S}}_a[f](z) - \beta\tilde{\mathbb{S}}_a[g](z) = 0$ for some $z = z_0 \in \mathbb{C} \setminus \mathbb{D}$, which is a contradiction. Therefore

$$|\tilde{\mathbb{S}}_a[f](z)| \leq |\tilde{\mathbb{S}}_a[g](z)| \text{ for } z \in \mathbb{C} \setminus \mathbb{D}.$$

□

LEMMA 2.3. *If $f \in \mathbb{P}_n$ with $|f(z)| \leq \mathbb{M}$ for $z \in \delta\mathbb{D}$. Then*

$$|\tilde{\mathbb{S}}_a[f](z)| \leq \mathbb{M}|\tilde{\mathbb{S}}_a[z^n]| \quad \textit{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Proof. Since $|f(z)| \leq \mathbb{M}$ for $z \in \delta\mathbb{D}$. If λ is a complex number with $|\lambda| > 1$. Then

$$|f(z)| < |\lambda \mathbb{M} z^n| \quad \text{for} \quad z \in \delta\mathbb{D}.$$

Since $\lambda\mathbb{M}z^n$ has all zeros in $\overline{\mathbb{D}}$, therefore by Rouche's theorem all zeros of $f(z) - \lambda\mathbb{M}z^n$ lie in $\overline{\mathbb{D}}$. Hence by Lemma 2.1, all zeros of $\tilde{\mathbb{S}}_a[f(z) - \lambda\mathbb{M}z^n]$ lie in $\overline{\mathbb{D}}$. Since $\tilde{\mathbb{S}}_a$ is linear, it follows that $\tilde{\mathbb{S}}_a[f](z) - \tilde{\mathbb{S}}_a[\lambda\mathbb{M}z^n]$ has all zeros in $\overline{\mathbb{D}}$.
This gives

$$|\tilde{\mathbb{S}}_a[f](z)| \leq \mathbb{M}|\tilde{\mathbb{S}}_a[z^n]| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}. \tag{10}$$

Because if this is not true, then there exists some $z_0 \in \mathbb{C} \setminus \mathbb{D}$, such that

$$|\tilde{\mathbb{S}}_a[f](z_0)| > \mathbb{M}|\tilde{\mathbb{S}}_a[z_0^n]|.$$

Choosing $\lambda = \dfrac{\tilde{\mathbb{S}}_a[f](z_0)}{\mathbb{M}\tilde{\mathbb{S}}_a[z_0^n]}$, so that $|\lambda| > 1$. With this choice of λ, we get a contradiction and hence (10) is true. □

Lemma 2.4. *If $f \in \mathbb{P}_n$, then for $z \in \mathbb{C} \setminus \mathbb{D}$*

$$|\tilde{\mathbb{S}}_a[f](z)| + |\tilde{\mathbb{S}}_a[g](z)| \leq \{|\tilde{\mathbb{S}}_a[z^n]| + n|a|\} \max_{|z|=1} |f(z)|, \tag{11}$$

where $g(z) = z^n \overline{f(\frac{1}{\bar{z}})}$.

Proof. Let $\mathbb{M} = \max_{z \in \delta\mathbb{D}} |f(z)|$, then $|f(z)| \leq \mathbb{M}$ for $z \in \overline{\mathbb{D}}$.
If λ is any real or complex number with $|\lambda| > 1$, then by Rouche's theorem

$$P(z) = f(z) - \lambda\mathbb{M}$$

does not vanish in $\overline{\mathbb{D}}$. Hence by Lemma 2.2

$$|\tilde{\mathbb{S}}_a[P](z)| \leq |\tilde{\mathbb{S}}_a[Q](z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D},$$

where

$$\begin{aligned} Q(z) &= z^n \overline{P(\frac{1}{\bar{z}})} \\ &= z^n \overline{f\left(\frac{1}{\bar{z}}\right)} - z^n \lambda\mathbb{M} \\ &= g(z) - \lambda\mathbb{M}z^n. \end{aligned}$$

That is

$$|\tilde{\mathbb{S}}_a[f](z) - \mathbb{M}\lambda\tilde{\mathbb{S}}_a[1]| \leq |\tilde{\mathbb{S}}_a[g](z) - \mathbb{M}\lambda\tilde{\mathbb{S}}_a[z^n]| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Using the fact $\tilde{\mathbb{S}}_a[1] = -na$, we get

$$|\tilde{\mathbb{S}}_a[f](z) - \mathbb{M}\lambda(-na)| \leq |\tilde{\mathbb{S}}_a[g](z) - \mathbb{M}\lambda\tilde{\mathbb{S}}_a[z^n]| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

This gives

$$|\tilde{\mathbb{S}}_a[f](z)| - |na\mathbb{M}\lambda| \leq |\tilde{\mathbb{S}}_a[g](z) - \mathbb{M}\lambda\tilde{\mathbb{S}}_a[z^n]| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Choosing argument of λ suitably, which is possible by Lemma 2.3, we get

$$|\tilde{\mathbb{S}}_a[f](z)| - n\mathbb{M}|a||\lambda| \leq \mathbb{M}|\lambda||\tilde{\mathbb{S}}_a[z^n]| - |\tilde{\mathbb{S}}_a[g](z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Making $|\lambda| \to 1$, we get

$$|\tilde{\mathbb{S}}_a[f](z)| + |\tilde{\mathbb{S}}_a[g](z)| \leq \{n|a| + |\tilde{\mathbb{S}}_a[z^n]\}\mathbb{M}.$$

This proves Lemma 2.4. □

We now prove the following result which is a compact generalization of inequalities (2) and (3).

Theorem 2.5. *If $f \in \mathbb{P}_n$ with $f(z) \neq 0$ in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Then*

$$|\tilde{\mathbb{S}}_a[f](z)| \geq |\tilde{\mathbb{S}}_a[z^n]| \min_{z \in \delta\mathbb{D}} |f(z)|. \tag{12}$$

Equivalently

$$|(1+az)f^{'}(z) - naf(z)| \geq n|z|^{n-1} \min_{z \in \delta\mathbb{D}} |f(z)|. \tag{13}$$

The result is best possible and equality holds for the polynomial $f(z) = cz^n; |c| \neq 0$.

Proof. If $f(z)$ has a zero on $\delta\mathbb{D}$, then there is nothing to prove as $\min_{z \in \delta\mathbb{D}} |f(z)| = 0$. Suppose all zeros of $f(z)$ lie in $\mathbb{D}$, then $\min_{z \in \delta\mathbb{D}} |f(z)| = m > 0$ and we have

$$m \leq |f(z)| \quad \text{for} \quad z \in \delta\mathbb{D}.$$

Equivalently for every λ with $|\lambda| < 1$, we have

$$|m\lambda z^n| < |f(z)| \quad \text{for} \quad z \in \delta\mathbb{D}. \tag{14}$$

Therefore by Rouche's theorem it follows that all zeros of $f(z) - \lambda m z^n$ lie in $\mathbb{D}$. This gives by Lemma 2.1 that all the zeros of $\tilde{\mathbb{S}}_a[f(z) - \lambda m z^n]$ and hence $\tilde{\mathbb{S}}_a[f](z) - m\lambda \tilde{\mathbb{S}}_a[z^n]$ lie in $\mathbb{D}$.
This implies

$$m|\tilde{\mathbb{S}}_a[z^n]| \leq |\tilde{\mathbb{S}}_a[f](z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}. \tag{15}$$

Because if this is not true then there exists a point $z_0 \in \mathbb{C} \setminus \mathbb{D}$, such that

$$m|\tilde{\mathbb{S}}_a[z_0^n]| > |\tilde{\mathbb{S}}_a[f](z_0)|.$$

We take $\lambda = \dfrac{\tilde{\mathbb{S}}_a[f](z_0)}{m\tilde{\mathbb{S}}_a[z_0^n]}$, so that $|\lambda| < 1$. For this value of λ, $\tilde{\mathbb{S}}_a[f](z) - m\lambda \tilde{\mathbb{S}}_a[z^n] = 0$ for some $z = z_0 \in \mathbb{C} \setminus \mathbb{D}$. This is a contradiction and hence we conclude

$$|\tilde{\mathbb{S}}_a[f](z)| \geq |\tilde{\mathbb{S}}_a[z^n]| \min_{z \in \delta\mathbb{D}} |f(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}. \tag{16}$$

This completes proof of Theorem 2.5. □

Remark 2.6. If we choose $a = 0$ in (13), we get

$$|f^{'}(z)| \geq n|z|^{n-1} \min_{z \in \delta\mathbb{D}} |f(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

This in particular gives inequality (2).
Next choosing $a = -\frac{1}{z}$ in inequality (13), we get for $z \in \mathbb{C} \setminus \mathbb{D}$

$$|f(z)| \geq |z|^n \min_{z \in \delta\mathbb{D}} |f(z)|.$$

Taking in particular $z = Re^{i\theta}$, $0 \leq \theta < 2\pi$, $R \geq 1$, we get for $z \in \delta\mathbb{D}$

$$|f(Rz)| \geq R^n \min_{z \in \delta\mathbb{D}} |f(z)|,$$

which is equivalent to (3).

The next result we prove, gives a compact generalization of inequalities (4) and (5).

THEOREM 2.7. *If $f \in \mathbb{P}_n$, with $f(z) \neq 0$ in $\mathbb{D}$. Then for $z \in \mathbb{C} \setminus \mathbb{D}$*

$$|\tilde{\mathbb{S}}_a[f](z)| \leq \frac{1}{2}\{|\tilde{\mathbb{S}}_a[z^n]| + n|a|\} \max_{z\in\delta\mathbb{D}} |f(z)|. \tag{17}$$

Or, equivalently

$$|(1+az)f^{'}(z) - naf(z)| \leq \frac{1}{2}\{n|z|^{n-1} + n|a|\} \max_{z\in\delta\mathbb{D}} |f(z)|. \tag{18}$$

The result is best possible and equality holds for the polynomials having all zeros on unit disk.

Proof. Note that $f(z)$ is a polynomial not vanishing inside $\mathbb{D}$. Therefore, if $g(z) = z^n\overline{f(\frac{1}{\bar{z}})}$, then by Lemma 2.2

$$2|\tilde{\mathbb{S}}_a[f](z)| \leq |\tilde{\mathbb{S}}_a[f](z)| + |\tilde{\mathbb{S}}_a[g](z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Using Lemma 2.4, we get

$$\begin{aligned} 2|\tilde{\mathbb{S}}_a[f](z)| &\leq |\tilde{\mathbb{S}}_a[f](z)| + |\tilde{\mathbb{S}}_a[g](z)| \\ &\leq \{n|a| + |\tilde{\mathbb{S}}_a[z^n]|\} \max_{z\in\delta\mathbb{D}} |f(z)|. \end{aligned}$$

That is

$$|\tilde{\mathbb{S}}_a[f](z)| \leq \frac{1}{2}\{|\tilde{\mathbb{S}}_a[z^n]| + n|a|\} \max_{z\in\delta\mathbb{D}} |f(z)|. \tag{19}$$

This proves Theorem 2.7. □

REMARK 2.8. If we choose $a = 0$ in inequality (18), we get

$$|f^{'}(z)| \leq \frac{n}{2}|z|^{n-1} \max_{z\in\delta\mathbb{D}} |f(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Choosing $a = -\frac{1}{z}$ in (18), we get

$$|f(z)| \leq \frac{1}{2}(|z|^n + 1) \max_{z\in\delta\mathbb{D}} |f(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Taking in particular $z = Re^{i\theta}$, $0 \leq \theta < 2\pi$, so that $|z| = R \geq 1$, we get for $z \in \delta\mathbb{D}$

$$|f(Rz)| \leq \frac{R^n + 1}{2} \max_{z\in\delta\mathbb{D}} |f(z)|.$$

As a refinement of Theorem 2.7, we next prove the following result which is a compact generalization of inequalities (6) and (7).

THEOREM 2.9. *If $f \in \mathbb{P}_n$ such that $f(z) \neq 0$ for $z \in \mathbb{D}$. Then for $z \in \mathbb{C} \setminus \mathbb{D}$*

$$|\tilde{\mathbb{S}}_a[f](z)| \leq \frac{1}{2}\{|\tilde{\mathbb{S}}_a[z^n]| + n|a|\} \max_{z\in\delta\mathbb{D}} |f(z)| - \{|\tilde{\mathbb{S}}_a[z^n]| - n|a|\} \min_{z\in\delta\mathbb{D}} |f(z)|. \tag{20}$$

Equivalently

$$|(1+az)f^{'}(z) - naf(z)| \leq \frac{1}{2}\{n|z|^{n-1} + n|a|\} \max_{z\in\delta\mathbb{D}} |f(z)| - \frac{1}{2}\{n|z|^{n-1} - n|a|\} \min_{z\in\delta\mathbb{D}} |f(z)|. \tag{21}$$

The result is best possible and equality holds for the polynomials having all zeros on unit disk.

Proof. If $f(z)$ has a zero on $\delta\mathbb{D}$, then $m = 0$ and the result follows from Theorem 2.7. We suppose that all the zeros of $f(z)$ lie in $\mathbb{C} \setminus \overline{\mathbb{D}}$, so that $m > 0$ and

$$m \leq |f(z)| \quad \text{for} \quad z \in \delta\mathbb{D}.$$

Therefore for every complex number β with $|\beta| < 1$, we have $|f(z)| > m|\beta|$. Hence by Rouche's theorem all zeros of $F(z) = f(z) - m\beta$ lie in $\mathbb{C} \setminus \overline{\mathbb{D}}$. We note that $F(z)$ has no zeros on $\delta\mathbb{D}$, because if for some $z = z_0$, with $z_0 \in \delta\mathbb{D}$ is a zero of $F(z)$, then

$$F(z_0) = f(z_0) - m\beta = 0.$$

This gives $|f(z_0)| = m|\beta| < m$, a contradiction.
Now if $G(z) = z^n \overline{F\left(\frac{1}{\overline{z}}\right)} = z^n \overline{f\left(\frac{1}{\overline{z}}\right)} - \overline{\beta} m z^n = g(z) - \overline{\beta} m z^n$, then all zeros of $G(z)$ lie in $\mathbb{D}$ and $|G(z)| = |F(z)|$ for $z \in \delta\mathbb{D}$. Therefore for every γ with $|\gamma| > 1$, the polynomial $F(z) - \gamma G(z)$ has all its zeros in $\mathbb{D}$. This gives by Lemma 2.1 all zeros of $\tilde{\mathbb{S}}_a[F(z) - \gamma G(z)]$ and hence $\tilde{\mathbb{S}}_a[F](z) - \gamma \tilde{\mathbb{S}}_a[G](z)$ lie in $\mathbb{D}$.
From this as before we conclude

$$|\tilde{\mathbb{S}}_a[F](z)| \leq |\tilde{\mathbb{S}}_a[G](z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Substituting for $F(z)$ and $G(z)$ and making use of the fact that $\tilde{\mathbb{S}}_a$ is linear and $\tilde{\mathbb{S}}_a[1] = -na$, we get

$$|\tilde{\mathbb{S}}_a[f](z) - m\beta(-na)| \leq |\tilde{\mathbb{S}}_a[g](z) - \overline{\beta} m \tilde{\mathbb{S}}_a[z^n]| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Choosing argument of β on right hand side suitably which is possible by Lemma 2.3 and making $|\beta| \to 1$, we get

$$|\tilde{\mathbb{S}}_a[f](z)| - n|a|m \leq |\tilde{\mathbb{S}}_a[g](z)| - m|\tilde{\mathbb{S}}_a[z^n]| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

This gives

$$|\tilde{\mathbb{S}}_a[f](z)| \leq |\tilde{\mathbb{S}}_a[g](z)| - \{\tilde{\mathbb{S}}_a[z^n]| - n|a|\}m \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}. \tag{22}$$

Inequality (22) along with Lemma 2.4, yields for $z \in \mathbb{C} \setminus \mathbb{D}$

$$\begin{aligned} 2|\tilde{\mathbb{S}}_a[f](z)| &\leq |\tilde{\mathbb{S}}_a[f](z)| + |\tilde{\mathbb{S}}_a[g](z)| - \{|\tilde{\mathbb{S}}_a[z^n]| - n|a|\}m \\ &\leq \{|\tilde{\mathbb{S}}_a[z^n]| + n|a|\} \max_{z \in \delta\mathbb{D}} |f(z)| - \{|\tilde{\mathbb{S}}_a[z^n]| - n|a|\} \min_{z \in \delta\mathbb{D}} |f(z)|. \end{aligned}$$

This proves Theorem 2.9 completely. □

Remark 2.10. Taking $a = 0$ in inequality (21), we get inequality (6) and if we take $a = -\frac{1}{z}$ in (21), we get inequality (7).

Definition 2.11. A polynomial $f \in \mathbb{P}_n$ is said to be a self-inversive polynomial, if $f(z) \equiv ug(z)$, where $u \in \delta\mathbb{D}$, and $g(z) = z^n \overline{f(\frac{1}{\overline{z}})}$.

Theorem 2.12. *If $f(z)$ is a self-inversive polynomial of degree n, then for $z \in \mathbb{C} \setminus \mathbb{D}$*

$$|\tilde{\mathbb{S}}_a[f](z)| \leq \frac{1}{2}\{|\tilde{\mathbb{S}}_a[z^n]| + n|a|\} \max_{z \in \delta\mathbb{D}} |f(z)|. \tag{23}$$

Equivalently

$$|(1 + az)f'(z) - naf(z)| \leq \frac{1}{2}\{n|z|^{n-1} + n|a|\} \max_{z \in \delta\mathbb{D}} |f(z)|. \tag{24}$$

The result is sharp and equality holds for the polynomial $f(z) = z^n + 1$.

Proof. Since $f(z)$ is a self-inversive polynomial. Therefore, we have

$$f(z) = g(z) = z^n \overline{f\left(\frac{1}{\overline{z}}\right)}.$$

Equivalently

$$\tilde{\mathbb{S}}_a[f](z) = \tilde{\mathbb{S}}_a[g](z).$$

Therefore by Lemma 2.4, we have

$$\begin{aligned} 2|\tilde{\mathbb{S}}_a[f](z)| &= |\tilde{\mathbb{S}}_a[f](z)| + |\tilde{\mathbb{S}}_a[g](z)| \\ &\leq \{|\tilde{\mathbb{S}}_a[z^n]| + n|a|\} \max_{z\in\delta\mathbb{D}} |f(z)|, \end{aligned}$$

from which the desired result follows. □

Remark 2.13. If we choose $a = 0$ in inequality (24), we get

$$|f'(z)| \leq \frac{n}{2}|z|^{n-1} \max_{z\in\delta\mathbb{D}} |f(z)| \quad \text{for} \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Next choosing $a = -\frac{1}{z}$ in (24), we obtain the following

Corollary 2.14. *If $f \in \mathbb{P}_n$ is a self-inversive polynomial, then for $z \in \mathbb{C} \setminus \mathbb{D}$*

$$|f(z)| \leq \frac{|z|^n + 1}{2} \max_{z\in\delta\mathbb{D}} |f(z)|.$$

The result is best possible and equality holds for polynomial $f(z) = z^n + 1$.

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References

[1] N. C. Ankeny and T. J. Rivlin, *On a theorem of S. Bernstein*, Pacific J. Math. **5** (1955), 849–852.
[2] A. Aziz and Q. M. Dawood, *Inequalities for a polynomial and its derivative*, J. Approx. Theory, **54** (1988), 306–313.
[3] S. N. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donńe, Memoires de l'Academie Royals de Belgique* **4** (1912), 1–103.
[4] S. Bernstein, *Sur la limitation des derivees des polynomes*, C. R. Acad. Sci. Paris. **190** (1930), 338–340.
[5] E. G. Ganenkova and V. V. Starkov, *The Möbius Transformation and Smirnov's Inequality for Polynomials*, Mathematical Notes **2** (2019), 216–226.
[6] E. G. Ganenkova and V. V. Starkov, *Variations on a theme of the Marden and Smirnov operators, differential inequalities for polynomials*, J. Math. Anal. Appl. **476** (2019), 696–714.
[7] E. Kompaneets and V. Starkov, *Generalization of Smirnov Operator and Differential inequalities for polynomials, Lobachevskii Journal of Mathematics*, **40**, (2019), 2043–2051.
[8] P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513.
[9] V. I. Smirnov and N. A. Lebedev, *Constructive theory of functions of a complex variable*, (Nauka, Moscow,1964) [Russian].

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