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ON THE LOWER LAYERS OF A $\mathbb{Z}_p$-EXTENSION

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Abstract. It is shown that the p-part of the class number of the lower layers of a cyclotomic $\mathbb{Z}_p$-extension can grow exponentially.

1. Introduction

Fix a prime number p and let k be a number field. Suppose that K is a $\mathbb{Z}_p$-extension of k, so $K = \cup_{n\geq 0} k_n$ with $k_n \subset k_{n+1}$ and $Gal(k_n/k) \simeq \mathbb{Z}/p^n\mathbb{Z}$. Denote by L_n the p-Hilbert class field of the n-th layer k_n, A_n the galois group $Gal(L_n/k_n)$, and write $L_K = \cup_{n\geq 0} L_n$. Iwasawa theory [1] shows that there exists integers $\mu \geq 0, \lambda \geq 0, \nu$ such that for sufficiently large n, one has

$$h_n := |A_n| = p^{\mu p^n + \lambda n + \nu}$$

It is well-known that μ-invariant vanishes when K is a cyclotomic $\mathbb{Z}_p$-extension of an abelian number field k. It implies that the exponent of h_n grows linearly for higher layers. In this paper, we prove that h_n can grow exponentially in lower layers of a cyclotomic $\mathbb{Z}_p$-extension and give an example of it.

2. Proof of Theorems

Let

$$Y_K = Gal(L_K/K).$$

By Nakayama lemma, one can show that Y_K is a finitely generated torsion $\Lambda = \mathbb{Z}_p[[Gal(K/k)]]$-module. The Iwasawa algebra Λ is isomorphic to the ring of the formal power series $\mathbb{Z}_p[[T]]$ in one variable over $\mathbb{Z}_p$. The isomorphism is given by identifying $1 + T$ with a topological generator γ of $Gal(K/k)$. A polynomial $P(T) \in \mathbb{Z}_p[T]$ is called distinguished if $P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$ with $p|a_i$ for $0 \leq i \leq n-1$. By the p-adic Weierstrass preparation theorem, every element $g(T)$ in $\mathbb{Z}_p[[T]]$ may be uniquely written in the form

$$g(T) = p^m U(T) P(T),$$

where m is a non-negative integer, $U(T)$ is a unit, and $P(T)$ is a distinguished polynomial.

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THEOREM 2.1. *Suppose K/k is a $\mathbb{Z}_p$-extension in which exactly one prime ramfies and totally ramifies in K/k. Then*

$$Gal(L_n/k_n) \simeq Y_K/((1+T)^{p^n}-1)Y_K.$$

Proof See [4].

THEOREM 2.2. *Let K be a $\mathbb{Z}_p$-extension of k with $\mu = 0$. Suppose that $h_0 = p$ and exactly one prime ramfies and totally ramifies in K/k. Then*

$$h_n = p^{p^n} \text{ for } n \text{ satisfying } p^n \leq \lambda.$$

Proof Since only one prime ramfies and totally ramifies in K/k and $Gal(L_0/k_0)$ is cyclic, we see that Y_K is a cyclic Λ-module by Theorem 2.1 and Nakayama lemma. Moreover μ is zero, hence

$$Y_K \simeq \mathbb{Z}_p[[T]]/(P(T))$$

By Theorem 2.1 and the condition in Theorem 2.2, we have

$$\mathbb{Z}/p\mathbb{Z} \simeq A_0 \simeq Y_K/TY_K \simeq Z_p[[T]]/(P(T),T)$$

Hence we may assume

$$P(0) = p.$$

For n satisfying $p^n \leq \lambda$,

$$P(T) - ((1+T)^{p^n}-1)T^{\lambda-p^n} = P(T) - T^{\lambda} + pTG(T)$$
$$= p + b_1T + \cdots + b_{\lambda-1}T^{\lambda-1} = pU(T),$$

where $p|b_i$ for $1 \leq i \leq \lambda - 1, G(T) \in \mathbb{Z}_p[T]$ and $U(T)$ is a unit in $\mathbb{Z}_p[T]$. Therefore we have

$$A_n \simeq Y_K/((1+T)^{p^n}-1)Y_K \simeq Z_p[[T]]/(P(T),(1+T)^{p^n}-1)$$
$$\simeq Z_p[[T]]/(p,(1+T)^{p^n}-1) \simeq Z_p[[T]]/(p,T^{p^n})$$

This completes the proof.

We give an example satisfying conditions of Theorem 2.2

EXAMPLE 1. *For $k = \mathbb{Q}(\sqrt{-53301})$ and $p = 3$, p ramifies in k, $\lambda = 11$ and $h_k = 264 = 3*8*11$. When K is the cyclotomic $\mathbb{Z}_3$-extension of k, all the assumptions in Theorem 2.2 are satisfied. So we see that*

$$h_0 = 3, h_1 = 3^3, h_2 = 3^9.$$

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