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GENERAL SOLUTION AND ULAM STABILITY OF GENERALIZED CQ FUNCTIONAL EQUATION

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Abstract. In this paper, we introduce the following cubic-quartic functional equation of the form

$$f(x+4y)+f(x-4y)=16\left[f(x+y)+f(x-y)\right]\pm 30f(-x)+\frac{5}{2}\left[f(4y)-64f(y)\right].$$

Further, we investigate the general solution and the Ulam stability for the above functional equation in non-Archimedean spaces by using the direct method.

1. Introduction

Jun and Kim [7] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

and they established the general solution and the Ulam stability for the functional equation (1.1). The function $f(x)=x^3$ satisfies the functional equation (1.1), which is thus called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic mapping. Now we introduce the cubic functional equation and quartic functional equation

$$f(x+4y)+f(x-4y)=16\left[f(x+y)+f(x-y)\right]+30f(-x)+\frac{5}{2}\left[f(4y)-64f(y)\right] \tag{1.2}$$

and

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)+24f(y)-6f(x). \tag{1.3}$$

It is easy to see that the function $f(x)=x^4$ is a solution of the functional equation (1.3). Thus, it is natural that (1.3) is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping.

In this section, we introduce the cubic-quartic functional equation of the form

$$f(x+4y)+f(x-4y)=16\left[f(x+y)+f(x-y)\right]\pm 30f(-x)+\frac{5}{2}\left[f(4y)-64f(y)\right] \tag{1.4}$$

Further, we investigate the general solution and the Ulam stability for the functional equation (1.4).

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By a non-Archimedean field we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r|=0$ if and only if $r=0, |rs|=|r||s|$, and $|r+s|\leq \max\{|r|,|s|\}$ for all $r,s\in K$. Clearly $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$.

Definition 1.1. Let X be a vector space over a scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\|: X\to K$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:
(i) $\|x\|=0$ if and only if $x=0$;
(ii) $\|rx\|=|r|\|x\|$ for all $r\in K, x\in X$;
(iii) The strong inequality (ultrametric); namely,

$$\|x+y\|\leq \max\{\|x\|,\|y\|\}$$

for all $x,y\in X$. Then $(X,\|\cdot\|)$ is called a non-Archimedean space.
Due to the fact that

$$\|x_m-x_n\|\leq \max\{\|x_{j+1}-x_j\|: m\leq j\leq n-1\} \qquad (n>m),$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. Furthermore, some of the research papers related to non-Archimedean spaces are very useful to develop this article such as [1–4, 10, 15] and some of the other papers are used to build this section (see [5, 6, 8, 9, 11–14, 16]).

2. General solution for the cubic-quartic functional equation (1.4)

In this section, we find out the general solution of the cubic-quartic functional equation (1.4).

Theorem 2.1. *If a mapping $f: X\to Y$ satisfies the functional equation (1.2), then the mapping $f: X\to Y$ satisfies the functional equation (1.1).*

Proof. Putting $x=y=0$ in (1.2), we get $f(0)=0$. Setting $y=0$ in (1.2), we obtain $f(-x)=-f(x)$ for all $x\in X$. Hence f is odd. Replacing (x,y) by $(0,x)$ in (1.2) we get

$$f(4x)=64f(x) \tag{2.5}$$

for all $x\in X$. So

$$f(x+4y)+f(x-4y)=16\,[f(x+y)+f(x-y)]-30f(x) \tag{2.6}$$

for all $x,y\in X$. Replacing x by $4x$ in (2.6), we obtain

$$f(4x+4y)+f(4x-4y)=16\,[f(4x+y)+f(4x-y)]-30f(4x) \tag{2.7}$$

for all $x,y\in X$. It follows from (2.5) and (2.7) that

$$f(4x+y)+f(4x-y)=4\,[f(x+y)+f(x-y)]+120f(4x) \tag{2.8}$$

for all $x,y\in X$. Replacing x by $x+y$ in (2.6), we obtain

$$f(x+5y)+f(x-3y)=16\,[f(x+2y)+f(x)]-30f(x+y) \tag{2.9}$$

for all $x,y\in X$. Replacing x by $x-y$ in (2.6), we obtain

$$f(x-5y)+f(x+3y)=16\,[f(x-2y)+f(x)]-30f(x-y) \tag{2.10}$$

for all $x,y\in X$. Adding (2.9) and (2.10), we get

$$\begin{aligned}&f(x+5y)+f(x-3y)+f(x-5y)+f(x+3y)\\ &=16\,[f(x+2y)+f(x-2y)]+32f(x)-30\,[f(x+y)+f(x-y)]\end{aligned} \tag{2.11}$$

for all $x, y \in X$. Further replacing y by $x+y$ in (2.6), we obtain

$$f(5x+4y)+f(-3x-4y)=16\,[f(2x+y)-f(y)]-30f(x) \tag{2.12}$$

for all $x, y \in X$ and replacing y by $-x+y$ in (2.6), we get

$$f(-3x+4y)+f(5x-4y)=16\,[f(y)-f(2x-y)]-30f(x) \tag{2.13}$$

for all $x, y \in X$. Adding (2.12) and (2.13), we get

$$\begin{aligned}&f(5x+4y)+f(5x-4y)+f(-3x+4y)+f(-3x-4y)\\&=16\,[f(2x+y)+f(2x-y)]-60f(x)\end{aligned} \tag{2.14}$$

for all $x, y \in X$. Interchanging x by y in (2.14), we get

$$\begin{aligned}&f(4x+5y)+f(-4x+5y)+f(4x-3y)+f(-4x-3y)\\&=16\,[f(x+2y)-f(x-2y)]-60f(y)\end{aligned} \tag{2.15}$$

for all $x, y \in X$. Simplifying (2.15) and using oddness, we have

$$\begin{aligned}&f(4x+5y)-f(4x-5y)+f(4x-3y)-f(4x+3y)\\&=16\,[f(x+2y)-f(x-2y)]-60f(y)\end{aligned} \tag{2.16}$$

for all $x, y \in X$. It follows from (2.9) and (2.10) that

$$\begin{aligned}&f(x+5y)-f(x-5y)+f(x-3y)-f(x+3y)\\&=16\,[f(x+2y)-f(x-2y)]-30\,[f(x+y)-f(x-y)]\end{aligned} \tag{2.17}$$

for all $x, y \in X$. Replacing x by $4x$ in (2.17), we obtain

$$\begin{aligned}&f(4x+5y)-f(4x-5y)+f(4x-3y)-f(4x+3y)\\&=16\,[f(4x+2y)-f(4x-2y)]-30\,[f(4x+y)-f(4x-y)]\end{aligned} \tag{2.18}$$

for all $x, y \in X$. By comparing (2.16) and (2.18), we obtain

$$\begin{aligned}&16\,[f(x+2y)-f(x-2y)-60f(x-2y)]\\&=16\,[f(4x+2y)-f(4x-2y)]-30\,[f(4x+y)-f(4x-y)]\end{aligned} \tag{2.19}$$

for all $x, y \in X$. Now by interchanging x and y in (2.19), we get

$$\begin{aligned}&16\,[f(2x+y)-f(2x-y)-60f(x)]\\&=16\,[f(2x+4y)+f(2x-4y)]-30\,[f(x+4y)+f(x-4y)]\end{aligned} \tag{2.20}$$

for all $x, y \in X$. It follows from (2.6) and (2.20) that

$$\begin{aligned}&f(2x+y)+f(2x-y)\\&=[f(2x+4y)+f(2x-4y)]-30\,[f(x+y)+f(x-y)]+60f(x)\end{aligned} \tag{2.21}$$

for all $x, y \in X$. Simplifying (2.21), we obtain

$$\begin{aligned}&f(2x+4y)+f(2x-4y)\\&=[f(2x+y)+f(2x-y)]+30\,[f(x+y)+f(x-y)]-60f(x)\end{aligned} \tag{2.22}$$

for all $x, y \in X$. Replacing x by $2x$ in (2.6), we obtain

$$f(2x+4y)+f(2x-4y)=16\,[f(2x+y)+f(2x-y)]-2400f(x) \tag{2.23}$$

for all $x, y \in X$. From (2.22) and (2.23), we get the desired equation (1.1). □

THEOREM 2.2. *If an even mapping $f: X \to Y$ satisfies the functional equation (1.4), then the mapping $f: X \to Y$ satisfies the functional equation (1.3).*

Proof. Putting $x = y = 0$ in (1.4), we get $f(0) = 0$. Replacing (x, y) by $(0, x)$ in (1.4) and using the evenness of f, we get

$$f(4x) = 256f(x) \tag{2.24}$$

for all $x \in X$. It follows from (2.24) and (1.4) that

$$f(x+4y) + f(x-4y) = 16\,[f(x+y) + f(x-y)] - 30f(x) + 480f(y) \tag{2.25}$$

for all $x, y \in X$. Replacing x by $2x$ in (2.25), we have

$$f(2x+4y) + f(2x-4y) = 16\,[f(2x+y) + f(2x-y)] - 480f(x) + 480f(y) \tag{2.26}$$

for all $x, y \in X$. Replacing x by $x + y$ in (2.25), we obtain

$$f(x+5y) + f(x-3y) = 16\,[f(x+2y) + f(x)] - 30f(x+y) + 480f(y) \tag{2.27}$$

for all $x, y \in X$. Replacing x by $x - y$ in (2.25), we obtain

$$f(x-5y) + f(x+3y) = 16\,[f(x-2y) + f(x)] - 30f(x-y) + 480f(y) \tag{2.28}$$

for all $x, y \in X$. Adding (2.8) and (2.28), we get

$$\begin{aligned} &f(x+5y) + f(x-3y) + f(x-5y) + f(x+3y) \\ &\quad = 16\,[f(x+2y) + f(x-2y)] + 32f(x) - 30\,[f(x+y) + f(x-y)] + 960f(y) \end{aligned} \tag{2.29}$$

for all $x, y \in X$. Replacing x by $4x$ in (2.29), we get

$$\begin{aligned} &f(4x+5y) + f(4x-3y) + f(4x-5y) + f(4x+3y) \\ &\quad = 16\,[f(4x+2y) + f(4x-2y)] + 32f(4x) - 30\,[f(4x+y) + f(4x-y)] + 960f(y) \end{aligned} \tag{2.30}$$

for all $x, y \in X$. Replacing y by $x + y$ in (2.25), we obtain

$$f(5x+4y) + f(-3x-4y) = 16\,[f(2x+y) - f(y)] - 30f(x) + 480f(x+y) \tag{2.31}$$

for all $x, y \in X$. Replacing y by $x - y$ in (2.25), we have

$$f(5x-4y) + f(-3x+4y) = 16\,[f(2x-y) + f(y)] - 30f(x) + 480f(x-y) \tag{2.32}$$

for all $x, y \in X$. Adding (2.31) and (2.32), we obtain

$$\begin{aligned} &f(5x+4y) + f(-3x-4y) + f(5x-4y) + f(-3x+4y) \\ &\quad = 16\,[f(2x+y) + f(2x-y)] + 32f(y) - 60f(x) + 480\,[f(x+y) + f(x-y)] \end{aligned} \tag{2.33}$$

for all $x, y \in X$. Interchanging x by y in (2.33) we get

$$\begin{aligned} &f(4x+5y) + f(4x+3y) + f(4x-5y) + f(4x-3y) \\ &\quad = 16\,[f(x+2y) + f(x-2y)] + 32f(y) - 60f(x) + 480\,[f(x+y) + f(x-y)] \end{aligned} \tag{2.34}$$

for all $x, y \in X$. It follows from (2.30) and (2.34) that

$$\begin{aligned} &16\,[f(4x+2y) + f(4x-2y)] + 32f(4x) - 60\,[f(4x+y) + f(4x-y)] + 960f(y) \\ &\quad = 16\,[f(x+2y) + f(x-2y)] + 32f(y) - 60f(x) + 480\,[f(x+y) + f(x-y)] \end{aligned} \tag{2.35}$$

for all $x, y \in X$. Simplifying (2.35), we have

$$\begin{aligned} &f(4x+2y) + f(4x-2y) - 60\,[f(x+y) + f(x-y)] - [f(x+2y) + f(x-2y)] \\ &\quad = 774f(x) - 120f(y) \end{aligned} \tag{2.36}$$

for all $x, y \in X$. Interchanging x by y in (2.26), we have

$$f(4x+2y) - f(4x-2y) = 16\,[f(x+2y) - f(x-2y)] - 480f(y) + 480f(x) \tag{2.37}$$

for all $x, y \in X$. It follows from (2.37) and (2.36) that

$$\begin{aligned} &16\,[f(x+2y)-f(x-2y)]-480f(y)+480f(x)-60\,[f(x+y)+f(x-y)] \\ &\quad -[f(x+2y)-f(x-2y)]=774f(x)-120f(y) \end{aligned} \tag{2.38}$$

for all $x, y \in X$. Simplifying (2.38), we get

$$15\,[f(x+2y)+f(x-2y)]-60\,[f(x+y)+f(x-y)]=-90f(x)+360f(y) \tag{2.39}$$

for all $x, y \in X$. Dividing (2.39) by 15, we get the required equation (1.3). □

3. Stability of the cubic functional equation (1.2)

In this section, assume that G is an additive group and X is a complete non-Archimedean normed space. Now before taking up the main subject, for a given mapping $f : G \to X$, we define the difference operator

$$Df(x,y)=f(x+4y)+f(x-4y)-16\,[f(x+y)+f(x-y)]+30f(-x)-\frac{5}{2}\,[f(4y)-64f(y)]$$

for all $x, y \in G$. We consider the following function inequality

$$\|Df(x,y)\| \le \varphi(x,y)$$

for an upper bound $\varphi : G \times G \to [0,\infty)$.

Theorem 3.1. *Let $\varphi : G \times G \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty}\frac{\varphi(4^n x,4^n y)}{|4|^{3n}}=0 \tag{3.40}$$

for all $x, y \in G$ and let for each $x \in G$ the following limit exists

$$\lim_{n\to\infty}\max\left\{\frac{\varphi(0,4^j x)}{|4|^{3j}} : 0\le j<n\right\}, \tag{3.41}$$

which is denoted by $\varphi_C(x)$. Suppose that $f : G \to X$ is an odd mapping satisfying

$$\|Df(x,y)\| \le \varphi(x,y) \tag{3.42}$$

for all $x, y \in G$. Then there exists a cubic mapping $C : G \to X$ such that

$$\|C(x)-f(x)\| \le \frac{1}{|4|^3}\varphi_C(x) \tag{3.43}$$

for all $x \in G$, and if, in addition,

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0,4^j x)}{|4|^{3j}} : i\le j<n+i\right\}=0$$

then C is the unique cubic mapping satisfying (3.43).

Proof. Replacing (x, y) by $(0, x)$ in (3.42), we get

$$\|f(4x)-64f(x)\| \le \varphi(0,x) \tag{3.44}$$

for all $x \in G$. It follows from (3.44) that

$$\left\|\frac{f(4x)}{4^3}-f(x)\right\| \le \frac{\varphi(0,x)}{4^3} \tag{3.45}$$

for all $x \in G$. Replacing x by $2^{2(n-1)}x$ in (3.45), we get

$$\left\|\frac{1}{4^{3n}}f(4^n x)-\frac{1}{4^{3(n-1)}}f(4^{n-1}x)\right\| \le \frac{\varphi(0,4^{n-1}x)}{|4^{3n}|} \tag{3.46}$$

for all $x \in G$. It follows from (3.46) and (3.40) that the sequence $\left\{\frac{f(4^n x)}{4^{3n}}\right\}$ is Cauchy. Since X is complete, we conclude that $\left\{\frac{f(4^n x)}{4^{3n}}\right\}$ is convergent. Set $C(x) := \lim_{n\to\infty} \frac{f(4^n x)}{4^{3n}}$. Using induction, one can show that

$$\|\frac{f(4^n x)}{4^{3n}} - f(x)\| \leq \frac{1}{|4^3|} \max\left\{\frac{\varphi(0, 4^i x)}{|4|^{3i}} : 0 \leq i < n\right\} \tag{3.47}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (3.47) and using (3.41) one obtains (3.43). By (3.40) and (3.42), we get

$$\|DC(x,y)\| = \lim_{n\to\infty} \frac{1}{|4^{3n}|}\|f(4^n x, 4^n y)\| \leq \lim_{n\to\infty} \frac{\varphi(4^n x, 4^n y)}{|4|^{3n}} = 0$$

for all $x, y \in G$. Therefore the mapping $C : G \to X$ satisfies (1.2).

To prove the uniqueness property of C, let D be another cubic mapping satisfying (3.43). Then

$$\begin{aligned}
\|C(x) - D(x)\| &= \lim_{i\to\infty} |4|^{-3i}\|C(4^i x) - D(4^i x)\| \\
&\leq \lim_{i\to\infty} |4|^{-3i} \max\left\{\|C(4^i x) - f(4^i x)\|, \|f(4^i x) - D(4^i x)\|\right\} \\
&\leq \frac{1}{|4|^3} \lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(0, 4^j x)}{|4|^{3j}} : i \leq j < n+i\right\}
\end{aligned}$$

for all $x \in G$. If

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(0, 4^j x)}{|4|^{3j}} : i \leq j < n+i\right\} = 0,$$

then $C = D$, and the proof is complete. □

4. Stability of the quartic functional equation (1.4)

In this section, assume that G is an additive group and X is a complete non-Archimedean normed space. Now before taking up the main subject, for a given mapping $f : G \to X$, we define the difference operator

$$Df(x,y) = f(x+4y) + f(x-4y) - 16\left[f(x+y) + f(x-y)\right] - 30f(-x) - \frac{5}{2}\left[f(4y) - 64f(y)\right]$$

for all $x, y \in G$. We consider the following function inequality

$$\|Df(x,y)\| \leq \varphi(x,y)$$

for an upper bound $\varphi : G \times G \to [0, \infty)$.

Theorem 4.1. *Let $\varphi : G \times G \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(4^n x, 4^n y)}{|4|^{4n}} = 0 \tag{4.48}$$

for all $x, y \in G$ and let for each $x \in G$ the following limit exists

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(0, 4^j x)}{|4|^{4j}} : 0 \leq j < n\right\}, \tag{4.49}$$

which is denoted by $\varphi_Q(x)$. Suppose that $f : G \to X$ is an even mapping satisfying

$$\|Df(x,y)\| \leq \varphi(x,y) \tag{4.50}$$

for all $x, y \in G$. Then there exists a quartic mapping $Q : G \to X$ such that

$$\|Q(x) - f(x)\| \le \frac{1}{|4|^4}\varphi_Q(x) \tag{4.51}$$

for all $x \in G$, and if, in addition,

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0,4^j x)}{|4|^{4j}} : i \le j < n+i\right\} = 0$$

then Q is the unique quartic mapping satisfying (4.51).

Proof. Replacing (x, y) by $(0, x)$ in (4.50), we get

$$\|f(4x) - 256f(x)\| \le \varphi(0, x) \tag{4.52}$$

for all $x \in G$. It follows from (4.52) that

$$\|\frac{f(4x)}{4^4} - f(x)\| \le \frac{\varphi(0,x)}{4^4} \tag{4.53}$$

for all $x \in G$. Replacing x by $2^{n-1}x$ in (4.53), we get

$$\left\|\frac{1}{4^{4n}}f(4^n x) - \frac{1}{4^{4(n-1)}}f(4^{n-1}x)\right\| \le \frac{\varphi(0,4^{n-1}x)}{|4^{4n}|} \tag{4.54}$$

for all $x \in G$. It follows from (4.54) and (4.48) that the sequence $\left\{\frac{f(4^n x)}{4^{4n}}\right\}$ is Cauchy. Since X is complete, we conclude that $\left\{\frac{f(4^n x)}{4^{4n}}\right\}$ is convergent. Set $Q(x) := \lim_{n\to\infty}\frac{f(4^n x)}{4^{4n}}$. Using induction, one can show that

$$\|\frac{f(4^n x)}{4^{4n}} - f(x)\| \le \frac{1}{|4^4|}\max\left\{\frac{\varphi(0,4^i x)}{|4|^{4i}} : 0 \le i < n\right\} \tag{4.55}$$

for all $n \in \mathbb{N}$ and all $x \in G$. By taking n to approach infinity in (4.55) and using (4.49) one obtains (4.51). By (4.48) and (4.50), we get

$$\|DQ(x,y)\| = \lim_{n\to\infty}\frac{1}{|4^{4n}|}\|f(4^n x, 4^n y)\| \le \lim_{n\to\infty}\frac{\varphi(4^n x,4^n y)}{|4|^{4n}} = 0$$

for all $x, y \in G$. Since f is even, we can easily show that Q is even. Thus the mapping $Q : G \to X$ satisfies (1.4). To prove the uniqueness property of Q, let R be another quartic mapping satisfying (4.51). Then

$$\begin{aligned}\|Q(x) - R(x)\| &= \lim_{i\to\infty}|4|^{-4i}\|Q(4^i x) - R(4^i x)\|\\ &\le \lim_{i\to\infty}|4|^{-4i}\max\left\{\|Q(4^i x) - f(4^i x)\|, \|f(4^i x) - R(4^i x)\|\right\}\\ &\le \frac{1}{|4|^4}\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0,4^j x)}{|4|^{4j}} : i \le j < n+i\right\}\end{aligned}$$

for all $x \in G$. If

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(0,4^j x)}{|4|^{4j}} : i \le j < n+i\right\} = 0,$$

then $Q = R$, and the proof is complete. □

5. Stability of the cubic-quartic functional equation (1.4)

In this section, assume that G is an additive group and X is a complete non-Archimedean normed space. Now before taking up the main subject, for a given mapping $f : G \to X$, we define the difference operator

$$Df(x,y) = f(x+4y)+f(x-4y)-16\left[f(x+y)+f(x-y)\right] \pm 30f(-x) - \frac{5}{2}\left[f(4y)-64f(y)\right]$$

for all $x, y \in G$. We consider the following function inequality

$$\|Df(x,y)\| \le \varphi(x,y)$$

for an upper bound $\varphi : G \times G \to [0,\infty)$.

THEOREM 5.1. *Let $\varphi : G \times G \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(4^n x, 4^n y)}{|4|^{3n}} = \lim_{n\to\infty} \frac{\varphi(4^n x, 4^n y)}{|4|^{4n}} = 0$$

for all $x, y \in G$ and let for each $x \in G$ the following limits exist

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(0,4^j x)}{|4|^{3j}} : 0 \le j < n\right\}, \text{ and } \lim_{n\to\infty} \max\left\{\frac{\varphi(0,4^j x)}{|4|^{4j}} : 0 \le j < n\right\}$$

denoted by $\varphi_C(x)$ and denoted by $\varphi_Q(x)$, respectively. Suppose that $f : G \to X$ is a mapping satisfying

$$\|Df(x,y)\| \le \varphi(x,y)$$

for all $x, y \in G$. Then there exist a cubic mapping $C : G \to X$ and a quartic mapping $Q : G \to X$ such that

$$\begin{aligned} &\|f(x)-C(x)-Q(x)\| \\ &\le \max\left\{\frac{1}{|2||4|^3}\max\{\varphi_C(x),\varphi_C(-x)\}, \frac{1}{|2||4|^4}\max\{\varphi_Q(x),\varphi_Q(-x)\}\right\} \end{aligned} \tag{5.56}$$

for all $x \in G$, and if, in addition,

$$\begin{aligned} \lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(0,4^j x)}{|4|^{3j}} : i \le j < n+i\right\} &= \lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(0,4^j x)}{|4|^{4j}} : i \le j < n+i\right\} \\ &= 0, \end{aligned}$$

then C is the unique cubic mapping and Q is the unique quartic mapping.

Proof. Let $f_0(x) = \frac{1}{2}\left[f(x)-f(-x)\right]$ for all $x \in G$. Then $f_0(0) = 0, f_0(-x) = -f_0(x)$, and

$$\|Df_0(x,y)\| \le \frac{1}{|2|}\max\{\varphi(x,y),\varphi(-x,-y)\}$$

for all $x, y \in G$. From Theorem 3.1, it follows that there exists a unique cubic mapping $C : G \to X$ satisfying

$$\|f_0(x)-C(x)\| \le \frac{1}{|2||4|^3}\max\{\varphi_C(x),\varphi_C(-x)\}$$

for all $x \in G$. Let $f_e(x) = \frac{1}{2}\left[f(x)-f(-x)\right]$ for all $x \in G$. Then $f_e(0) = 0, f_e(-x) = f_e(x)$, and

$$\|Df_e(x,y)\| \le \frac{1}{2}\max\{\varphi(x,y),\varphi(-x,-y)\}$$

for all $x, y \in G$. From Theorem 4.1, it follows that there exists a unique quartic mapping $Q : G \to X$ satisfying

$$\|f_e(x) - Q(x)\| \leq \frac{1}{|2||4|^4} \max\{\varphi_Q(x), \varphi_Q(-x)\}$$

for all $x \in G$. Thus we get the desired inequality (5.56) □

6. Conclusion

In this paper, we have introduced the cubic-quartic functional equation (1.4) and we have investigated the general solution and have proved the Ulam stability for the functional equation (1.4) in non-Archimedean spaces by using the direct method.

Declarations

Availablity of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Fundings
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Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

References

[1] M. Eshaghi Gordji, R. Khodabakhsh, S. Jung and H. Khodaei, *AQCQ-Functional equation in non-Archimedean normed spaces*, Abstr.. Appl. Anal. **2010** (2010), Article ID 741942.

[2] M. Eshaghi Gordji, H. Khodaei and R. Khodabakhsh, *General quartic-cubic-quadratic functional equation in non-Archimedean normed spaces*, UPB Sci. Bull. Ser. A, **72** (2010), no. 3, 69–84.

[3] M. Eshaghi Gorgji and M. B. Savadkouhi, *Stability of cubic and quartic functional equations in non-Archimedean spaces,* Acta Appl. Math. **110** (2010), 1321–1329.

[4] M. Eshaghi Gordji and M. B. Savadkouhi, *Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces,* Appl. Math. Lett. **23** (2010), 1198–1202.

[5] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.

[6] V. Govindan, S. Murthy and M. Saravanan, *Solution and stability of new type of (aaq,bbq,caq,daq) mixed type functional equation in various normed spaces: Using two different methods,* Int. J. Math. Appl. **5** (2017), 187–211.

[7] K. Jun and H. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. **274** (2002), 267–278.

[8] S. Jung, D. Popa and M. Th. Rassias, *On the stability of the linear functional equation in a single variable on complete metric spaces*, J. Global Optim. **59** (2014), 13–16.

[9] Y. Lee, S. Jung and M. Th. Rassias, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequal. **12** (2018), 43–61.

[10] D. Mihet, *The stability of the additive Cauchy functional equation in non- Archimedean fuzzy normed spaces*, Fuzzy Sets Syst. **161** (2010), 2206–2212.

[11] R. Murali, S. Pinelas and V. Vithya, *The stability of viginti unus functional equation in various spaces*, Global J. Pure Appl. Math. **13** (2017), 5735–5759.

[12] S. Murthy, V. Govindhan, *General solution and generalized HU (Hyers-Ulam) statbility of new dimension cubic functional equation in fuzzy ternary Banach algebras: Using two different methods*, Int. J. Pure Appl. Math. **113** (2017), no. 6, 394–403.

[13] P. Narasimman, K. Ravi and S. Pinelas, *Stability of Pythagorean mean functional equation,* Global J. Math. **4** (2015), 398–411.

[14] C. Park, *Additive ρ-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.

[15] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.

[16] S. Pinelas, V. Govindan and K. Tamilvanan, *Stability of non-additive functional equation*, IOSR J. Math. **14** (2018), no. 2, 70–78.

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