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CONDITIONAL FORUIER-FEYNMAN TRANSFORM AND CONVOLUTION PRODUCT FOR A VECTOR VALUED CONDITIONING FUNCTION

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Abstract. Let $C_0[0,T]$ denote the Wiener space, the space of continuous functions $x(t)$ on $[0,T]$ such that $x(0)=0$. Define a random vector $Z_{\vec{e},k}: C_0[0,T] \to \mathbb{R}^k$ by

$$Z_{\vec{e},k}(x) = (\int_0^T e_1(t)dx(t), \ldots, \int_0^T e_k(t)dx(t))$$

where $e_j \in L_2[0,T]$ with $e_j \neq 0$ a.e., $j = 1, \ldots, k$. In this paper we study the conditional Fourier-Feynman transform and a conditional convolution product for a cylinder type functionals defined on $C_0[0,T]$ with a general vector valued conditioning functions $Z_{\vec{e},k}$ above which need not depend upon the values of x at only finitely many points in $(0,T]$ rather than a conditioning function $X(x) = (x(t_1), \ldots, x(t_n))$ where $0 < t_1 < \ldots < t_n = T$. In particular we show that the conditional Fourier-Feynman transform of the conditional convolution product is the product of conditional Fourier-Feynman transforms.

1. Introduction

Let $C_0[0,T]$ denote the Wiener space, the space of real valued continuous functions x on $[0,T]$ such that $x(0)=0$. Let $z_h(x,t) = \int_0^t h(s)dx(s)$ be the Gaussian process with $h(\neq 0$ a.e.$)$ in $L_2[0,T]$ and the integral $\int_0^t h(s)dx(s)$ denote the Paley-Wiener-Zygmund stochastic integral [3, 9].

Note that z_h is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0,T]} z_h(x,t)z_h(x,s)dm(x) = \int_0^{min\{s,t\}} h^2(u)du$$

where the left hand side of above denotes the Wiener integral. Of course if $h \equiv 1$ on $[0,T]$, then $z_h(x,t) = x(t)$ is the standard Wiener process. For convenience, throughout this paper, we will let $z_h(x,T) = z_h(x)$.

In [7], the authors condider a general vector valued conditioning functions to study the conditional integral transforms and convolutions. In [1], Cameron and Storvick defined a Fourier-Feynman transform of functionals on $C_0[0,T]$. Chung and Skoug

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introduced the concept of a conditional Feynman integral [5], while Park and Skoug introduced the concept of a conditional Fourier-Feynman transforms and a conditional convolution for functionals defined on $C_0[0,T]$ with the conditioning function $X(x) = \int_0^T h(s)dx(s)$ where $h \in L_2[0,T]$ [10]. In [2], Chang, et al. developed the conditional Fourier Feynman transform and convolution product over Wiener paths in abstract Wiener space, concerned an conditioning funcion which depend upon the value of x at some finitely many points in $(0,T]$; that is, $X_k : C_0[0,T] \to \mathbb{R}^k$ with $X_k(x) = (x(t_1),\ldots,x(t_k))$, where $0 < t_1 < t_2 < \cdots < t_k = T$ for any fixed positive integer k.

In this paper we study the conditional Fourier-Feynman transform and a conditional convolution product for a cylinder type functionals on $C_0[0,T]$ with a general vector valued conditoning functions of the form $Z_{\vec{h},k}(x) = (z_{h_1}(x),\ldots,z_{h_k}(x))$ which need not depend upon the value of x in $C_0[0,T]$ at only finitely many points in $(0,T]$ rather than a conditioning function $X(x) = (x(t_1),\ldots,x(t_n))$ where $0 < t_1 < \ldots < t_n = T$. And we show that the conditional Fourier-Feynman transform of the conditional convolution product is the product of conditional Fourier-Feynman transforms.

2. Definitions and preliminaries

In this section we introduce a conditional Wiener integral, conditional Fourier-Feynman transform and conditional convolution product for a general vector valued conditoning function.

Let $\mathcal{H}$ be an infinite dimensional subspace of $L_2[0,T]$ with a complete orthonormal basis $\{e_j\}$. For each $k \in \mathbb{N}$ let $\mathcal{H}_k$ be a subspace of $\mathcal{H}$ spanned by $\{e_1,e_2,\ldots,e_k\}$ and let $Z_{\vec{e},k} : C_0[0,T] \to \mathbb{R}^k$ be the conditioning function defined by

$$Z_{\vec{e},k}(x) = (z_{e_1}(x),\ldots,z_{e_k}(x)). \tag{2.1}$$

Further, for $h \in L_2[0,T]$, let

$$\mathcal{P}_k h(t) = \sum_{j=1}^{k}(h,e_j)e_j(t) \tag{2.2}$$

be the orthogonal projection from $L_2[0,T]$ onto the subspace generated by $\{e_1,e_2,\ldots,e_k\}$ where $(\cdot,\cdot)$ denotes the inner product on the real Hilbert space $L_2[0,T]$. Then we see that $h - \mathcal{P}_k h$ is orthogonal to e_j, $j = 1,\ldots,k$. For convenience, let

$$x_k(t) = \int_0^T \mathcal{P}_k \mathcal{I}_{[0,t]}(s)dx(s) = \sum_{j=1}^{k} z_{e_j}(x)\int_0^t e_j(s)ds \tag{2.3}$$

and

$$\vec{\xi_k}(t) = \sum_{j=1}^{k}\xi_j(e_j,\mathcal{I}_{[0,t]}) = \sum_{j=1}^{k}\xi_j\int_0^t e_j(s)ds \tag{2.4}$$

where $x \in C_0[0,T]$, $\vec{\xi} = (\xi_1,\ldots,\xi_k) \in \mathbb{R}^k$ and $\mathcal{I}_{[0,t]}$ is the indicator function of the interval [0,t].

From [9], we can see that the process $\{x(t) - x_k(t), 0 \le t \le T\}$ and $z_{e_j}(x)$ are stochastically independent for $j = 1,\ldots,k$. Also for $0 \le t \le T$ as an immediate consequence of above, two processes $\{x(t) - x_k(t)\}$ and $\{x_k(t)\}$ are independent.

Let $F : C_0[0,T] \to \mathbb{C}$ be integrable functional and let $Z_{\vec{e},k}$ be a random vector on $C_0[0,T]$ given by (2.1). Then we have the conditional Wiener integral $E[F|Z_{\vec{e},k}]$ given $Z_{\vec{e},k}$ from a well-known probability theory. For a more detailed survey of the conditional Wiener integrals see [4, 9,10,11].

In [9], Park and Skoug gave a useful simple formula to express conditional Wiener integrals in terms of ordinary Wiener integrals ($E[F]$); namely for the conditioning function $Z_{\vec{e},k}(x)$ given by (2.1),

$$E_x[F(x)|Z_{\vec{e},k}(x)](\vec{\xi}) = E_x[F(x - x_k + \vec{\xi}_k)] \tag{2.5}$$

for $P_{Z_{\vec{e},k}}-a.e.\ \vec{\xi} \in \mathbb{R}^k$, where $P_{Z_{\vec{e},k}}$ is the probability distribution of $Z_{\vec{e},k}$ on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$.

In this paper we shall be concerned exclusively with $Z_{\vec{e},k}(x)$ given by (2.1) for the the conditioning function.

For $\lambda > 0$, and $\vec{\xi} \in \mathbb{R}^k$, suppose $E[F(\lambda^{-\frac{1}{2}}\cdot)|Z_{\vec{e},k}(\lambda^{-\frac{1}{2}}\cdot)](\vec{\xi})$ exists.

From (2.5) we have

$$\begin{aligned} &E_x[F(\lambda^{-\frac{1}{2}}x)|Z_{\vec{e},k}(\lambda^{-\frac{1}{2}}x)](\vec{\xi}) \\ =&E_x[F(\lambda^{-\frac{1}{2}}(x - x_k) + \vec{\xi}_k)] \end{aligned} \tag{2.6}$$

for $a.e.\vec{\xi} \in \mathbb{R}^k$.

If, for $\vec{\xi} \in \mathbb{R}^k$, $E_x[F(\lambda^{-\frac{1}{2}}(x - x_k) + \vec{\xi}_k)]$ has the analytic extension $J_\lambda(\vec{\xi})$ on $\mathbb{C}_+ \equiv \{\lambda \in \mathbb{C}|Re\lambda > 0\}$, then we write

$$J_\lambda(\vec{\xi}) = E_x^{anw_\lambda}[F(x)|Z_{\vec{e},k}(x)](\vec{\xi}) \tag{2.7}$$

for $\lambda \in \mathbb{C}_+$.

In this case, we call $J_\lambda(\vec{\xi})$ a conditional analytic Wiener integral of F given $Z_{\vec{e},k}$.

For non zero real number q and $\vec{\xi} \in \mathbb{R}^k$, if the limit

$$\lim_{\lambda \to -iq} E_x^{anw_\lambda}[F(x)|Z_{\vec{e},k}(x)](\vec{\xi}) \tag{2.8}$$

exists, where λ approaches to $-iq$ through $\mathbb{C}_+$, then we write

$$\lim_{\lambda \to -iq} E_x^{anw_\lambda}[F(x)|Z_{\vec{e},k}(x)](\vec{\xi}) = E_x^{anf_q}[F(x)|Z_{\vec{e},k}(x)](\vec{\xi}). \tag{2.9}$$

In this case, we call $E_x^{anf_q}[F(x)|Z_{\vec{e},k}(x)](\vec{\xi})$ a conditional analytic Feynman integral of F given $Z_{\vec{e},k}$.

Next we state the definitions of the L_1 analytic Fourier-Feynman transform and the convolution product given in [7]. For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, let

$$T_\lambda(F)(y) = E_x^{anw_\lambda}[F(y + x)]. \tag{2.10}$$

We define L_1 analytic Fourier-Feynman transform , $T_q^{(1)}(F)$ of F, by the formula

$$T_q^{(1)}(F)(y) = \lim_{\lambda \to -iq} T_\lambda(F)(y) \tag{2.11}$$

if it exists $\lambda \in \mathbb{C}_+$. We note that $T_q^{(1)}(F)$ is only defined for s-a.e. $y \in C_0[0,T]$. Also if $T_q^{(1)}(F)$ exists and $F \approx G$, then $T_q^{(1)}(G)$ exsits and $T_q^{(1)}(G) \approx T_q^{(1)}(F)$.

We define the convolution product $(F * G)_\lambda$ by

$$(F*G)_\lambda(y)=\begin{cases}E_x^{anw_\lambda}[F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})], & \lambda\in\mathbb{C}_+\\ E_x^{anf_q}[F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})], & \lambda=-iq, q\in\mathbb{R}, q\neq 0\end{cases} \tag{2.12}$$

if it exists.

REMARK 2.1. The convolution product above is commutative, that is to say, $(F*G)_\lambda=(G*F)_\lambda$ for all $\lambda\in\mathbb{C}$ with $\lambda\neq 0$ and $Re\lambda\geq 0$. When $\lambda=-iq(q\neq 0)$ we denote $(F*G)_\lambda$ by $(F*G)_q$.

Next we define conditional Fourier-Feynman transform and the conditional convolution product.

DEFINITION 2.2. Let F and G be defined on $C_0[0,T]$ and $Z_{\vec{e},k}$ be given by (2.1). For $\lambda\in\mathbb{C}_+$, $y\in C_0[0,T]$ and $\vec{\xi}\in\mathbb{R}^k$, let

$$T_\lambda(F|Z_{\vec{e},k})(y,\vec{\xi})=E_x^{anw_\lambda}(F(y+x)|Z_{\vec{e},k})(\vec{\xi}) \tag{2.13}$$

if it exists. For nonzero real number q, we define the conditional Fourier-Feynman transform(if it exists) of F given $Z_{\vec{e},k}$ by the formula

$$T_q^{(1)}(F|Z_{\vec{e},k})(y,\vec{\xi})=\lim_{\lambda\to -iq}T_\lambda(F|Z_{\vec{e},k})(y,\vec{\xi}) \tag{2.14}$$

where λ approach to $-iq$ through $\mathbb{C}_+$ and we define the conditional convolution product of F and G given $Z_{\vec{e},k}$ by the formula

$$((F*G)_\lambda|Z_{\vec{e},k})(y,\vec{\xi})=\begin{cases}E_x^{anw_\lambda}[F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})|Z_{\vec{e},k}](\vec{\xi}), & \lambda\in\mathbb{C}_+\\ E_x^{anf_q}[F(\frac{y+x}{\sqrt{2}})G(\frac{y-x}{\sqrt{2}})|Z_{\vec{e},k}](\vec{\xi}), & \lambda=-iq.\end{cases} \tag{2.15}$$

Under rather mild conditions on F and G, we have the following Theorems 2.3 and 2.4. Our next theorem shows that the analytic Fourier-Feynman transform of the convolution product is the product of analytic Fourier-Feynman transforms.

THEOREM 2.3. *Assume $T_q^{(1)}(F)$, $T_q^{(1)}(G)$ and $T_q^{(1)}((F*G)_q)$ all exists at $q\in\mathbb{R}-\{0\}$. Then*

$$T_q^{(1)}((F*G)_q)(y)=T_q^{(1)}(F)(\frac{y}{\sqrt{2}})T_q^{(1)}(G)(\frac{y}{\sqrt{2}}) \tag{2.16}$$

for s-a.e. $y\in C_0[0,T]$.

Proof. In view of (2.11), (2.10) it will suffice to show that $T_\lambda((F*G)_\lambda)(y)=T_\lambda(F)(\frac{y}{\sqrt{2}})T_\lambda(G)(\frac{y}{\sqrt{2}})$ for $\lambda>0$. But for all $\lambda>0$,

$$\begin{aligned}T_\lambda((F*G)_\lambda)(y)=&E_x[((F*G)_\lambda(y+\lambda^{-\frac{1}{2}}x)]\\ =&E_x[E_w[F(\frac{y+\lambda^{-\frac{1}{2}}x+\lambda^{-\frac{1}{2}}w}{\sqrt{2}})G(\frac{y+\lambda^{-\frac{1}{2}}x-\lambda^{-\frac{1}{2}}w}{\sqrt{2}})]]\\ =&E_x[E_w[[F(\frac{y+\lambda^{-\frac{1}{2}}(x+w)}{\sqrt{2}})G(\frac{y+\lambda^{-\frac{1}{2}}(x-w)}{\sqrt{2}})]].\end{aligned} \tag{2.17}$$

But $\frac{x+w}{\sqrt{2}}$ and $\frac{x-w}{\sqrt{2}}$ are independent Gaussian processes and each is equivalent to x. Hence

$$\begin{aligned} T_\lambda((F*G)_\lambda)(y) =& E_x[F(\frac{y}{\sqrt{2}}+\lambda^{-\frac{1}{2}}x)]E_x[G(\frac{y}{\sqrt{2}}+\lambda^{-\frac{1}{2}}x)] \\ =& T_\lambda(F)(\frac{y}{\sqrt{2}})T_\lambda(G)(\frac{y}{\sqrt{2}}) \end{aligned} \tag{2.18}$$

which concludes the proof of Theorem 2.3.

□

Next theorem shows that the conditional Fourier-Feynman transform of the conditional convolution product is the product of conditional Fourier-Feynman transforms.

THEOREM 2.4. *Assume* $T_q^{(1)}(((F*G)_q|Z_{\vec{e},k})(\cdot,\vec{\xi})|Z_{\vec{e},k})(y,\vec{\eta})$, $T_q^{(1)}(F|Z_{\vec{e},k})$ *and* $T_q^{(1)}(G|Z_{\vec{e},k})$ *all exists at* $q\in\mathbb{R}-\{0\}$. *Then*

$$\begin{aligned} & T_q^{(1)}(((F*G)_q|Z_{\vec{e},k})(\cdot,\vec{\xi})|Z_{\vec{e},k})(y,\vec{\eta}) \\ =& T_q^{(1)}(F|Z_{\vec{e},k})(\frac{y}{\sqrt{2}},\frac{\vec{\eta}+\vec{\xi}}{\sqrt{2}})T_q^{(1)}(G|Z_{\vec{e},k})(\frac{y}{\sqrt{2}},\frac{\vec{\eta}-\vec{\xi}}{\sqrt{2}}) \end{aligned} \tag{2.19}$$

for s-a.e. $y\in C_0[0,T]$.

Proof. Using the same process used in the proof of Theorem 2.3, we only need to consider the case where $\lambda>0$. From (2.5), (2.13) and (2.15) we observe that for all $\lambda>0$,

$$\begin{aligned} & T_\lambda(((F*G)_\lambda|Z_{\vec{e},k})(\cdot,\vec{\xi})|Z_{\vec{e},k})(y,\vec{\eta}) \\ =& E_x\left[((F*G)_\lambda|Z_{\vec{e},k})(y+\lambda^{-\frac{1}{2}}(x-x_k)+\vec{\eta}_k,\vec{\xi})\right] \\ =& E_x[E_w[F(\frac{1}{\sqrt{2}}(y+(\vec{\eta}_k+\vec{\xi}_k)+\lambda^{-\frac{1}{2}}(x-x_k+w-w_k))) \\ & G(\frac{1}{\sqrt{2}}(y+(\vec{\eta}_k-\vec{\xi}_k)+\lambda^{-\frac{1}{2}}(x-x_k-w+w_k)))]]. \end{aligned} \tag{2.20}$$

Now, $x-x_k+w-w_k$ and $x-x_k-w+w_k$ are independent processes as we can be seen by checking their covariance functions. Hence the expectation of FG equals the product of the expectations and so using (2.15) and (2.13) we see that

$$\begin{aligned} & T_\lambda(((F*G)_\lambda|Z_{\vec{e},k})(\cdot,\vec{\xi})|Z_{\vec{e},k})(y,\vec{\eta}) \\ =& E_x[E_w[F(\frac{1}{\sqrt{2}}(y+(\vec{\eta}_k+\vec{\xi}_k)+\lambda^{-\frac{1}{2}}(x-x_k+w-w_k))) \\ & G(\frac{1}{\sqrt{2}}(y+(\vec{\eta}_k-\vec{\xi}_k)+\lambda^{-\frac{1}{2}}(x-x_k-w+w_k)))]]. \end{aligned} \tag{2.21}$$

Now, $\frac{x+w}{\sqrt{2}}$ is equivalent to x and so is $\frac{x-w}{\sqrt{2}}$.

Hence for $\lambda>0$,

$$
\begin{aligned}
&T_\lambda(((F*G)_\lambda|Z_{\vec{e},k})(\cdot,\vec{\xi})|Z_{\vec{e},k})(y,\vec{\eta}) \\
=&E_x[F(\frac{y}{\sqrt{2}}+\lambda^{-\frac{1}{2}}x)|Z_{\vec{e},k}(\lambda^{-\frac{1}{2}}x)=\frac{\vec{\eta}+\vec{\xi}}{\sqrt{2}}] \\
&E_x[G(\frac{y}{\sqrt{2}}+\lambda^{-\frac{1}{2}}x)|Z_{\vec{e},k}(\lambda^{-\frac{1}{2}}x)=\frac{\vec{\eta}-\vec{\xi}}{\sqrt{2}}] \\
=&T_\lambda(F|Z_{\vec{e},k})(\frac{y}{\sqrt{2}},\frac{\vec{\eta}+\vec{\xi}}{\sqrt{2}})T_\lambda(G|Z_{\vec{e},k})(\frac{y}{\sqrt{2}},\frac{\vec{\eta}-\vec{\xi}}{\sqrt{2}}).
\end{aligned}
\tag{2.22}
$$

□

3. Conditional Fourier-Feynman transform and convolution for the cylinder type functionals

Now we describe the class of functionals that we work with in this paper. Let $\{\theta_1,\theta_2,\ldots\}$ be a complete orthonormal set of $\mathbb{R}$-valued functions in $L_2[0,T]$ and assume that each θ_j is of bounded variation on $[0,T]$. Then for each $y\in C_0[0,T]$ and $j=1,2,\ldots$, the Riemann-Stieltjes integral $\langle\theta_j,y\rangle\equiv\int_0^T\theta_j(t)dy(t)$ exists.

For $0\le\sigma<1$, let E_σ be the space of all cylinder type functionals $F:C_0[0,T]\to\mathbb{C}$ of the form

$$F(y)=f(\langle\theta_1,y\rangle,\ldots,\langle\theta_n,y\rangle)=f(\langle\vec{\theta},y\rangle) \tag{3.1}$$

for some positive integer n, where $f(\lambda_1,\ldots,\lambda_n)=f(\vec{\lambda})$ is an entire function of n complex variables $\lambda_1,\ldots,\lambda_n$ of exponential type; that is to say

$$|f(\vec{\lambda})|\le A_F\exp\{B_F\sum_{j=1}^n|\lambda_j|^{1+\sigma}\} \tag{3.2}$$

for some positive constants A_F and B_F.

Now let $\{u_1-\mathcal{P}_ku_1,\ldots,u_m-\mathcal{P}_ku_m\}$ be a maximal independent subset of $\{\theta_1-\mathcal{P}_k\theta_1,\ldots,\theta_n-\mathcal{P}_k\theta_n\}$ with $m\le n$ if it exists, where $\mathcal{P}_k$ is the orthogonal projection given by (2.2). Let $\{\phi_1,\ldots,\phi_m\}$ be the orthonormal set obtained from $\{u_1-\mathcal{P}_ku_1,\ldots,u_m-\mathcal{P}_ku_m\}$ using Gram-Schmidt orthonormalization process. Then we can find $m\times n$ matrix $A_{m_n}=(a_{i,j})$ with

$$\vec{\theta}-\mathcal{P}_k\vec{\theta}=(\sum_{j=1}^m a_{j,1}\phi_j,\ldots,\sum_{j=1}^m a_{j,n}\phi_j)=\vec{\phi}A_{m_n} \tag{3.3}$$

where $\vec{\theta}=(\theta_1,\ldots,\theta_n)$ and $\vec{\theta}-\mathcal{P}_k\vec{\theta}=(\theta_1-\mathcal{P}_k\theta_1,\ldots,\theta_n-\mathcal{P}_k\theta_n)$.

In our next theorem we show that the conditional Fourier-Feynman transform of functionals from E_σ for the general vector valued conditioning function $Z_{\vec{e},k}$ is an element of E_σ.

Theorem 3.1. *Let $F\in E_\sigma$ and $Z_{\vec{e},k}$ be given by (3.1) and (2.1), respectively. Then for each nonzero real number q, conditional Fourier-Feynman transform $T_q^{(1)}(F|Z_{\vec{e},k})$ exists, belongs to E_σ and is given by*

$$T_q^{(1)}(F|Z_{\vec{e},k})(y,\vec{\eta}) \tag{3.4}$$
$$=(\frac{q}{2\pi i})^{\frac{m}{2}}\int_{\mathbb{R}^m} f(\vec{v}A_{m_n}+\langle\vec{\theta},\vec{\eta}_k\rangle+\langle\vec{\theta},y\rangle)\exp\Big\{-\frac{q}{2i}\|\vec{v}\|^2\Big\}d\vec{v}$$

where $\|\vec{v}\|^2=\sum_{j=1}^m v_j^2$ *and* $\vec{v}=(v_1,\dots,v_m)$.

Proof. For $\lambda>0$, and $\vec{\eta}\in\mathbb{R}^k$,

$$\begin{aligned}(3.5)\qquad &T_\lambda(F|Z_{\vec{e},k})(y,\vec{\eta})\\ =&E_x[F(y+\lambda^{-\frac{1}{2}}(x-x_k)+\vec{\eta}_k)\\ =&E_x[f(\langle\vec{\theta},y+\lambda^{-\frac{1}{2}}(x-x_k)+\vec{\eta}_k\rangle)\\ =&E_x[f(\langle\vec{\theta},y\rangle+\langle\vec{\theta},\vec{\eta}_k\rangle+\lambda^{-\frac{1}{2}}\langle\vec{\theta},x-x_k\rangle)].\end{aligned}$$

From [8] we can see

$$\langle\vec{\theta},x-x_k\rangle=\langle\vec{\theta}-\mathcal{P}_k\vec{\theta},x\rangle=\langle\vec{\phi}A_{m_n},x\rangle$$

$\vec{\phi}=(\phi_1,\dots,\phi_m)$ and $A_{m_n}=(a_{ij})_{m\times n}$. Using Gaussian process property, Wiener integration formula and the change of variables, we see that for $\lambda>0$

$$\begin{aligned}(3.6)\qquad &T_\lambda(F|Z_{\vec{e},k})(y,\vec{\eta})\\ =&E_x[f(\lambda^{-\frac{1}{2}}\langle\vec{\phi}A_{m_n},x\rangle+\langle\vec{\theta},\vec{\eta}_k\rangle+\langle\vec{\theta},y\rangle)]\\ =&(\frac{\lambda}{2\pi})^{\frac{m}{2}}\int_{\mathbb{R}^m} f(\vec{u}A_{m_n}+\langle\vec{\theta},\vec{\eta}_k\rangle+\langle\vec{\theta},y\rangle)\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^m u_j^2\Big\}d\vec{u}.\end{aligned}$$

where $\vec{u}=(u_1,\dots,u_m)$.

But the last expression above is an analytic function of λ in $\mathbb{C}_+$ and is a bounded continuous function of $\lambda\in\mathbb{C}_+$. Hence $T_q^{(1)}(F|Z_{\vec{e},k})$ exists and is given by (3.4).

If we let $T_\lambda(F|Z_{\vec{e},k})(y,\vec{\eta})=h_{\lambda,\vec{\eta}}(\langle\vec{\theta},y\rangle)$ where

$$h_{\lambda,\vec{\eta}}(\vec{\gamma})=(\frac{\lambda}{2\pi})^{\frac{m}{2}}\int_{\mathbb{R}^m} f(\vec{u}A_{m_n}+\langle\vec{\theta},\vec{\eta}_k\rangle+\vec{\gamma})\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^m u_j^2\Big\}d\vec{u}$$

then by [6, Theorem 3.15] $h_{\lambda,\vec{\eta}}(\vec{\gamma})$ is an entire function. Furthermore using the integration $\int_{\mathbb{R}} e^{-\alpha v^2+\beta v}dv<\infty$ for $\alpha>0$ and $\beta\in\mathbb{R}$, we can see, $h_{\lambda,\vec{\eta}}(\vec{\gamma})$ is an element of E_σ as a function of $\vec{\gamma}$ [8]. Hence for $\lambda>0$, $T_\lambda(F|Z_{\vec{e},k})\in E_\sigma$ as a function of y. □

COROLLARY 3.2. *Let* $F\in E_\sigma$ *and* $Z_{\vec{e},k}$ *be as in Theorem 3.1. If* $\{\theta_1,\theta_2,\dots,\theta_n,e_1,\dots,e_k\}$ *is an orthonormal set of functions in* $L_2[0,T]$, *then*

$$T_q^{(1)}(F|Z_{\vec{e},k})(y,\vec{\eta})=T_q^{(1)}(F)(y). \tag{3.7}$$

Proof. As we noted in the proof of Theorem 2.3, we only need to consider the case when $\lambda>0$. Using definition of the Riemann-Stieltjes integral and the condition of $\{\theta_1,\theta_2,\dots,\theta_n,e_1,\dots,e_k\}$above together with the notations (2.3) and (2.4) we obtain

$$\langle\theta_j,x_k\rangle=0=\langle\theta_j,\vec{\eta}_k\rangle$$

for each $j = 1, \dots n$. Then from (3.5) and (2.10), we have

$$T_\lambda(F|Z_{\vec{e},k})(y,\vec{\eta}) = E_x[f(\langle\vec{\theta},y\rangle + \lambda^{-\frac{1}{2}}\langle\vec{\theta},x\rangle) = T_\lambda(F)(y)$$

for $\lambda > 0$ and $\vec{\eta} \in \mathbb{R}^k$. □

In Theorem 3.2 below we obtain the formula for conditional convolution product of functionals for the general vector valued conditioning function.

THEOREM 3.3. *Let $F_j \in E_\sigma$ be given by (3.1) with corresponding entire functions f_j for $j = 1, 2$ and let $Z_{\vec{e},k}$ be given by (2.1). Then for each nonzero real number q, conditional convolution product $((F_1 * F_2)_q|Z_{\vec{e},k})$ exists, belongs to E_σ and is given by*

$$\begin{aligned}&((F_1 * F_2)_q|Z_{\vec{e},k})(y,\vec{\eta})\\ =&(\frac{q}{2\pi i})^{\frac{m}{2}}\int_{\mathbb{R}^m} f_1(\frac{\langle\vec{\theta},y\rangle + q\vec{v}A_{m_n} + q\langle\vec{\theta},\vec{\eta}_k\rangle}{\sqrt{2}})\\ &f_2(\frac{\langle\vec{\theta},y\rangle - q\vec{v}A_{m_n} - q\langle\vec{\theta},\vec{\eta}_k\rangle}{\sqrt{2}})\exp\Big\{-\frac{q}{2i}\|\vec{v}\|^2\Big\}d\vec{v}\end{aligned} \tag{3.8}$$

where $\|\vec{v}\| = \sum_{j=1}^{m} v_j^2$ and $\vec{v} = (v_1, \dots, v_m)$ for $y \in C_0[0,T]$.

Proof. For $\lambda > 0$, and a.e. $\vec{\eta} \in \mathbb{R}^k$,

$$\begin{aligned}&((F_1 * F_2)_\lambda|Z_{\vec{e},k})(y,\vec{\eta}) = E_x^{anw_\lambda}[F_1(\frac{y+x}{\sqrt{2}})F_2(\frac{y-x}{\sqrt{2}})|Z_{\vec{e},k} = \vec{\eta}]\\ =&E_x[F_1(\frac{1}{\sqrt{2}}(y + \lambda^{-\frac{1}{2}}(x - x_k) + \vec{\eta}_k))F_2(\frac{1}{\sqrt{2}}(y - \lambda^{-\frac{1}{2}}(x - x_k) - \vec{\eta}_k))]\\ =&E_x[f_1(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y\rangle + \lambda^{-\frac{1}{2}}\langle\vec{\theta},x - x_k\rangle + \langle\vec{\theta},\vec{\eta}_k\rangle))\\ &f_2(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y\rangle - \lambda^{-\frac{1}{2}}\langle\vec{\theta},x - x_k\rangle - \langle\vec{\theta},\vec{\eta}_k\rangle))]\\ =&E_x[f_1(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y\rangle + \lambda^{-\frac{1}{2}}\langle\vec{\phi}A_{m_n},x\rangle + \langle\vec{\theta},\vec{\eta}_k\rangle))\\ &f_2(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y\rangle - \lambda^{-\frac{1}{2}}\langle\vec{\phi}A_{m_n},x\rangle - \langle\vec{\theta},\vec{\eta}_k\rangle))]\\ =&(\frac{\lambda}{2\pi})^{\frac{m}{2}}\int_{\mathbb{R}^m} f_1(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y\rangle + \vec{v}A_{m_n} + \langle\vec{\theta},\vec{\eta}_k\rangle))\\ &f_2(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y\rangle - \vec{v}A_{m_n} - \langle\vec{\theta},\vec{\eta}_k\rangle))\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{m} v_j^2\Big\}d\vec{v}.\end{aligned} \tag{3.9}$$

But (3.8) now follows directly from (3.9) since the last expression in (3.9) above an entire function of λ in $\mathbb{C}_+$.

If we let $((F_1 * F_2)_\lambda|Z_{\vec{e},k})(y,\vec{\eta}) = L_{\lambda,\vec{\eta}}(\langle\vec{\theta},y\rangle)$ where

$$\begin{aligned}L_{\lambda,\vec{\eta}}(\vec{\gamma}) =&(\frac{\lambda}{2\pi})^{\frac{m}{2}}\int_{\mathbb{R}^m} f_1(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y\rangle + \vec{v}A_{m_n} + \langle\vec{\theta},\vec{\eta}_k\rangle))\\ &f_2(\frac{1}{\sqrt{2}}(\langle\vec{\theta},y\rangle - \vec{v}A_{m_n} - \langle\vec{\theta},\vec{\eta}_k\rangle))\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{m} v_j^2\Big\}d\vec{v}\end{aligned} \tag{3.10}$$

then by [6, Theorem 3.15] $L_{\lambda,\vec{\eta}}(\vec{\gamma})$ is an entire function. Furthermore as a function of $\vec{\gamma}$ $L_{\lambda,\vec{\eta}}(\vec{\gamma})$is an element of E_σ([8], Theorem 2.6). Hence for $\lambda > 0$, $((F_1 * F_2)_q|Z_{\vec{e},k}) \in E_\sigma$ as a functionof y.

□

In view of Theorems 3.1 and 3.3 above, conditional Fourier-Feynman transforms and conditional convolutions of functionals from E_σ for the general vector valued conditioning function are also belong to E_σ. Then by Theorem 2.4, we have the following result.

THEOREM 3.4. *Let F_1 and F_2 be as in Theorem 3.2. Then for each nonzero real number q,*

$$\begin{aligned}&T_q^{(1)}((F_1 * F_2)_q|Z_{\vec{e},k})(\cdot,\vec{\eta}_1)|Z_{\vec{e},k})(y,\vec{\eta}_2)\\ =&T_q^{(1)}(F_1|Z_{\vec{e},k})(\frac{y}{\sqrt{2}},\frac{\vec{\eta}_2+\vec{\eta}_1}{\sqrt{2}})T_q^{(1)}(F_2|Z_{\vec{e},k})(\frac{y}{\sqrt{2}},\frac{\vec{\eta}_2-\vec{\eta}_1}{\sqrt{2}})\end{aligned} \tag{3.11}$$

for s-a.e. $y \in C_0[0,T]$.

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