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ON HOM-LIE TRIPLE SYSTEMS AND INVOLUTIONS OF HOM-LIE ALGEBRAS

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Abstract. In this paper we mainly establish a relationship between involutions of multiplicative Hom-Lie algebras and Hom-Lie triple systems. We show that the −1-eigenspace of any involution on any multiplicative Hom-Lie algebra becomes a Hom-Lie triple system and we construct some examples of Hom-Lie triple systems using some involutions of some classical Hom-Lie algebras.

1. Introduction

Let $\mathbb{K}$ be an arbitrary field of characteristic 0. Lie triple systems are subspaces of any Lie algebra which are closed under the ternary composition $[[x, y], z]$. They were first noted by E. Cartan in his work on geodesic submanifolds [3]. From the algebraic point of view, Lie triple systems were studied by N. Jacobson [6, 7] and Lister [8].
In general, Lie triple systems have natural embeddings into certain canonical Lie algebras called "standard" and "universal" embeddings, and any Lie triple system can be shown to arise precisely as the −1-eigenspace of an involution on some Lie algebra [5].
The Hom-Lie algebras structures arose first in deformation of Lie algebras of vector fields. The notion of Hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov in [4] as part of a study of deformations of the Witt and Virasoro algebras. The notion of Hom-Lie triple system generalizing Lie triple system to a situation where the trilinear law is twisted by a linear map was introduced by D. Yau in [10].
The purpose of this paper consists in giving a relationship between involutions of multiplicative Hom-Lie algebras and Hom-Lie triple systems.
The paper is organised as follows. In section 2, we recall some basic definitions and properties of Hom-Lie algebras and Hom-Lie triple systems. We derive new Hom-Lie triple systems from a given multiplicative Hom-Lie triple system and we construct Hom–Lie triple systems involving elements of the centroid of Lie triple systems. In section 3, we show that there exists a connection between involutions of multiplicative Hom-Lie algebras and Hom-Lie triple systems with some examples.

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2. Hom-Lie triple systems

2.1. Preliminaries on Hom-Lie algebras.

Definition 2.1. [2] A Hom-Lie algebra is a triple $(\mathcal{G}, [,], \alpha)$ consisting of a vector space $\mathcal{G}$ over $\mathbb{K}$, a skew-symmetric bilinear map $[,] : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and a linear map $\alpha : \mathcal{G} \to \mathcal{G}$ satisfying the following Hom-Jacobi identity :

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \text{ for all } x, y, z \in \mathcal{G}. \tag{1}$$

Moreover, if $\alpha([x, y]) = [\alpha(x), \alpha(y)]$, for all $x, y \in \mathcal{G}$, the Hom-Lie algebra $(\mathcal{G}, [,], \alpha)$ is said to be multiplicative.

Definition 2.2. Let $(\mathcal{G}, [,], \alpha)$ and $(\mathcal{G}', [,]', \alpha')$ be two Hom-Lie algebras.
A map $f : \mathcal{G} \longrightarrow \mathcal{G}'$ is called a morphism of Hom-Lie algebras if
$f([x, y]) = [f(x), f(y)]'$ and $f(\alpha(x)) = \alpha'(f(x))$, for all $x, y \in \mathcal{G}$.

Definition 2.3. Let $(\mathcal{G}, [,], \alpha)$ be a Hom-Lie algebra.

1. A Hom-Lie subalgebra of $(\mathcal{G}, [,], \alpha)$ is a subspace $\mathcal{H}$ of $\mathcal{G}$ such that for all $x, y \in \mathcal{H}, [x, y] \in \mathcal{H}$ and $\alpha(x) \in \mathcal{H}$.
2. An ideal of $(\mathcal{G}, [,], \alpha)$ is a subspace $\mathcal{I}$ of $\mathcal{G}$ such that for all $x \in \mathcal{I}$ and for all $y \in \mathcal{G}, [x, y] \in \mathcal{I}$ and $\alpha(x) \in \mathcal{I}$.

The following theorem can be found in [2].

Theorem 2.4. *Let $(\mathcal{G}, [,])$ be a Lie algebra. Let $\alpha : \mathcal{G} \longrightarrow \mathcal{G}$ be an endomorphism of the Lie algebra $(\mathcal{G}, [,])$. Let $[,]_\alpha : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ be the map defined by $[x, y]_\alpha = \alpha([x, y])$, for all $x, y \in \mathcal{G}$. Then $(\mathcal{G}, [,]_\alpha, \alpha)$ is a multiplicative Hom-Lie algebra.*

In what follows, using the theorem 2.4, we construct examples of Hom-Lie algebras from classical Lie algebras.

Example 2.5. **Case of the Lie algebra $\mathcal{S}l(n)$**
Let us consider the Lie algebra $(\mathcal{S}l(n), [,])$ consisting of the square matrices X of order n with elements in $\mathbb{K}$ such that $tr(X) = 0$. We have

$$\mathcal{S}l(n) = \{X \in \mathcal{M}_n(\mathbb{K}); tr(X) = 0\}.$$

The map $[,]$ is defined by : for all $X, Y \in \mathcal{S}l(n), [X, Y] = XY - YX$.
Denote by $Gl(n)$ the set of invertible matrices of order n with elements in $\mathbb{K}$. We have

$$Gl(n) = \{X \in \mathcal{M}_n(\mathbb{K}); det(X) \neq 0\}.$$

Let $A \in Gl(n)$. Define the map

$$\alpha : \mathcal{S}l(n) \longrightarrow \mathcal{S}l(n), X \mapsto A^{-1}XA.$$

Let us show that α is an endomorphism of the Lie algebra $(\mathcal{S}l(n), [,])$.
For all $X \in \mathcal{S}l(n)$, we have $tr(X) = 0$ and

$$tr(\alpha(X)) = tr(A^{-1}XA) = tr(A^{-1}AX) = tr(I_nX) = tr(X) = 0.$$

That means for all $X \in \mathcal{S}l(n), \alpha(X) \in \mathcal{S}l(n)$.
Next, for all $X, Y \in \mathcal{S}l(n)$ and for all $k \in \mathbb{K}$, we have

$$\alpha(X + kY) = A^{-1}(X + kY)A = A^{-1}XA + kA^{-1}YA = \alpha(X) + k\alpha(Y).$$

That proves the linearity of α.
Moreover, For all $X, Y \in \mathcal{S}l(n)$, we have,

$$\begin{aligned}
[\alpha(X), \alpha(Y)] &= \alpha(X)\alpha(Y) - \alpha(Y)\alpha(X) \\
&= A^{-1}XAA^{-1}YA - A^{-1}YAA^{-1}XA \\
&= A^{-1}XI_nYA - A^{-1}YI_nXA \\
&= A^{-1}XYA - A^{-1}YXA \\
&= A^{-1}(XY - YX)A \\
&= A^{-1}[X, Y]A \\
&= \alpha([X, Y]).
\end{aligned}$$

So, the map α is an endomorphism of the Lie algebra $(\mathcal{S}l(n), [,])$. Therefore $(\mathcal{S}l(n), [,]_\alpha, \alpha)$ is a multiplicative Hom-Lie algebra where $\alpha(X) = A^{-1}XA$ and $[X, Y]_\alpha = \alpha([X, Y]) = A^{-1}XYA - A^{-1}YXA,$ for all $X, Y \in \mathcal{S}l(n)$.

Example 2.6. **Case of the Lie algebra $\mathcal{S}o(n)$**
Let us consider the Lie algebra $(\mathcal{S}o(n), [,])$ consisting of the skew-symmetric matrices of order n with elements in $\mathbb{K}$. We have

$$\mathcal{S}o(n) = \left\{X \in \mathcal{M}_n(\mathbb{K}); X^t = -X\right\}.$$

The map $[,]$ is defined by : for all $X, Y \in \mathcal{S}o(n), [X, Y] = XY - YX$.
Denote by $O(n)$ the set of orthogonal matrices of order n with elements in $\mathbb{K}$. We have

$$O(n) = \left\{X \in \mathcal{M}_n(\mathbb{K}); X^tX = XX^t = I_n\right\}.$$

Let $A \in O(n)$. Define the map

$$\alpha : \mathcal{S}o(n) \longrightarrow \mathcal{S}o(n), X \mapsto A^tXA.$$

Let us show that α is an endomorphism of the Lie algebra $(\mathcal{S}o(n), [,])$.
Let $X \in \mathcal{S}o(n)$. Then $X^t = -X$. Since, for all matrices M and N in $\mathcal{M}_n(\mathbb{K})$ we have $(MN)^t = N^tM^t$ and $(M^t)^t = M$, then it follows

$$(\alpha(X))^t = (A^tXA)^t = A^tX^t(A^t)^t = A^t(-X)A = -A^tXA = -\alpha(X).$$

That means for all $X \in \mathcal{S}o(n), \alpha(X) \in \mathcal{S}o(n)$.
Next, for all $X, Y \in \mathcal{S}o(n)$ and for all $k \in \mathbb{K}$, we have

$$\alpha(X + kY) = A^t(X + kY)A = A^tXA + kA^tYA = \alpha(X) + k\alpha(Y).$$

That proves the linearity of α.
Moreover, for all $X, Y \in \mathcal{S}o(n)$, we have

$$\begin{aligned}
[\alpha(X), \alpha(Y)] &= \alpha(X)\alpha(Y) - \alpha(Y)\alpha(X) \\
&= A^tXAA^tYA - A^tYAA^tXA \\
&= A^tXI_nYA - A^tYI_nXA \\
&= A^tXYA - A^tYXA \\
&= A^t(XY - YX)A \\
&= A^t[X, Y]A \\
&= \alpha([X, Y]).
\end{aligned}$$

So the map α is an endomorphism of the Lie algebra $(\mathcal{S}o(n), [,])$. Therefore $(\mathcal{S}o(n), [,]_\alpha, \alpha)$ is a multiplicative Hom-Lie algebra where $\alpha(X) = A^tXA$ and $[X, Y]_\alpha = \alpha([X, Y]) = A^tXYA - A^tYXA,$ for all $X, Y \in \mathcal{S}o(n)$.

2.2. Hom-Lie triple systems.

Definition 2.7. [1] A Lie triple system is a couple $(T, [,,])$ consisting of a vector space T over $\mathbb{K}$ and a trilinear map $[,,] : T \times T \times T \to T$ satisfying

1. $[x, y, z] = -[y, x, z]$,
2. $[x, y, z] + [y, z, x] + [z, x, y] = 0$,
3. $[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]$,
   for all $x, y, z, u, v \in T$.

Definition 2.8. [1] A Hom-Lie triple system is a triple $(T, [,,], \alpha)$ consisting of a vector space T over $\mathbb{K}$, a trilinear map $[,,] : T \times T \times T \to T$ and a linear map $\alpha : T \to T$ satisfying

1. $[x, y, z] = -[y, x, z]$,
2. $[x, y, z] + [y, z, x] + [z, x, y] = 0$,
3. $[\alpha(u), \alpha(v), [x, y, z]] = [[u, v, x], \alpha(y), \alpha(z)] + [\alpha(x), [u, v, y], \alpha(z)] + [\alpha(x), \alpha(y), [u, v, z]]$,
   for all $x, y, z, u, v \in T$.

Moreover, if $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$, for all $x, y, z \in T$, then $(T, [,,], \alpha)$ is called a multiplicative Hom-Lie triple system.

When α is the identity map, we recover the classical Lie triple system. So Lie triple systems are examples of Hom-Lie triple systems.

Definition 2.9. Let $(T, [,,], \alpha)$ and $(T', [,,]', \alpha')$ be two Hom-Lie triple systems. A linear map $f : T \longrightarrow T'$ is called morphism of Hom-Lie triple systems if for all $x, y, z \in T, f([x, y, z]) = [f(x), f(y), f(z)]'$ and $f(\alpha(x)) = \alpha'(f(x))$.

Definition 2.10. Let $(T, [,,], \alpha)$ be a Hom-Lie triple system.

1. A Hom-Lie triple subsystem of T is a subspace S of T such that for all $x, y, z \in S$, $[x, y, z] \in S$ and $\alpha(x) \in S$.
2. An ideal of T is a subspace I of T such that for all $x \in I$ and for all $y, z \in T, [x, y, z] \in I$ and $\alpha(x) \in I$.

Theorem 2.11. *Let $(T, [,,])$ be a Lie triple system, $\alpha : T \longrightarrow T$ a morphism of the Lie triple systems T. Then $(T, [,,]_\alpha, \alpha)$ is a Hom-Lie triple system where $[,,]_\alpha = \alpha \circ [,,]$.*

Proof. As the map α is linear and the map $[,,]$ is trilinear then the map $\alpha \circ [,,]$ is trilinear. So the map $[,,]_\alpha$ is trilinear.
$i)$ For all $x, y, z \in T$, we have

$$[x, y, z]_\alpha = \alpha([x, y, z]) \qquad = \alpha(-[y, x, z]) = -\alpha([y, x, z]) \qquad = -[y, x, z]_\alpha.$$

$ii)$ For all $x, y, z \in T$, we have

$$\begin{aligned}
[x, y, z]_\alpha + [y, z, x]_\alpha + [z, x, y]_\alpha &= \alpha([x, y, z]) + \alpha([y, z, x]) + \alpha([z, x, y]) \\
&= \alpha([x, y, z] + [y, z, x] + [z, x, y]) \\
&= \alpha(0) \\
&= 0.
\end{aligned}$$

iii) For all $x, y, z, u, v \in T$, we have

$$\begin{aligned}
&[\alpha(u), \alpha(v), [x, y, z]_\alpha]_\alpha \\
&= \alpha([\alpha(u), \alpha(v), \alpha([x, y, z])]) \\
&= \alpha \circ \alpha([u, v, [x, y, z]]) \\
&= \alpha \circ \alpha([[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]) \\
&= \alpha \circ \alpha([[u, v, x], y, z]) + \alpha \circ \alpha([x, [u, v, y], z]) + \alpha \circ \alpha([x, y, [u, v, z]]) \\
&= \alpha([\alpha([u, v, x]), \alpha(y), \alpha(z)]) + \alpha([\alpha(x), \alpha([u, v, y]), \alpha(z)]) \\
&\quad + \alpha([\alpha(x), \alpha(y), \alpha([u, v, z])]) \\
&= [[u, v, x]_\alpha, \alpha(y), \alpha(z)]_\alpha + [\alpha(x), [u, v, y]_\alpha, \alpha(z)]_\alpha + [\alpha(x), \alpha(y), [u, v, z]_\alpha]_\alpha.
\end{aligned}$$

Therefore $(T, [, ,]_\alpha, \alpha)$ is a Hom-Lie triple system. □

It is well-known that, any subspace of a Lie algebra $(\mathcal{G}, [,])$ closed under the ternary product $[x, y, z] = [[x, y], z]$, is a Lie triple system relative to $[, ,]$. But, for an arbitrary Hom-Lie algebra $(\mathcal{G}, [,], \alpha)$, it is not natural to construct a Hom-Lie triple system without some conditions on the map α. The following theorem can be found in [9].

Theorem 2.12. [9] *Let $(\mathcal{G}, [,], \alpha)$ be a multiplicative Hom-Lie algebra. Then $(\mathcal{G}, [, ,], \alpha^2)$ is a multiplicative Hom-Lie triple system where* $[x, y, z] = [[x, y], \alpha(z)]$, *for all $x, y, z \in \mathcal{G}$.*

Remark 2.13. The fact that the Hom-Lie algebra $(\mathcal{G}, [,], \alpha)$ is multiplicative, is necessary in the theorem 2.12.

Corollary 2.14. *Let $(\mathcal{G}, [,], \alpha)$ be a multiplicative Hom-Lie algebra. Then, any subspace T of $\mathcal{G}$ closed under the ternary product $[x, y, z] = [[x, y], \alpha(z)]$ and α^2, determines a multiplicative Hom-Lie triple system $(T, [, ,], \alpha^2)$.*

Proof. Let $(\mathcal{G}, [,], \alpha)$ be a multiplicative Hom-Lie algebra. Let T be a subspace of $\mathcal{G}$ closed under the ternary product $[x, y, z] = [[x, y], \alpha(z)]$ and α^2. By the theorem 2.12, $(\mathcal{G}, [, ,], \alpha^2)$ is a multiplicative Hom-Lie triple system. So, T becomes a Hom-Lie triple subsystem of $(\mathcal{G}, [, ,], \alpha^2)$. Therefore $(T, [, ,], \alpha^2)$ is a multiplicative Hom-Lie triple system. □

We give here some examples of Hom-Lie triple systems by using multiplicative Hom-Lie algebras as in the theorem 2.12.

Example 2.15. **Case of $\mathcal{S}l(n)$**
Consider the multiplicative Hom-Lie algebra $(\mathcal{S}l(n), [,]_\alpha, \alpha)$ given in the example 2.5, where $\alpha(X) = A^{-1}XA$, $[X, Y]_\alpha = A^{-1}XYA - A^{-1}YXA$, for all X, $Y \in \mathcal{S}l(n)$ and $A \in GL(n)$.
For all $X \in \mathcal{S}l(n)$, we have

$$\alpha^2(X) = \alpha(\alpha(X)) = \alpha(A^{-1}XA) = A^{-1}(A^{-1}XA)A = (A^{-1})^2XA^2.$$

For all $X, Y, Z \in \mathcal{S}l(n)$, we have

$$\begin{aligned}[X,Y,Z]_\alpha &= [[X,Y]_\alpha, \alpha(Z)]_\alpha \\ &= [A^{-1}XYA - A^{-1}YXA, A^{-1}ZA] \\ &= [A^{-1}XYA, A^{-1}ZA] - [A^{-1}YXA, A^{-1}ZA] \\ &= A^{-1}(A^{-1}XYA)(A^{-1}ZA)A - A^{-1}(A^{-1}ZA)(A^{-1}XYA)A \\ &\quad - A^{-1}(A^{-1}YXA)(A^{-1}ZA)A + A^{-1}(A^{-1}ZA)(A^{-1}YXA)A \\ &= (A^{-1})^2XYZA^2 - (A^{-1})^2ZXYA^2 - (A^{-1})^2YXZA^2 + (A^{-1})^2ZYXA^2.\end{aligned}$$

By the theorem 2.12, the triple $(\mathcal{S}l(n), [, ,]_\alpha, \alpha^2)$ is a multiplicative Hom-Lie triple system where for all $X, Y, Z \in \mathcal{S}l(n)$, $\alpha^2(X) = (A^{-1})^2XA^2$ and $[X,Y,Z]_\alpha = (A^{-1})^2XYZA^2 - (A^{-1})^2ZXYA^2 - (A^{-1})^2YXZA^2 + (A^{-1})^2ZYXA^2$.

Example 2.16. **Case of $\mathcal{S}o(n)$**
Consider the multiplicative Hom-Lie algebra $(\mathcal{S}o(n), [,]_\alpha, \alpha)$ given in the example 2.6, where $\alpha(X) = A^tXA$, $[X,Y]_\alpha = A^tXYA - A^tYXA$, for all X, $Y \in \mathcal{S}o(n)$ and $A \in O(n)$.
For all $X \in \mathcal{S}o(n)$, we have

$$\alpha^2(X) = \alpha(\alpha(X)) = \alpha(A^tXA) = A^t(A^tXA)A = (A^t)^2XA^2.$$

For all $X, Y, Z \in \mathcal{S}o(n)$, we have

$$\begin{aligned}[X,Y,Z]_\alpha &= [[X,Y]_\alpha, \alpha(Z)]_\alpha \\ &= [A^tXYA - A^tYXA, A^tZA] \\ &= [A^tXYA, A^tZA] - [A^tYXA, A^tZA]\end{aligned}$$

$$\begin{aligned}&= A^t(A^tXYA)(A^tZA)A - A^t(A^tZA)(A^tXYA)A \\ &\quad - A^t(A^tYXA)(A^tZA)A + A^t(A^tZA)(A^tYXA)A \\ &= (A^t)^2XYZA^2 - (A^t)^2ZXYA^2 - (A^t)^2YXZA^2 + (A^t)^2ZYXA^2.\end{aligned}$$

By the theorem 2.12, the triple $(\mathcal{S}o(n), [, ,]_\alpha, \alpha^2)$ is a multiplicative Hom-Lie triple system where for all $X, Y, Z \in \mathcal{S}o(n)$, $\alpha^2(X) = (A^t)^2XA^2$ and $[X,Y,Z]_\alpha = (A^t)^2XYZA^2 - (A^t)^2ZXYA^2 - (A^t)^2YXZA^2 + (A^t)^2ZYXA^2$.

We may also derive new Hom-Lie triple systems from a given multiplicative Hom-Lie triple system. This procedure allows to generate a sequence of multiplicative Hom-Lie triple systems starting with any multiplicative Hom-Lie triple system.
Let $(T, [, ,], \alpha)$ be a multiplicative Hom-Lie triple system and n be a positive integer. Let the map $[, ,]^{(n)} : T \times T \times T \longrightarrow T$ defined by $[, ,]^{(n)} = \alpha^n \circ [, ,]$. We have the following theorem.

Theorem 2.17. *The triple $(T, [, ,]^{(n)}, \alpha^{n+1})$ is a multiplicative Hom-Lie triple system, called the n^{th} derived Hom-Lie triple system of T.*
In particular for $n = 0$ we have the multiplicative Hom-Lie triple system $(T, [, ,], \alpha)$.

Proof. Let $n \in \mathbb{N}$. It is obvious that the maps $[, ,]^{(n)}$ and α^{n+1} are respectively trilinear and linear.
$i)$ For all $x, y, z \in T$, we have

$$[x,y,z]^{(n)} = \alpha^n([x,y,z]) = \alpha^n(-[y,x,z]) = -[y,x,z]^{(n)}.$$

ii) For all $x, y, z \in T$, we have

$$\begin{aligned} [x,y,z]^{(n)} + [y,z,x]^{(n)} + [z,x,y]^{(n)} &= \alpha^n([x,y,z]) + \alpha^n([y,z,x]) + \alpha^n([z,x,y]) \\ &= \alpha^n([x,y,z] + [y,z,x] + [z,x,y]) \\ &= \alpha^n(0) \\ &= 0. \end{aligned}$$

iii) By using the fact that the Hom-Lie triple system $(T, [,,], \alpha)$ is multiplicative and the linearity of the map α, we have for all $x, y, z, u, v \in T$,

$$\begin{aligned} &[\alpha^{n+1}(u), \alpha^{n+1}(v), [x,y,z]^{(n)}]^{(n)} \\ &= \alpha^n([\alpha^{n+1}(u), \alpha^{n+1}(v), \alpha^n([x,y,z])]) \\ &= \alpha^{2n}([\alpha(u), \alpha(v), [x,y,z]]) \\ &= \alpha^{2n}([[u,v,x], \alpha(y), \alpha(z)] + [\alpha(x), [u,v,y], \alpha(z)] + [\alpha(x), \alpha(y), [u,v,z]]) \\ &= \alpha^{2n}([[u,v,x], \alpha(y), \alpha(z)]) + \alpha^{2n}([\alpha(x), [u,v,y], \alpha(z)]) + \alpha^{2n}([\alpha(x), \alpha(y), [u,v,z]]) \\ &= \alpha^n([\alpha^n([u,v,x]), \alpha^{n+1}(y), \alpha^{n+1}(z)]) + \alpha^n([\alpha^{n+1}(x), \alpha^n([u,v,y]), \alpha^{n+1}(z)]) \\ &\quad + \alpha^n([\alpha^{n+1}(x), \alpha^{n+1}(y), \alpha^n([u,v,z])]) \\ &= [[u,v,x]^{(n)}, \alpha^{n+1}(y), \alpha^{n+1}(z)]^{(n)} + [\alpha^{n+1}(x), [u,v,y]^{(n)}, \alpha^{n+1}(z)]^{(n)} \\ &\quad + [\alpha^{n+1}(x), \alpha^{n+1}(y), [u,v,z]^{(n)}]^{(n)}. \end{aligned}$$

Therefore $(T, [,,]^{(n)}, \alpha^{n+1})$ is a multiplicative Hom-Lie triple system. □

In the following we construct Hom–Lie triple systems involving elements of the centroid of Lie triple systems.

Definition 2.18. [11] Let $(T, [,,])$ be a Lie triple system. The centroide of $(T, [,,])$ is the set denoted by $Cent(T)$ and defined by

$$Cent(T) = \{\alpha \in End(T); \alpha([x,y,z]) = [\alpha(x), y, z], \text{ for all } x, y, z \in T\}.$$

Remark 2.19. For any Lie triple system $(T, [,,])$, if $\alpha \in Cent(T)$ then we have $\alpha([x,y,z]) = [x, \alpha(y), z] = [x, y, \alpha(z)]$, for all $x, y, z \in T$.
Hence, $\alpha \in Cent(T) \Leftrightarrow \alpha([x,y,z]) = [\alpha(x), y, z] = [x, \alpha(y), z] = [x, y, \alpha(z)]$, for all $x, y, z \in T$.

Theorem 2.20. *Let $(T, [,,])$ be a Lie triple system, $\alpha \in Cent(T)$ and k, $n \in \mathbb{N}$. Define the map $[,,]^n_\alpha$ by $[x,y,z]^n_\alpha = [\alpha^n(x), y, z]$, for all $x, y, z \in T$. Then $(T, [,,]^n_\alpha, \alpha^k)$ is a Hom-Lie triple system.*

Proof. It is obvious that the maps α^n and α^k are linears. Since $\alpha \in Cent(T)$, it follows that

$$[x,y,z]^n_\alpha = [\alpha^n(x), y, z] = \alpha^n([x,y,z]), \text{ for all } x, y, z \in T.$$

So $[,,]^n_\alpha = \alpha^n \circ [,,]$. Therefore $[,,]^n_\alpha$ is a trilinear map.
i) For all $x, y, z \in T$, we have

$$[x,y,z]^n_\alpha = \alpha^n([x,y,z]) \qquad = \alpha^n(-[y,x,z]) = -\alpha^n([y,x,z]) \qquad = -[y,x,z]^n_\alpha.$$

ii) For all $x, y, z \in T$, we have

$$\begin{aligned} [x,y,z]^n_\alpha + [y,z,x]^n_\alpha + [z,x,y]^n_\alpha &= \alpha^n([x,y,z]) + \alpha^n([y,z,z]) + \alpha^n([z,x,z]) \\ &= \alpha^n([x,y,z] + [y,z,x] + [z,x,y]) \\ &= \alpha^n(0) \\ &= 0. \end{aligned}$$

iii) For all $x, y, z, u, v \in T$, we have

$[\alpha^k(u), \alpha^k(v), [x, y, z]^n_\alpha]^n_\alpha$
$= \alpha^n([\alpha^k(u), \alpha^k(v), \alpha^n([x, y, z])])$
$= \alpha^{2n+2k}([u, v, [x, y, z]])$
$= \alpha^{2n+2k}([[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]])$
$= \alpha^{2n+2k}([[u, v, x], y, z]) + \alpha^{2n+2k}([x, [u, v, y], z]) + \alpha^{2n+2k}([x, y, [u, v, z]])$
$= \alpha^n([\alpha^n([u, v, x]), \alpha^k(y), \alpha^k(z)]) + \alpha^n([\alpha^k(x), \alpha^n([u, v, y]), \alpha^k(z)])$
$\quad + \alpha^n([\alpha^k(x), \alpha^k(y), \alpha^n([u, v, z])])$
$= [[u, v, x]^n_\alpha, \alpha^k(y), \alpha^k(z)]^n_\alpha + [\alpha^k(x), [u, v, y]^n_\alpha, \alpha^k(z)]^n_\alpha + [\alpha^k(x), \alpha^k(y), [u, v, z]^n_\alpha]^n_\alpha$.
Thus $(T, [, ,]^n_\alpha, \alpha^k)$ is a Hom-Lie triple system. □

3. Involutions of Hom-Lie algebras and Hom-Lie triple systems

By the corollary 2.14, we see that any subspace of a multiplicative Hom-Lie algebra $(\mathcal{G}, [,], \alpha)$ closed under the map α^2 and the ternary product $[x, y, z] = [[x, y], \alpha(z)]$, is a multiplicative Hom-Lie triple system. We use this process to establish a connection between involutions of multiplicative Hom-Lie algebras and Hom-Lie triple systems. Start by recalling the definition of an involution of Hom-Lie algebra.

Definition 3.1. A linear map $\theta : \mathcal{G} \to \mathcal{G}$ is an involution of a Hom-Lie algebra $(\mathcal{G}, [,], \alpha)$ if

1. $\theta([x, y]) = [\theta(x), \theta(y)]$, for all $x, y \in \mathcal{G}$;
2. $\theta \circ \alpha = \alpha \circ \theta$;
3. $\theta \circ \theta = id_{\mathcal{G}}$.

Theorem 3.2. *Let $(\mathcal{G}, [,], \alpha)$ be a multiplicative Hom-Lie algebra. Let θ be an involution of $(\mathcal{G}, [,], \alpha)$. Define $\mathcal{G}^-_\theta = \{x \in \mathcal{G}; \theta(x) = -x\}$. Then $(\mathcal{G}^-_\theta, [, ,], \alpha^2)$ is a multiplicative Hom-Lie triple system where $[x, y, z] = [[x, y], \alpha(z)]$, for all $x, y, z \in \mathcal{G}$.*

Proof. As $\mathcal{G}^-_\theta$ is the -1-eigenspace of θ in $\mathcal{G}$, then $\mathcal{G}^-_\theta$ is a subspace of $\mathcal{G}$. So, we just need to show that $\mathcal{G}^-_\theta$ is closed under the maps $[, ,]$ and α^2.
Let $x, y, z \in \mathcal{G}^-_\theta$. We have $\theta([x, y, z]) = \theta([[x, y], \alpha(z)])$. Since θ is an involution of $(\mathcal{G}, [,], \alpha)$, then for all $u, v \in \mathcal{G}$, $\theta([u, v]) = [\theta(u), \theta(v)]$ and $\theta \circ \alpha = \alpha \circ \theta$. We have also $\theta(x) = -x, \theta(y) = -y$ et $\theta(z) = -z$. It follows that

$$\theta([x, y, z]) = [[-x, -y], \alpha(-z)] = -[[x, y], \alpha(z)] = -[x, y, z].$$

That means $[x, y, z] \in \mathcal{G}^-_\theta$.
Let $x \in \mathcal{G}^-_\theta$. Then $\theta(x) = -x$. As $\theta \circ \alpha = \alpha \circ \theta$, it comes that, $\theta(\alpha^2(x)) = \alpha^2(\theta(x)) = \alpha^2(-x) = -\alpha^2(x)$. That means $\alpha^2(x) \in \mathcal{G}^-_\theta$.
By the corollary 2.14, $(\mathcal{G}^-_\theta, [, ,], \alpha^2)$ is a multiplicative Hom-Lie triple system. □

Proposition 3.3. *Let $(\mathcal{G}, [,])$ be a Lie algebra. Let θ be an involution of $(\mathcal{G}, [,])$ and α an endomorphism of $(\mathcal{G}, [,])$ such that $\theta \circ \alpha = \alpha \circ \theta$. Define $\mathcal{G}^-_\theta = \{x \in \mathcal{G}; \theta(x) = -x\}$. Then the triples $(\mathcal{G}^-_\theta, [, ,]^1_\alpha, \alpha)$ and $(\mathcal{G}^-_\theta, [, ,]^2_\alpha, \alpha^2)$ are multiplicative Hom-Lie triple systems where $[x, y, z]^1_\alpha = \alpha([[x, y], z])$ and $[x, y, z]^2_\alpha = \alpha^2([[x, y], z])$ for all $x, y, z \in \mathcal{G}$. The triple $(\mathcal{G}^-_\theta, [, ,]^2_\alpha, \alpha^2)$ is the first derived Hom-Lie triple system of $(\mathcal{G}^-_\theta, [, ,]^1_\alpha, \alpha)$.*

Proof. In the one hand, the vector space $\mathcal{G}_\theta^-$ with $[,,]$ is a Lie triple system as the -1-eigenspace of the involution θ of the Lie algebra $\mathcal{G}$ where
$[x,y,z]=[[x,y],z]$, for all $x,y,z\in\mathcal{G}$.
For all $x\in\mathcal{G}_\theta^-$, we have

$$\theta(\alpha(x))=\alpha(\theta(x))=\alpha(-x)=-\alpha(x);$$

that implies $\alpha(x)\in\mathcal{G}_\theta^-$. So $\mathcal{G}_\theta^-$ is closed under α.
Moreover, for all $x,y,z\in\mathcal{G}_\theta^-$, we have $\alpha([x,y,z]) = [\alpha(x),\alpha(y),\alpha(z)]$. So α is an endomorphism of the Lie triple system $(\mathcal{G}_\theta^-,[,,])$. By using the theorem 2.11, the triple $(\mathcal{G}_\theta^-,[,,]_\alpha^1,\alpha)$ is multiplicative Hom-Lie triple system.
In the other hand, we know by theorem 2.4, that $(\mathcal{G},[,]_\alpha=\alpha\circ[,],\alpha)$ is a multiplicative Hom-Lie algebra. Since $\theta\circ\theta=id_{\mathcal{G}}$, $\theta\circ\alpha=\alpha\circ\theta$ and

$$\theta([x,y]_\alpha) = \theta(\alpha([x,y])) = \alpha(\theta([x,y])) = \alpha([\theta(x),\theta(y)]) = [\theta(x),\theta(y)]_\alpha,$$

then θ is also an involution of the multiplicative Hom-Lie algebra
$(\mathcal{G},[,]_\alpha,\alpha)$. Moreover, we have for all $x,y,z\in\mathcal{G}$

$$[x,y,z]_\alpha^2 = \alpha^2([[x,y],z]) = \alpha([\alpha([x,y]),\alpha(z)]) = [[x,y]_\alpha,\alpha(z)]_\alpha.$$

So, by using the theorem 3.2, the triple $(\mathcal{G}_\theta^-,[,,]_\alpha^2,\alpha^2)$ is a multiplicative Hom-Lie triple system. Since $[,,]_\alpha^2=\alpha\circ[,,]_\alpha^1$, therefore $(\mathcal{G}_\theta^-,[,,]_\alpha^2,\alpha^2)$ is the first derived Hom-Lie triple system of $(\mathcal{G}_\theta^-,[,,]_\alpha^1,\alpha)$. □

In what follows, we give some examples of construction of Hom-Lie triple systems with involutions of classical Hom-Lie algebras.

Example 3.4. Let n be a positive integer such that $n\geq 2$. Let n_1 and n_2 be two positive integers such that $n_1+n_2=n$ and $0<n_1,n_2<n$.
Put $J=\begin{pmatrix} I_{n_1} & 0\\ 0 & -I_{n_2}\end{pmatrix}$. It is clear that $J^2=I_n$.
Let A be a matrix in $GL(n)$ such that $AJ=JA$.
Consider in the example 2.5, the multiplicative Hom-Lie algebra
$(\mathcal{S}l(n),[,]_\alpha,\alpha)$ where $\alpha(X)=A^{-1}XA$ and $[X,Y]_\alpha=A^{-1}XYA-A^{-1}YXA$,
for all $X,Y\in\mathcal{S}l(n)$. Define the map $\theta:\mathcal{S}l(n)\longrightarrow\mathcal{S}l(n)$, $X\mapsto JXJ$. The map θ is an involution of the Hom-Lie algebra $(\mathcal{S}l(n),[,]_\alpha,\alpha)$. Indeed, for all $X\in\mathcal{S}l(n)$, we have,

$$tr(\theta(X)) = tr(JXJ) = tr(JJX) = tr(I_nX) = tr(X) = 0.$$

That means for all $X\in\mathcal{S}l(n),\theta(X)\in\mathcal{S}l(n)$.
Also, for all $X,Y\in\mathcal{S}l(n)$ and for all $k\in\mathbb{K}$, we have,

$$\theta(X+kY) = J(X+kY)J = JXJ+kJYJ = \theta(X)+k\theta(Y).$$

That proves the linearity of θ.
Moreover for all X in $\mathcal{S}l(n)$, we have

$$\theta^2(X) = \theta(\theta(X)) = \theta(JXJ) = J^2XJ^2 = X.$$

That means $\theta^2\equiv id_{\mathcal{S}l(n)}$. By using the fact that $AJ=JA$, we have for all $X\in\mathcal{S}l(n)$,

$$\theta(\alpha(X)) = \theta(A^{-1}XA) = JA^{-1}XAJ = A^{-1}JXJA = \alpha(JXJ) = \alpha(\theta(X)).$$

So $\theta(\alpha(X)) = \alpha(\theta(X)),$ for all $X \in \mathcal{S}l(n)$.
By using the fact that $J^2 = I_n$, we have for all $X, Y \in \mathcal{S}l(n)$,

$$\begin{aligned}\theta([X,Y]_\alpha) &= \theta(A^{-1}XYA - A^{-1}YXA)\\ &= J(A^{-1}XYA - A^{-1}YXA)J\\ &= JA^{-1}XYAJ - JA^{-1}YXAJ\\ &= A^{-1}JXI_nYJA - A^{-1}JYI_nXJA\\ &= A^{-1}(JXJ)(JYJ)A - A^{-1}(JYJ)(JXJ)A\\ &= A^{-1}\theta(X)\theta(Y)A - A^{-1}\theta(Y)\theta(X)A\\ &= [\theta(X), \theta(Y)]_\alpha.\end{aligned}$$

So $\theta([X,Y]_\alpha) = [\theta(X), \theta(Y)]_\alpha,$ for all $X, Y \in \mathcal{S}l(n)$.
Therefore the map θ is an involution of the Hom-Lie algebra $(\mathcal{S}l(n), [,]_\alpha, \alpha)$. By the theorem 3.2, the -1-eigenspace of θ in $\mathcal{S}l(n)$ defined by

$$\mathcal{S}l(n)^-_\theta = \{X \in \mathcal{S}l(n); \theta(X) = -X\} = \{X \in \mathcal{S}l(n); JX = -XJ\}$$

is a Hom-Lie triple system relative to α^2 and $[,,]_\alpha$ where

$$[X,Y,Z]_\alpha = [[X,Y]_\alpha, \alpha(Z)]_\alpha \text{ for all } X,Y,Z \in \mathcal{S}l(n).$$

Example 3.5. Let A be a matrix in $O(n)$. Then $A^t = A^{-1}$. It follows that $A = (A^{-1})^t = (A^t)^{-1}$.
Consider in the example 2.5, the multiplicative Hom-Lie algebra $(\mathcal{S}l(n), [,]_\alpha, \alpha)$ where $\alpha(X) = A^{-1}XA$ and $[X,Y]_\alpha = A^{-1}XYA - A^{-1}YXA$, for all $X, Y \in \mathcal{S}l(n)$. Define the map $\theta : \mathcal{S}l(n) \longrightarrow \mathcal{S}l(n)$, $X \mapsto -X^t$. The map θ is an involution of the Hom-Lie algebra $(\mathcal{S}l(n), [,]_\alpha, \alpha)$. Indeed, for all $X \in \mathcal{S}l(n)$, we have,

$$tr(\theta(X)) \;= tr(-X^t) \;= -tr(X^t) \;= -tr(X) \;= 0.$$

That means for all $X \in \mathcal{S}l(n), \theta(X) \in \mathcal{S}l(n)$.
Also, for all $X, Y \in \mathcal{S}l(n)$ and for all k in $\mathbb{K}$, we have,

$$\theta(X + kY) \;= -(X + kY)^t \;= -X^t + k(-Y^t) \;= \theta(X) + k\theta(Y).$$

That proves the linearity of θ.
Moreover for all $X \in \mathcal{S}l(n)$, we have

$$\theta^2(X) \;= \theta(\theta(X)) \;= \theta(-X^t) \;= -(-X^t)^t \;= X.$$

That means $\theta^2 \equiv id_{\mathcal{S}l(n)}$.
For all $X \in \mathcal{S}l(n)$, we have

$$\theta(\alpha(X)) \;= -(A^{-1}XA)^t \;= -A^tX^t(A^{-1})^t \;= A^{-1}(-X^t)A \;= \alpha(\theta(X)).$$

So $\theta(\alpha(X)) = \alpha(\theta(X)),$ for all $X \in \mathcal{S}l(n)$.
For all $X, Y \in \mathcal{S}l(n)$, we have

$$\begin{aligned}\theta([X,Y]_\alpha) &= \theta(A^{-1}XYA - A^{-1}YXA)\\ &= -(A^{-1}XYA - A^{-1}YXA)^t\\ &= -(A^{-1}XYA)^t + (A^{-1}YXA)^t\\ &= -A^tY^tX^t(A^{-1})^t + A^tX^tY^t(A^{-1})^t\\ &= -A^{-1}Y^tX^tA + A^{-1}X^tY^tA\\ &= -A^{-1}(-Y^t)(-X^t)A + A^{-1}(-X^t)(-Y^t)A\\ &= -A^{-1}\theta(Y)\theta(X)A + A^{-1}\theta(X)\theta(Y)A\\ &= A^{-1}\theta(X)\theta(Y)A - A^{-1}\theta(Y)\theta(X)A\\ &= [\theta(X), \theta(Y)]_\alpha.\end{aligned}$$

So $\theta([X,Y]_\alpha) = [\theta(X), \theta(Y)]_\alpha,$ for all $X, Y \in \mathcal{S}l(n)$.
Therefore the map θ is an involution of the Hom-Lie algebra $(\mathcal{S}l(n), [,]_\alpha, \alpha)$. By the theorem 3.2, the -1-eigenspace of θ in $\mathcal{S}l(n)$ defined by

$$\mathcal{S}l(n)^-_\theta = \{X \in \mathcal{S}l(n); \theta(X) = -X\} = \left\{X \in \mathcal{S}l(n); X^t = X\right\}$$

is a Hom-Lie triple system relative to α^2 and $[,,]_\alpha$ where

$$[X,Y,Z]_\alpha = [[X,Y]_\alpha, \alpha(Z)]_\alpha, \text{ for all } X, Y, Z \in \mathcal{S}l(n).$$

The vector space $\mathcal{S}l(n)^-_\theta$ consists of symmetric matrices X of order n with elements in $\mathbb{K}$ such that $tr(X) = 0$.

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