Korean J. Math. **30** (2022), No. 2, pp. 205–229
http://dx.doi.org/10.11568/kjm.2022.30.2.205

COMMON FIXED POINT THEOREMS FOR COMPLEX-VALUED MAPPINGS WITH APPLICATIONS

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Abstract. The aim of this paper is to obtain some results which belong to fixed point theory such as strong convergence, rate of convergence, stability, and data dependence by using the new Jungck-type iteration method for a mapping defined in complex-valued Banach spaces. In addition, some of these results are supported by nontrivial numerical examples. Finally, it is shown that the sequence obtained from the new iteration method converges to the solution of the functional integral equation in complex-valued Banach spaces. The results obtained in this paper may be interpreted as a generalization and improvement of the previously known results.

1. Introduction and Preliminaries

Fixed point theory is a dynamic field that has been studied theoretically and practically by many researchers. Iteration methods are used to show the existence or uniqueness of fixed points of certain mapping classes [31, 32, 34]. In addition, fixed point theory has become a useful tool in demonstrating the solution of some differential and integral equations [4, 10, 13, 33].

In the fixed point theory, many studies have been done in different spaces, such as metric spaces [40], b-metric spaces [9, 23], quasi metric spaces [41], G-metric spaces [25], convex metric spaces [11], partially ordered metric spaces [24, 38], hyperbolic spaces [2, 19], CAT spaces [12] and elliptic valued metric spaces [28]. Complex-valued Banach spaces, one of them, were introduced in 2011 by Azam et al. [5].

The authors in [5], have achieved an extension of the Banach fixed point theorem for complex-valued metric spaces. After that, many researchers have obtained some remarkable and applicable results by studying this theory [1, 18, 26, 36]. Recall the definition of partial order $\precsim$ on $\mathbb{C}$ as follows [5]:
Let $\mathbb{C}$ be the set of complex numbers and $w_1, w_2 \in \mathbb{C}$. The $\precsim$ symbol has the following condition:
$w_1 \precsim w_2$ if only if $Re(w_1) \leq Re(w_2)$ and $Im(w_1) \leq Im(w_2)$.
It means that $w_1 \precsim w_2$, if the following conditions are satisfied:

i. $Re(w_1) = Re(w_2)$, $Im(w_1) < Im(w_2)$

Received January 11, 2022. Revised May 6, 2022. Accepted May 7, 2022.
2010 Mathematics Subject Classification: 47H10, 54H25, 30E10.
Key words and phrases: convergence, stability, data dependence, iteration method, functional-integral equation.
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ii. $Re(w_1) < Re(w_2)$, $Im(w_1) = Im(w_2)$
iii. $Re(w_1) < Re(w_2)$, $Im(w_1) < Im(w_2)$
iv. $Re(w_1) = Re(w_2)$, $Im(w_1) = Im(w_2)$

Definition 1.1. [5]: Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies in the following conditions:

i. $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$
ii. $d(x, y) = d(y, x)$ for all $x, y \in X$
iii. $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex-valued metric on X and (X, d) is called a complex-valued metric space.

Definition 1.2. [26]: Let X be a linear space over a field $\mathbb{K}$, where $\mathbb{K} = \mathbb{C}$ (the set of complex numbers). A complex-valued norm on X is a complex-valued function $\|\cdot\| : X \to \mathbb{C}$ satisfying in the following conditions:

i. $\|x\| = 0$ iff $x = 0$, $x \in X$
ii. $\|\delta x\| = |\delta| \cdot \|x\|$, $\delta \in \mathbb{C}$, $x \in X$
iii. $\|x - y\| \precsim \|x - z\| + \|z - y\|$ for all $x, y, z \in X$.

Then, $\|\cdot\|$ is called a complex-valued norm on X and $(X; \|\cdot\|)$ is called a complex-valued linear normed space. If every Cauchy sequence is convergent in $(X, \|\cdot\|)$, then $(X, \|\cdot\|)$ is called a complex-valued Banach space.

Example 1.3. [27]: Let $E = [0, 1]$ and define $\|\cdot\| : E \to \mathbb{C}$ by

$$\|x - y\| = |x - y| e^{\frac{i\pi}{3}}. \tag{1}$$

Then, $(E, \|\cdot\|)$ is a complex-valued Banach space.

Example 1.4. [27]: Let $E = [0, 1]$ and define $\|\cdot\| : E \to \mathbb{C}$ by

$$\|x - y\| = |x - y| i. \tag{2}$$

Then, $(E, \|\cdot\|)$ is a complex-valued Banach space.

Example 1.5. [26]: Let $C[a, b]$ be all continuous complex-valued functions defined on $[a, b]$ and define $\|\cdot\| : C[a, b] \to \mathbb{C}$ by

$$\|x - y\|_\infty = \max_{s \in [a,b]} |x(s) - y(s)| e^{it} \tag{3}$$

where $x, y \in C[a, b]$, $t \in [0, \frac{\pi}{2}]$. Then, $(C[a, b], \|\cdot\|)$ is a complex-valued Banach space.

Let X be Banach space, Y an arbitrary set and $S, T{:}Y \to X$ such that $T(Y) \subseteq S(Y)$. For $x_0 \in Y$, the following iteration method is called Jungck iteration method [15]:

$$Sx_{n+1} = f(T, x_n) \tag{4}$$

for all $n \in \mathbb{N}$. After the sequence has been defined by Jungck, many iteration methods have been introduced in this context [3, 8, 14, 16, 17, 20].

Jungck-SP iteration method [8], Jungck-CR iteration method [14], and Jungck Agarwal iteration method [16] are given below respectively:

$$\begin{cases} Sx_{n+1} = (1 - \alpha_n) Sy_n + \alpha_n Ty_n \\ Sy_n = (1 - \beta_n) Sz_n + \beta_n Tz_n \\ Sz_n = (1 - \gamma_n) Sx_n + \gamma_n Tx_n \end{cases} \tag{5}$$

and

$$(6)\qquad \begin{cases} Su_{n+1} = (1-\alpha_n)\, Sv_n + \alpha_n Tv_n \\ Sv_n = (1-\beta_n)\, Tu_n + \beta_n Tw_n \\ Sw_n = (1-\gamma_n)\, Su_n + \gamma_n Tu_n \end{cases}$$

and

$$(7)\qquad \begin{cases} x_{n+1} = (1-\alpha_n)\, Tx_n + \alpha_n Ty_n \\ Sy_n = (1-\beta_n)\, Sx_n + \beta_n Tx_n \end{cases}$$

in which $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \subset [0,1]$.

The iteration methods defined in [21] and [22] respectively, are as follows:

$$(8)\qquad \begin{cases} x_{n+1} = \frac{(1-\alpha_n)}{k} Tx_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) Ty_n \\ y_n = \frac{(1-\beta_n)}{k} x_n + \left(1 - \frac{(1-\beta_n)}{k}\right) Tx_n \end{cases}$$

and

$$(9)\qquad \begin{cases} x_{n+1} = Tu_n \\ u_n = \frac{(1-\alpha_n)}{k} Tx_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) Ty_n \\ y_n = Tz_n \\ z_n = \frac{(1-\beta_n)}{k} x_n + \left(1 - \frac{(1-\beta_n)}{k}\right) Tx_n \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ and $k \in \mathbb{N}$.
By inspiring the above iteration methods (8) and (9), we have introduced new Jungck type iteration methods as follows:

$$(10)\qquad \begin{cases} Sx_{n+1} = \frac{(1-\alpha_n)}{k} Tx_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) Ty_n \\ Sy_n = \frac{(1-\beta_n)}{k} Sx_n + \left(1 - \frac{(1-\beta_n)}{k}\right) Tx_n \end{cases}$$

and

$$(11)\qquad \begin{cases} Sx_{n+1} = Tu_n \\ Su_n = \frac{(1-\alpha_n)}{k} Tx_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) Ty_n \\ Sy_n = Tz_n \\ Sz_n = \frac{(1-\beta_n)}{k} Sx_n + \left(1 - \frac{(1-\beta_n)}{k}\right) Tx_n \end{cases}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ and $k \in \mathbb{N}$.

Definition 1.6. [30]: Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real numbers with the same limit c. We say that $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$ to c, if

$$\lim_{n\to\infty} \frac{d(a_n, c)}{d(b_n, c)} = 0.$$

Definition 1.7. [14]: Let X be a nonempty set and S,T:$X \to X$ be mappings. If Tx=Sx, then $x \in X$ is called coincidence point of T and S. If x=Tx=Sx, then $x \in X$ is called common fixed point of T and S. If p=Tx=Sx for some $x \in X$, then p is called the point of coincidence of T and S. If TSx=STx for all $x \in X$, then a pair (S,T) is called commuting. If TSx=STx whenever Tx=Sx for some $x \in X$, then a pair (S,T) is called weakly compatible.

DEFINITION 1.8. [18]: For two non-self mappings $S, T : E \to E$ of complex-valued metric space (E, d) satisfying the following condition:

i. $T(E) \subseteq S(E)$

ii.

$$d(Tx, Ty) \precsim \lambda \cdot d(Sx, Sy) + \mu \cdot \left(\frac{d(Sx, Tx) \cdot d(Sy, Ty)}{1 + d(Sx, Sy)} \right) \tag{12}$$

in which λ and μ are non-negative constants with $\lambda + \mu < 1$.

iii. $S(E)$ is a complete subspace of E.

Then S and T have a coincidence point. Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

The mapping (12) has been given in normed spaces as follows:

DEFINITION 1.9. $(E, \|\cdot\|)$ is a complex-valued Banach space. For two nonself mappings $S, T : E \to X$ satisfying the following condition:

$$\|Tx - Ty\| \precsim \lambda \cdot \|Sx - Sy\| + \mu \cdot \left(\frac{\|Sx - Tx\| \cdot \|Sy - Ty\|}{1 + \|Sx - Sy\|} \right) \tag{13}$$

in which λ and μ are non-negative constants with $\lambda + \mu < 1$.

EXAMPLE 1.10. Let $E = [0, 1]$ and define operators S, T:$E \to \mathbb{C}$ by $Sx = Tx = \frac{x}{8}$ with a coincidence point $p = 0$. Suppose $\lambda = \frac{1}{8}, \mu = \frac{1}{3}$, $k = 40$. By using norm (1) given by Example 1.3 one can see that S, T satisfy the condition (13). Since

$$\begin{aligned} \|Tx - Ty\| &\precsim \lambda \cdot \|Sx - Sy\| + \mu \cdot \left(\frac{\|Sx - Tx\| \cdot \|Sy - Ty\|}{1 + \|Sx - Sy\|} \right) \\ &= \|\frac{x}{8} - \frac{y}{8}\| + \left(\frac{\|\frac{x}{8} - \frac{x}{8}\| \cdot \|\frac{y}{8} - \frac{y}{8}\|}{1 + \|\frac{x}{8} - \frac{y}{8}\|} \right) \\ &= \frac{1}{8} \cdot |x - y| e^{\frac{i\pi}{3}} \\ &= \frac{1}{8} \cdot \|x - y\| \end{aligned}$$

It is clear that we have

$$\|Tx - Ty\| \precsim \frac{1}{8} \cdot \|x - y\|. \tag{14}$$

DEFINITION 1.11. [35]: Let $S, T : E \to X$, $T(Y) \subseteq S(Y)$ and $p = Tx = Sx$. For any $x_0 \in Y$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$ generated by the iteration method $Sx_{n+1} = f(T, x_n)$ converges to p. Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set

$$\epsilon_n = d(Sy_{n+1}, f(T, y_n)) \tag{15}$$

for all $n \in \mathbb{N}$. Then the iteration method $f(T, x_n)$ will be called (S, T)-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} \|Sy_n - p\| = 0$.

DEFINITION 1.12. [37]: Let T, $S : X \to X$ be two operators. S is called an approximate operator of T for all $x \in X$ and a fixed $\varepsilon > 0$ if $\|Tx - Sx\| \leq \varepsilon$.

Definition 1.13. [17]: Let $(S,T),(\tilde{S},\tilde{T}):Y\to X$ be nonself-mapping pairs on an arbitrary set Y such that $T(Y)\subseteq S(Y)$ and $\tilde{T}(Y)\subseteq\tilde{S}(Y)$. We say that the pair $(\tilde{S},\tilde{T})$ is an approximate mapping pair of (S,T) if for all $x\in Y$ and for fixed $\epsilon_1\geq 0,\epsilon_2\geq 0$, we have

$$\left\|Tx-\tilde{T}x\right\|\leq\epsilon_1,\left\|Sx-\tilde{S}x\right\|\leq\epsilon_2.$$

.

Definition 1.14. [6]: Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be sequences in X. We say that $\{y_n\}_{n=0}^{\infty}$ is an approximate sequence of $\{x_n\}_{n=0}^{\infty}$ if, for any $k\in\mathbb{N}$, there exists $\varepsilon(k)$ such that

$$\|x_n-y_n\|\leq\varepsilon(k)\text{, for all }n\geq k.$$

Definition 1.15. [7]: Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be sequences in X. We say that these sequences are equivalent if

$$\lim_{n\to\infty}\|x_n-y_n\|=0.$$

Lemma 1.16. [37]: *Let $\{a_n\}_{n=1}^{\infty}$ be a nonnegative real sequence and there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$ satisfying the following condition:*

$$a_{n+1}\leq(1-\mu_n)a_n+\mu_n\eta_n,$$

where $\mu_n\in(0,1)$ such that $\sum\limits_{n=1}^{\infty}\mu_n=\infty$ and $\eta_n\geq 0$. Then the following inequality holds:

$$0\leq\lim_{n\to\infty}\sup a_n\leq\lim_{n\to\infty}\sup\eta_n.$$

Lemma 1.17. [39]: *Let $\{b_n\}_{n=0}^{\infty}$ and $\{d_n\}_{n=0}^{\infty}$ be nonnegative real sequences satisfying the following inequality:*

$$b_{n+1}\leq(1-r_n)\,b_n+d_n$$

where $r_n\in(0,1)$ for all $n\in\mathbb{N}$, $\sum\limits_{n=0}^{\infty}r_n=\infty$ and $\lim_{n\to\infty}\frac{d_n}{r_n}=0$. Then $b_n\to 0$ as $n\to\infty$.

2. Convergence and Rate of Convergence

In this section, we show that the new Jungck type iteration method (11) converges to the unique common fixed point of complex-valued mappings S and T given in (13). Furthermore, we prove that the iteration method (11) converges faster than the iteration method (6) for this mapping. Finally, we give an example for comparison of the speed of convergence among various iteration methods in the literature.

2.1. Convergence Theorems.

Theorem 2.1. *Let E be nonempty closed convex subset of a complex-valued Banach space $(X,\|.\|)$ and $S,T{:}E{\to}X$ satisfy condition (13). Assume that $T(E)\subseteq S(E)$, $S(E)\subseteq X$, $Tx_p{=}Sx_p{=}p$. Let $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence (11) with $\sum_{n=0}^{\infty}\alpha_n{=}\infty$. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges to p. Moreover, p is a unique common fixed point of S and T provided that $E=X$. Also, S and T are weakly compatible.*

Proof. By using iterative sequence (11) and condition (13), we have

$$\begin{aligned} \|Sx_{n+1}-p\| &\precsim \lambda \| \ Su_n - Sx_p\| \\ &+ \mu \left(\frac{\|Su_n - Tu_n\| \cdot \|Sx_p - Tx_p\|}{1 + \|Su_n - Sx_p\|} \right) \\ &= \lambda \| \ Su_n - Sx_p\| \end{aligned} \tag{16}$$

and

$$\begin{aligned} \|Sy_n - p\| &\precsim \lambda \| \ Sz_n - Sx_p\| \\ &+ \mu \left(\frac{\|Sz_n - Tz_n\| \cdot \|Sx_p - Tx_p\|}{1 + \|Sz_n - Sx_p\|} \right) \\ &= \lambda \| \ Sz_n - Sx_p\| \end{aligned} \tag{17}$$

and

$$\begin{aligned} \|Sz_n - p\| &\precsim \frac{(1-\beta_n)}{k} \|Sx_n - p\| + \left(1 - \frac{(1-\beta_n)}{k}\right) \|Tx_n - p\| \\ &\precsim \frac{(1-\beta_n)}{k} \|Sx_n - \ p\| \\ &+ \left(1 - \frac{(1-\beta_n)}{k}\right) (\lambda \| \ Sx_n - Sx_p\| \\ &+ \mu \left(\frac{\|Sx_n - Tx_n\| \cdot \|Sx_p - Tx_p\|}{1 + \|Sx_n - Sx_p\|} \right)) \\ &\precsim \left(1 - \frac{\beta_n (1-\lambda)}{k}\right) \|Sx_n - p\| \end{aligned} \tag{18}$$

and

$$\begin{aligned} \|Su_n - p\| &\precsim \frac{(1-\alpha_n)}{k} \|Tx_n - p\| + \left(1 - \frac{(1-\alpha_n)}{k}\right) \|Ty_n - p\| \\ &\precsim \frac{(1-\alpha_n)}{k} (\lambda \| \ Sx_n - Sx_p\| \\ &+ \ \mu \left(\frac{\|Sx_n - Tx_n\| \cdot \|Sx_p - Tx_p\|}{1 + \|Sx_n - Sx_p\|} \right)) \\ &+ \left(1 - \frac{(1-\alpha_n)}{k}\right) (\lambda \| \ Sy_n - Sx_p\| \\ &+ \mu \left(\frac{\|Sy_n - Ty_n\| \cdot \|Sx_p - Tx_p\|}{1 + \|Sy_n - Sx_p\|} \right)) \\ &= \lambda \frac{(1-\alpha_n)}{k} \| \ Sx_n - Sx_p\| \\ &+ \lambda \left(1 - \frac{(1-\alpha_n)}{k}\right) \| \ Sy_n - Sx_p\| \\ &\precsim \lambda \left(1 - \frac{\alpha_n (1-\lambda)}{k}\right) \|Sx_n - p\| \end{aligned} \tag{19}$$

Substituting (17), (18), and (19) into (16), we have

$$\|Sx_{n+1}-p\| \precsim \lambda^2\left(1-\frac{\alpha_n(1-\lambda)}{k}\right)\|Sx_n-p\|. \tag{20}$$

By induction, we obtain

$$\|Sx_{n+1}-p\| \precsim \lambda^2\left(1-\frac{\alpha_n(1-\lambda)}{k}\right)\|Sx_n-p\|$$
$$\|Sx_n-p\| \precsim \lambda^2\left(1-\frac{\alpha_n(1-\lambda)}{k}\right)\|Sx_{n-1}-p\|$$
$$\|Sx_{n-1}-p\| \precsim \lambda^2\left(1-\frac{\alpha_n(1-\lambda)}{k}\right)\|Sx_{n-2}-p\|$$

and

$$\begin{aligned}\|Sx_{n+1}-p\| &\precsim \lambda^{2(n+1)}\prod_{i=0}^{n}\left(1-\frac{\alpha_i(1-\lambda)}{k}\right)\|Sx_0-p\|\\ &\precsim \lambda^{2(n+1)}\prod_{i=0}^{n}e^{\left(1-\frac{\alpha_i(1-\lambda)}{k}\right)}\|Sx_0-p\| \\ &\precsim \lambda^{2(n+1)}\frac{1}{e^{\frac{(1-\lambda)\sum_{i=0}^{n}\alpha_i}{k}}}\|Sx_0-p\|\end{aligned} \tag{21}$$

Taking the limit on both sides of (21) and using $\lambda^2<1$, we obtain

$$\lim_{n\to\infty}\|Sx_n-p\|=0. \tag{22}$$

We prove that p is a unique common fixed point of S and T. Assume that there exist another point of coincide q of the pair (S,T). Then, there exists x_q, which is an element of the set of coincidence points of S and T, such that $Sx_q=Tx_q=q$.
Then

$$\begin{aligned}0 \precsim \|p-q\| = \|Tx_p-Tx_q\| &\precsim \lambda\,\|Sx_p-Sx_q\| + \mu\left(\frac{\|Sx_p-Tx_p\|\cdot\,\|Sx_q-Tx_q\|}{1+\|Sx_p-Sx_q\|}\right)\\ &\precsim \lambda\,\|Sx_p-Sx_q\| = \lambda\,\|p-q\|\end{aligned}$$

Therefore,

$$0 \precsim \|p-q\| = \|Tx_p-Tx_q\| \precsim \lambda\,\|p-q\|$$

which implies that $p=q$. Also, S and T are weakly compatible and $Sx_p=Tx_p=p$, then $Tp=TTx_p=TSx_p=STx_p$ implies $Tp=Sp$. Hence, Tp is a point of coincidence of the pair (S,T) and because point of coincidence is unique, then $Tp=p$. So, $Sp=Tp=p$ and thus p is a unique common fixed point of S and T. □

The following theorem indicates that the convergence result can be obtained without the $\sum_{n=0}^{\infty}\alpha_n=\infty$ condition for the sequence of $\{\alpha_n\}_{n=0}^{\infty}\subset[0,1]$:

Theorem 2.2. *Let S,T be the same as in Theorem 2.1 with $Tx_p=Sx_p=p$. Let $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence (11) with $\{\alpha_n\}_{n=0}^{\infty},\{\beta_n\}_{n=0}^{\infty}\subset[0,1]$. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges to p. Moreover, p is a unique common fixed point of S and T provided that $E=X$. Also, S and T are weakly compatible.*

Proof. The proof is similar to that of Theorem 2.1. Consider the following inequality

$$\|Sx_{n+1}-p\| \precsim \lambda^{2(n+1)}\prod_{i=0}^{n}\left(1-\frac{\alpha_i(1-\lambda)}{k}\right)\|Sx_0-p\|. \tag{23}$$

Since $\left(1-\frac{\alpha_i(1-\lambda)}{k}\right)<1$, (for $i=0,1,2,...n$), $\lambda<1$ and $\{\alpha_n\}_{n=0}^{\infty},\{\beta_n\}_{n=0}^{\infty}\subset[0,1]$ for all $n\in\mathbb{N}$, we have obtained in the following

$$\|Sx_{n+1}-p\| \precsim \lambda^{2(n+1)}\|Sx_0-p\| \tag{24}$$

Taking the limit in the inequality (24), it can be seen that $\lim\limits_{n\to\infty}\|Sx_n-p\|=0$. Hence the condition $\sum_{n=0}^{\infty}\alpha_n=\infty$ is unnecessary. □

THEOREM 2.3. *Let S,T be the same as in Theorem 2.1 with $Tx_p=Sx_p=p$. Suppose that $\{Su_n\}_{n=0}^{\infty}$ be iterative sequence (6) with $\sum_{n=0}^{\infty}\alpha_n=\infty$. Then, $\{Su_n\}_{n=0}^{\infty}$ converges to p. Moreover, p is a unique common fixed point of S and T provided that $E=X$. Also S and T are weakly compatible.*

Proof. The proof is similar to that of Theorem 2.1. From iterative method (6) and (13), we obtain

$$\|Su_{n+1}-p\| \precsim \lambda[1-\alpha_n(1-\lambda)]\|Su_n-p\| \tag{25}$$

By using (25), we get

$$\|Su_{n+1}-p\| \precsim \lambda^{(n+1)}\prod_{i=0}^{n}[1-\alpha_i(1-\lambda)]\|Su_0-p\|. \tag{26}$$

Taking the limit both side of the inequality (26) and using Lemma 1.17, it can be seen that $\lim\limits_{n\to\infty}\|Su_n-p\|=0$.
It can be shown similarly to the proof in Theorem 2.1 that p is a unique common fixed point. Hence we omit it. □

The following theorem indicates that the convergence result can be obtained without the $\sum_{n=0}^{\infty}\alpha_n=\infty$ condition for the sequence of $\{\alpha_n\}_{n=0}^{\infty}\subset[0,1]$:

THEOREM 2.4. *Let S,T be the same as in Theorem 2.1 with $Tx_p=Su_p=p$. Let $\{Su_n\}_{n=0}^{\infty}$ be iterative sequence (6) with $\{\alpha_n\}_{n=0}^{\infty},\{\beta_n\}_{n=0}^{\infty}\subset[0,1]$. Then, $\{Su_n\}_{n=0}^{\infty}$ converges to p. Moreover, p is a unique common fixed point of S and T provided that $E=X$. Also, S and T are weakly compatible.*

Proof. The proof is similar to that of Theorem 2.1. Consider the following inequality

$$\|Su_{n+1}-p\| \precsim \lambda[1-\alpha_n(1-\lambda)]\|Su_n-p\| \tag{27}$$

Since $1-\alpha_n(1-\lambda)<1,\lambda<1$ and $\{\alpha_n\}_{n=0}^{\infty},\{\beta_n\}_{n=0}^{\infty},\{\gamma_n\}_{n=0}^{\infty}\subset[0,1]$ for all $n\in\mathbb{N}$, we have obtained in the following

$$\|Su_{n+1}-p\| \precsim \lambda^{(n+1)}\|Su_0-p\| \tag{28}$$

Taking the limit the inequality (28), it can be seen that $\lim\limits_{n\to\infty}\|Su_n-p\|=0$. Hence, the condition $\sum_{n=0}^{\infty}\alpha_n=\infty$ is unnecessary. □

2.2. Convergence Rate Analysis. In this section, we prove that the new Jungck-type iteration method (11) has a better convergence speed than the iteration method (6). We also give some numerical examples to show the efficiency of the new Jungck-type iteration method (11).

THEOREM 2.5. *Let S,T be the same as in Theorem 2.1 with $Tx_p{=}Sx_p{=}p$. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, and $\{\gamma_n\}_{n=0}^{\infty}$ be real sequences in $[0,1]$ satisfying $\alpha_1 \leq \alpha_n < 1$. For given $x_0 = u_0 \in E$, consider the iterative sequences $\{Sx_n\}_{n=0}^{\infty}$ and $\{Su_n\}_{n=0}^{\infty}$ defined by iteration method (11) and iteration method (6) respectively. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges to p faster than $\{Su_n\}_{n=0}^{\infty}$.*

Proof. From iteration methods (11) and (6), we obtain the following inequalties respectively,

$$\|Sx_{n+1} - p\| \precsim \lambda^{3(n+1)} \prod_{i=0}^{n} \left(1 + \frac{(1-\alpha_i)(1-\lambda)}{k\lambda}\right) \|Sx_0 - p\| \tag{29}$$

and

$$\|Su_{n+1} - p\| \precsim \lambda^{(n+1)} \prod_{i=0}^{n} [1 - \alpha_i(1-\lambda)] \|Su_0 - p\|. \tag{30}$$

Applying assumption to (29) and (30) respectively, we obtain

$$\|Sx_{n+1} - p\| \precsim \lambda^{3(n+1)} \|Sx_0 - p\| \left(1 + \frac{(1-\alpha_1)(1-\lambda)}{k\lambda}\right)^{n+1}, \tag{31}$$

and

$$\|Su_{n+1} - p\| \precsim \lambda^{(n+1)} \|Su_0 - p\| [1 - \alpha_1(1-\lambda)]^{n+1}. \tag{32}$$

Define

$$a_n = \lambda^{3(n+1)} \|Sx_0 - p\| \left(1 + \frac{(1-\alpha_1)(1-\lambda)}{k\lambda}\right)^{n+1}$$
$$b_n = \lambda^{n+1} \|Su_0 - p\| [1 - \alpha_1(1-\lambda)]^{n+1}$$

and

$$\psi_n = \frac{a_n}{b_n} = \frac{\lambda^{3(n+1)} \|Sx_0 - p\| \left(1 + \frac{(1-\alpha_1)(1-\lambda)}{k\lambda}\right)^{n+1}}{\lambda^{n+1} \|Su_0 - p\| [1 - \alpha_1(1-\lambda)]^{n+1}}$$

$$= \left[\lambda^2 \left(\frac{1 + \frac{(1-\alpha_1)(1-\lambda)}{k\lambda}}{1 - \alpha_1(1-\lambda)}\right)\right]^{n+1}$$

Since $k \in \mathbb{N}$, $\lambda \in (0,1)$ and $\alpha_1 \leq 1$ we have

$$\begin{aligned} \lambda^2 \left(1 + \frac{(1-\alpha_1)(1-\lambda)}{k\lambda}\right) &< \left(\lambda + \frac{(1-\alpha_1)(1-\lambda)}{k}\right) \\ &\leq (\lambda + (1-\alpha_1)(1-\lambda)) \\ &= (1 - \alpha_1(1-\lambda)) \end{aligned}$$

That is $\psi_n < 1$. Therefore $\lim_{n\to\infty} \psi_n = 0$. From Definition 1.6, we obtain that $\{Sx_n\}_{n=1}^{\infty}$converges to p faster than $\{Su_n\}_{n=1}^{\infty}$. □

Remark 2.6. We can reconstruct Theorem 2.5 using Theorem 2.2 and Theorem 2.4. In this case, it can be seen that condition $\alpha_1 \leq \alpha_n \leq 1$ is unnecessary. From the same process in the proof of Theorem 2.5, we obtain

$$\psi_n = \frac{a_n}{b_n} = \frac{\lambda^{2(n+1)} \|Sx_0 - p\|}{\lambda^{n+1} \|Su_0 - p\|} = \lambda^{n+1}.$$

That is $\psi_n < 1$. Therefore $\lim_{n\to\infty} \psi_n = 0$. From Definition 1.6, we obtain that $\{Sx_n\}_{n=1}^{\infty}$converges to p faster than $\{Su_n\}_{n=1}^{\infty}$.

Example 2.7. Let $E = [0, 1]$ and define operators S, T:$E \to \mathbb{C}$ by $Sx = \frac{x}{3}$, $Tx = \frac{x}{9}$ with unique common fixed point $x_p = 0$. Let $\lambda = \frac{1}{2}$, $\mu = \frac{9}{64}$, $k = 40$, and $\alpha_n = \beta_n = \gamma_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$. By using norm (1) given in Example 1.3, it can be seen that S and T satisfy the condition (13). By taking the initial point $0.5 \in E$ for all Jungck type iteration methods mentioned in this paper, we can get the following table.

Table 1. Comparison rate of convergence among some iterative methods for the initial point 0.5

Iteration Steps	Iteration Method (11)	Iteration Method 10	Iteration Method (7)	Iteration Method (6)	Iteration Method (5)
1	0.50000000000000	0.50000000000000	0.50000000000000	0.50000000000000	0.50000000000000
2	0.00694251543210	0.05831597222222	0.09259259259259	0.13888888888889	0.14814814814815
3	0.00009994746154	0.00690792288237	0.02222730274856	0.04286694101509	0.06970482141950
4	0.00000146441154	0.00082457332947	0.00591699031501	0.01369360615760	0.04033843832147
5	0.00000002168052	0.00009887550413	0.00166377001747	0.00444281444224	0.02625882933104
6	0.00000000032319	0.00001189214445	0.00048383984047	0.00145351336691	0.01844241511316
7	0.00000000000484	0.00000143339442	0.00014393464642	0.00047791255828	0.01365905682552
8	0.00000000000007	0.00000017304940	0.00004352190553	0.00015764476749	0.01052095175623
9	0.00000000000000	0.00000002091785	0.00001332212980	0.00005211575990	0.00835187071032
⋮	⋮	⋮	⋮	⋮	⋮
16	0.00000000000000	0.00000000000001	0.00000000405106	0.00000006957238	0.00268613394733
17	0.00000000000000	0.00000000000000	0.00000000129441	0.00000002313040	0.00238234896661
⋮	⋮	⋮	⋮	⋮	⋮
28	0.00000000000000	0.00000000000000	0.00000000000001	0.00000000000004	0.00088501223256
29	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000001	0.00082536926532
30	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000	0.00077155835937
⋮	⋮	⋮	⋮	⋮	⋮
279	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000	0.00000894565898
280	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000	0.00000888213975

The following inferences can be seen from the Table 1.

- Iteration method (11) reached to $x_p = 0$ at the 9th step,
- Iteration method (10) reached to $x_p = 0$ at the 17th step,
- Iteration method (7) reached to $x_p = 0$ at the 29th step,
- Iteration method (6) reached to $x_p = 0$ at the 30th step,
- Iteration method (5) reached to $x_p = 0$ after 280th step.

As a result of these data, we can say that the iterative sequence (11) converges to x_p faster than the iterative sequences mentioned in the Table 1.

Example 2.8. Let $E = [1.5, 2.5]$ and define operators S, T:$E \to [54, 150]$ by $Sx = 24x^2$, $Tx = x^4 - 16x + 112$ with a coincidence point $x_p = 2$. Let $k = 40$ and

$\alpha_n = \beta_n = \gamma_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$. By using norm (2) given in Example 1.4, it can be seen that S and T satisfy condition (13). By taking the initial point $1.5 \in E$ for all Jungck type iterations mentioned in this paper, we can get the following table.

Table 2. Comparison rate of convergence among some iterative methods for the initial point 1.5

Iteration Steps	Iteration Method (11)	Iteration Method (10)	Iteration Method (7)	Iteration Method (6)	Iteration Method (5)
1	1,50000000000000	1,50000000000000	1,50000000000000	1,50000000000000	1,50000000000000
2	1,99978186529065	1,96405535251285	1,98313445464299	1,97084706544253	1,92414360280142
3	1,99999971980879	1,99382800620470	1,99820174201912	1,99576886918276	1,97222100856849
4	1,99999999962204	1,99888589901988	1,99977566913639	1,99933555113960	1,98632602352750
5	1,99999999999948	1,99979641935165	1,99996989128173	1,99989305177446	1,99211189792208
6	2,00000000000000	1,99996261914998	1,99999577905271	1,99998259058559	1,99497101700627
⋮	⋮	⋮	⋮	⋮	⋮
16	2,00000000000000	1,99999999999823	1,99999999999997	1,99999999999973	1,99955826358893
17	2,00000000000000	1,99999999999967	2,00000000000000	1,99999999999996	1,99962010453220
18	2,00000000000000	1,99999999999994	2,00000000000000	1,99999999999999	1,99967047124433
19	2,00000000000000	1,99999999999999	2,00000000000000	2,00000000000000	1,99971196249360
20	2,00000000000000	2,00000000000000	2,00000000000000	2,00000000000000	1,99974649229432
⋮	⋮	⋮	⋮	⋮	⋮
280	2,00000000000000	2,00000000000000	2,00000000000000	2,00000000000000	0,00204729288472

The following inferences can be seen from the Table 2

- Iteration method (11) reached to $x_p = 2$ at the 6th step,
- Iteration method (10) reached to $x_p = 2$ at the 20th step,
- Iteration method (7) reached to $x_p = 2$ at the 17th step,
- Iteration method (6) reached to $x_p = 2$ at the 19th step,
- Iteration method (5) reached to $x_p = 2$ after 280th step.

As a result of these data, we can say that the iterative sequence (11) converges to x_p faster than the iterative sequences mentioned in the Table 2.

3. (S,T)-Stablity of New Jungck Type Iteration Method

In this section, we analyze the stablity of the iteration method (11) with respect to operator of (S,T).

Theorem 3.1. *Let S,T be the same as in Theorem 2.1 with $Tx_p{=}Sx_p{=}p$. Suppose that $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence generated by (11) with the condition $\sum_{n=0}^{\infty} \alpha_n{=}\infty$, converges to p. Then, the iteration method (11) is (S,T)-stable.*

Proof. Let $\{S\omega_n\}_{n=0}^{\infty}$ be an arbitrary sequence and $\lim\limits_{n\to\infty} \epsilon_n = 0$ such that $\epsilon_n = \|S\omega_{n+1} - f(T, \omega_n)\|$. We have

$$\begin{cases} S\omega_{n+1} = T\upsilon_n \\ S\upsilon_n = \frac{(1-\alpha_n)}{k} T\omega_n + \left(1 - \frac{(1-\alpha_n)}{k}\right) T\vartheta_n \\ S\vartheta_n = T\varpi_n \\ S\varpi_n = \frac{(1-\beta_n)}{k} S\omega_n + \left(1 - \frac{(1-\beta_n)}{k}\right) T\omega_n. \end{cases} \tag{33}$$

We will show that $\lim_{n\to\infty} S\omega_n = p$. We get

$$\begin{aligned}\|T\upsilon_n - Tx_p\| &\precsim \lambda \| S\upsilon_n - Sx_p\| \\ &+ \mu\left(\frac{\|S\upsilon_n - T\upsilon_n\| \cdot \|Sx_p - Tx_p\|}{1+\|S\upsilon_n - Sx_p\|}\right) \\ &= \lambda \| S\upsilon_n - Sx_p\|\end{aligned} \tag{34}$$

and by doing calculations similar to the inequality (34), we have

$$\|T\vartheta_n - Tx_p\| \precsim \lambda \| S\vartheta_n - Sx_p\| \tag{35}$$

and

$$\|S\vartheta_n - Sx_p\| = \|T\varpi_n - Sx_p\| \precsim \lambda \| S\varpi_n - Sx_p\| \tag{36}$$

and

$$\|T\omega_n - Tx_p\| \precsim \lambda \| S\omega_n - Sx_p\| . \tag{37}$$

Using the iteration method (33) and the inequalities (35), (36), (37), (38), we get

$$\begin{aligned}\| S\varpi_n - Sx_p\| &\precsim \frac{(1-\beta_n)}{k} \|S\omega_n - Sx_p\| \\ &+ \left(1 - \frac{(1-\beta_n)}{k}\right) \|T\omega_n - Tx_p\| \\ &\precsim \frac{(1-\beta_n)}{k} \| S\omega_n - Sx_p\| \\ &+ \lambda\left(1 - \frac{(1-\beta_n)}{k}\right) \|S\omega_n - Sx_p\| \\ &\precsim \left(1 - \frac{\beta_n(1-\lambda)}{k}\right) \|S\omega_n - Sx_p\|\end{aligned} \tag{38}$$

and

$$\begin{aligned}\| S\upsilon_n - Sx_p\| &\precsim \frac{(1-\alpha_n)}{k} \|T\omega_n - Tx_p\| \\ &+ \left(1 - \frac{(1-\alpha_n)}{k}\right) \|T\vartheta_n - Tx_p\| \\ &\precsim \lambda\frac{(1-\alpha_n)}{k} \| S\omega_n - Sx_p\| \\ &+ \lambda\left(1 - \frac{(1-\alpha_n)}{k}\right) \| S\vartheta_n - Sx_p\| \\ &\precsim \lambda\frac{(1-\alpha_n)}{k} \| S\omega_n - Sx_p\| \\ &+ \lambda^2\left(1 - \frac{(1-\alpha_n)}{k}\right) \| S\varpi_n - Sx_p\| \\ &\precsim \lambda\left(\lambda\left(1 - \frac{(1-\alpha_n)}{k}\right)\left(1 - \frac{\beta_n(1-\lambda)}{k}\right)\right. \\ &\left.+ \frac{(1-\alpha_n)}{k}\right) \|S\omega_n - Sx_p\| .\end{aligned} \tag{39}$$

Combining (34), (38) and (39), we have

$$\begin{aligned}\|S\omega_{n+1}-Sx_p\| &\precsim \|S\omega_{n+1}-T\upsilon_n\|+\|T\upsilon_n-Sx_p\|\\ &\precsim \epsilon_n+\|T\upsilon_n-Tx_p\|\\ &\precsim \epsilon_n+\lambda\,\|\,S\upsilon_n-Sx_p\|\\ &\precsim \epsilon_n+\lambda^2\left(\frac{(1-\alpha_n)}{k}+\lambda\left(1-\frac{(1-\alpha_n)}{k}\right)\right)\\ &\cdot\|S\omega_n-Sx_p\|\\ &\precsim \epsilon_n+\lambda^2\left(1-\frac{\alpha_n\,(1-\lambda)}{k}\right)\|S\omega_n-p\|\,.\end{aligned}\tag{40}$$

Taking the limit both side of the inequality (40) and using Lemma 1.17, we get $\lim_{n\to\infty}\|S\omega_n-p\|=0$.

Suppose that $\lim_{n\to\infty}\|S\omega_n-p\|=0$. By using the inequalities (34), (39), we obtain

$$\begin{aligned}\epsilon_n &= \|S\omega_{n+1}-T\upsilon_n\|\\ &\precsim \|S\omega_{n+1}-p\|+\|T\upsilon_n-Tx_p\|\\ &\precsim \|S\omega_{n+1}-p\|+\lambda\,\|\,S\upsilon_n-Sx_p\|\\ &\precsim \|S\omega_{n+1}-p\|\\ &+\lambda^2\left(\frac{(1-\alpha_n)}{k}+\lambda\left(1-\frac{(1-\alpha_n)}{k}\right)\left(1-\frac{\beta_n\,(1-\lambda)}{k}\right)\right)\|S\omega_n-Sx_p\|\,.\end{aligned}$$

Taking the limit both side of the above inequality, we have $\lim_{n\to\infty}\epsilon_n=0$. □

EXAMPLE 3.2. Let $E=[2,6]$ and define $\|\cdot\|:E\to\mathbb{C}$ by using norm (2) given by Example 1.4. If we take operators T and S as $Tx=x+12$ and $Sx=x^2$, respectively, it can be seen that these operators satisfy the condition (13) with a coincidence point $p=4$. Assume that $\alpha_n=\beta_n=\frac{1}{n+1}$ for all $n\in\mathbb{N}$, and $k=40$. If the iteration method (11) generated through these operators and control sequences, we obtain

$$\begin{aligned}z_n &= \sqrt{\frac{480+468n+40x_n+39nx_n+nx_n^2}{40+40n}}\\ y_n &= \sqrt{12+z_n}\\ u_n &= \sqrt{\frac{n(12+x_n)}{40(1+n)}+(1-\frac{n}{40(1+n)})\left(12+\sqrt{12+z_n}\right)}\\ x_{n+1} &= \sqrt{12+\sqrt{\frac{n(12+x_n)}{40(1+n)}+(1-\frac{n}{40(1+n)})\left(12+\sqrt{12+z_n}\right)}}\end{aligned}\tag{41}$$

Also, we take the sequence of $\{S\omega_n\}_{n=0}^{\infty}$ as $S\omega_n=\left(4+\frac{1}{2n+1}\right)^2$ for all $n\in\mathbb{N}$, then we have $\lim_{n\to\infty}\|Sx_n-S\omega_n\|=0$. Hence, $\{S\omega_n\}_{n=0}^{\infty}$ is approximate sequence of $\{Sx_n\}_{n=0}^{\infty}$ according to Definition 1.14. The following equations can be obtained similar to the processes performed in (41) for $\{S\omega_n\}_{n=0}^{\infty}$:

$$\omega_{n+1} = \sqrt{12 + \sqrt{\frac{n(17+32n)}{40(1+n)(1+2n)}} + \frac{1}{40}\left(39 + \frac{1}{1+n}\right)B} \tag{42}$$

in which

$$B = \sqrt{12 + \frac{1}{2\sqrt{5}}\sqrt{\frac{340+1664n+2607n^2+1280n^3}{(2+n)(1+2n)^2}}}.$$

Let $\epsilon_n = \|S\omega_{n+1} - f(T,\omega_n)\|$. Then, we get

$$\lim_{n\to\infty}\left\|\left(4+\frac{1}{2n+3}\right)^2 - 12 - \frac{1}{2\sqrt{5}}\sqrt{\frac{340+1664n+2607n^2+1280n^3}{(2+n)(1+2n)^2}}\right\| = 0$$

Consequently, $\lim_{n\to\infty}\epsilon_n = 0$.

4. Data Dependence

THEOREM 4.1. *Let $(\tilde{S},\tilde{T}) : Y \to X$ be an approximate operator pair of the pair $(S,T) : Y \to X$ satisfies condition (13). Assume that $T(E) \subseteq S(E)$, $S(E) \subseteq \mathrm{X}$ is a complex-valued Banach space and there exist $Tx_p{=}Sx_p{=}p$ and $\tilde{T}x_p{=}\tilde{S}x_p{=}\tilde{p}$. Let $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence generated by (11) with the condition $\frac{1}{2} \leq \frac{\alpha_n}{k}$ and sequence of $\left\{\tilde{S}\omega_n\right\}_{n=0}^{\infty}$ defined by*

$$\begin{cases} \tilde{S}\omega_{n+1} = \tilde{T}\upsilon_n \\ \tilde{S}\upsilon_n = \frac{(1-\alpha_n)}{k}\tilde{T}\omega_n + \left(1 - \frac{(1-\alpha_n)}{k}\right)\tilde{T}\vartheta_n \\ \tilde{S}\vartheta_n = \tilde{T}\varpi_n \\ \tilde{S}\varpi_n = \frac{(1-\beta_n)}{k}\tilde{S}\omega_n + \left(1 - \frac{(1-\beta_n)}{k}\right)\tilde{T}\omega_n. \end{cases} \tag{43}$$

with $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}\subset[0,1]$.Assume that $\{Sx_n\}_{n=0}^{\infty}$ and $\left\{\tilde{S}w_n\right\}_{n=0}^{\infty}$ converge to p and $\tilde{p}$, respectively. Then, we have following estimate:

$$\|p - \tilde{p}\| \precsim \frac{8\epsilon}{1-\lambda}$$

where $\epsilon = max\{\epsilon_1, \epsilon_2\}$ is a fixed number.

Proof. By using iteration methods (11) and (43), Definition 1.13, and the condition (13) we obtain

$$\begin{aligned} \left\|Sx_{n+1} - \tilde{S}w_{n+1}\right\| &= \left\|Tu_n - \tilde{T}\upsilon_n\right\| \\ &\precsim \|Tu_n - T\upsilon_n\| + \left\|T\upsilon_n - \tilde{T}\upsilon_n\right\| \\ &\precsim \|Tu_n - T\upsilon_n\| + \epsilon_1 \end{aligned} \tag{44}$$

and

$$\begin{aligned}\|Tx_n - T\omega_n\| &\precsim \lambda \|Sx_n - S\omega_n\| \\ &+ \mu \left(\frac{\|Sx_n - Tx_n\| \cdot \|S\omega_n - T\omega_n\|}{1 + \|Sx_n - S\omega_n\|} \right) \\ &\precsim \lambda \left\| Sx_n - \tilde{S}\omega_n \right\| + \lambda \left\| S\omega_n - \tilde{S}\omega_n \right\| + \mu A_1 \\ &\precsim \lambda \left\| Sx_n - \tilde{S}\omega_n \right\| + \lambda\epsilon_2 + \mu A_1\end{aligned} \tag{45}$$

in which $A_1 = \left(\frac{\|Sx_n - Tx_n\| \cdot \|S\omega_n - T\omega_n\|}{1 + \|Sx_n - S\omega_n\|} \right)$.
Similarly,

$$\begin{aligned}\|Ty_n - T\vartheta_n\| &\precsim \lambda \|Sy_n - S\vartheta_n\| \\ &+ \mu \left(\frac{\|Sy_n - Ty_n\| \cdot \|S\vartheta_n - T\vartheta_n\|}{1 + \|Sy_n - S\vartheta_n\|} \right) \\ &\precsim \lambda \left\| Sy_n - \tilde{S}\vartheta_n \right\| + \lambda \left\| \tilde{S}\vartheta_n - S\vartheta_n \right\| + \mu A_2 \\ &\precsim \lambda \left\| Sy_n - \tilde{S}\vartheta_n \right\| + \lambda\epsilon_2 + \mu A_2\end{aligned} \tag{46}$$

in which $A_2 = \left(\frac{\|Sy_n - Ty_n\| \cdot \|S\vartheta_n - T\vartheta_n\|}{1 + \|Sy_n - S\vartheta_n\|} \right)$.
By using the condition (13)

$$\begin{aligned}\left\| Sy_n - \tilde{S}\vartheta_n \right\| &= \left\| Tz_n - \tilde{T}\varpi_n \right\| = \|Tz_n - T\varpi_n\| + \left\| T\varpi_n - \tilde{T}\varpi_n \right\| \\ &\precsim \|Tz_n - T\varpi_n\| + \epsilon_1 \\ &\precsim \lambda \|Sz_n - S\varpi_n\| \\ &+ \mu \left(\frac{\|Sz_n - Tz_n\| \cdot \|S\varpi_n - T\varpi_n\|}{1 + \|Sz_n - S\varpi_n\|} \right) + \epsilon_1 \\ &= \lambda \|Sz_n - S\varpi_n\| + \mu A_4 + \epsilon_1 \\ &\precsim \lambda \left\| Sz_n - \tilde{S}\varpi_n \right\| + \lambda \left\| S\varpi_n - \tilde{S}\varpi_n \right\| + \mu A_4 + \epsilon_1 \\ &\precsim \lambda \left\| Sz_n - \tilde{S}\varpi_n \right\| + \lambda\epsilon_2 + \mu A_4 + \epsilon_1\end{aligned} \tag{47}$$

in which $A_4 = \left(\frac{\|Sz_n - Tz_n\| \cdot \|S\varpi_n - T\varpi_n\|}{1 + \|Sz_n - S\varpi_n\|} \right)$.
Similarly,

$$\begin{aligned}\|Tu_n - T\upsilon_n\| &\precsim \lambda \|Su_n - S\upsilon_n\| \\ &+ \mu \left(\frac{\|Su_n - Tu_n\| \cdot \|S\upsilon_n - T\upsilon_n\|}{1 + \|Su_n - S\upsilon_n\|} \right) \\ &\precsim \lambda \left\| Su_n - \tilde{S}\upsilon_n \right\| + \lambda \left\| \tilde{S}\upsilon_n - S\upsilon_n \right\| + \mu A_3 \\ &= \lambda \left\| Su_n - \tilde{S}\upsilon_n \right\| + \lambda\epsilon_2 + \mu A_3\end{aligned} \tag{48}$$

in which $A_3 = \left(\frac{\|Su_n - Tu_n\| \cdot \|S\upsilon_n - T\upsilon_n\|}{1+\|Su_n - S\upsilon_n\|}\right)$.
Combining the condition (13) and (45), we obtain

$$
\begin{aligned}
\left\|Sz_n - \tilde{S}\varpi_n\right\| &\precsim \frac{(1-\beta_n)}{k}\left\|Sx_n - \tilde{S}\omega_n\right\| \\
&+ \left(1 - \frac{(1-\beta_n)}{k}\right)\|Tx_n - \tilde{T}\omega_n\| \\
&\precsim \frac{(1-\beta_n)}{k}\left\|Sx_n - \tilde{S}\omega_n\right\| \\
&+ \left(1 - \frac{(1-\beta_n)}{k}\right)\|Tx_n - T\omega_n\| \\
&+ \left(1 - \frac{(1-\beta_n)}{k}\right)\epsilon_1 \\
&\precsim \frac{(1-\beta_n)}{k}\left\|Sx_n - \tilde{S}\omega_n\right\| \\
&+ \lambda\left(1 - \frac{(1-\beta_n)}{k}\right)\left\|Sx_n - \tilde{S}\omega_n\right\| \\
&+ \mu\left(1 - \frac{(1-\beta_n)}{k}\right)A_1 \\
&+ \left(1 - \frac{(1-\beta_n)}{k}\right)\epsilon_1 + \lambda\left(1 - \frac{(1-\beta_n)}{k}\right)\epsilon_2 \\
&= \left(1 - \frac{\beta_n(1-\lambda)}{k}\right)\left\|Sx_n - \tilde{S}\omega_n\right\| \\
&+ \mu\left(1 - \frac{(1-\beta_n)}{k}\right)A_1 \\
&+ \left(1 - \frac{(1-\beta_n)}{k}\right)\epsilon_1 + \lambda\left(1 - \frac{(1-\beta_n)}{k}\right)\epsilon_2.
\end{aligned}
\tag{49}
$$

By doing calculations similar to the inequality (49), we have

$$
\begin{aligned}
\left\|Su_n - \tilde{S}\upsilon_n\right\| &\precsim \lambda\frac{(1-\alpha_n)}{k}\left\|Sx_n - \tilde{S}\omega_n\right\| \\
&+ \lambda\epsilon_2 + \mu\frac{(1-\alpha_n)}{k}A_1 + \epsilon_1 \\
&+ \lambda\left(1 - \frac{(1-\alpha_n)}{k}\right)\left\|Sy_n - \tilde{S}\vartheta_n\right\| \\
&+ \mu\left(1 - \frac{(1-\alpha_n)}{k}\right)A_2
\end{aligned}
\tag{50}
$$

Substituting (47), (48), (49), and (50) into (44), we have

$$
\begin{aligned}
\left\|Sx_{n+1}-\tilde{S}w_{n+1}\right\| &\precsim \lambda\left\|Su_n-\tilde{S}v_n\right\|+\epsilon_1+\lambda\epsilon_2+\mu A_3\\
&\precsim \lambda^2\left(1-\frac{\alpha_n(1-\lambda)}{k}\right)\left\|Sx_n-\tilde{S}\omega_n\right\|\\
&+\left(1+\lambda+\lambda^2\left(1-\frac{(1-\alpha_n)}{k}\right)\right.\\
&\left.+\lambda^3\left(1-\frac{(1-\alpha_n)}{k}\right)\right)(\epsilon_1+\epsilon_2)\\
&+\lambda\mu\left(\frac{(1-\alpha_n)}{k}+\lambda^2\left(1-\frac{(1-\alpha_n)}{k}\right)\right.\\
&\left.\times\left(1-\frac{(1-\beta_n)}{k}\right)\right)A_1\\
&+\lambda\mu\left(1-\frac{(1-\alpha_n)}{k}\right)A_2+\mu A_3\\
&+\lambda^2\mu\left(1-\frac{(1-\alpha_n)}{k}\right)A_4
\end{aligned}\tag{51}
$$

Also, we have

$$
\begin{aligned}
&\left(1+\lambda+\lambda^2\left(1-\frac{(1-\alpha_n)}{k}\right)+\lambda^3\left(1-\frac{(1-\alpha_n)}{k}\right)\left(1-\frac{(1-\beta_n)}{k}\right)\right)(\epsilon_1+\epsilon_2)\\
&=(1+\lambda)\left[1+\lambda^2\left(1-\frac{(1-\alpha_n)}{k}\right)\right](\epsilon_1+\epsilon_2)\leq 4(\epsilon_1+\epsilon_2)
\end{aligned}
$$

from hypothesis, we obtain

$$
1-\frac{\alpha_n}{k}\leq\frac{\alpha_n}{k}.
$$

Hence, from (51) and the above inequality, we have

$$
\begin{aligned}
\left\|Sx_{n+1}-\tilde{S}w_{n+1}\right\| &\precsim \lambda^2\left(1-\frac{\alpha_n(1-\lambda)}{k}\right)\left\|Sx_n-\tilde{S}\omega_n\right\|\\
&+\frac{\alpha_n(1-\lambda)}{k}\left(\frac{8(\epsilon_1+\epsilon_2)+2\mu\left(A_1+A_2+A_3+A_4\right)}{(1-\lambda)}\right)
\end{aligned}
$$

Denote that,

$$
\begin{aligned}
a_n &= \left\|Sx_n-\tilde{S}\omega_n\right\|,\\
\mu_n &= \frac{\alpha_n}{k}(1-\lambda)\in(0,1),\\
\eta_n &= \frac{8(\epsilon_1+\epsilon_2)+2\mu\left(A_1+A_2+A_3+A_4\right)}{(1-\lambda)}.
\end{aligned}
$$

It follows from Lemma 1.16 that

$$
\begin{aligned}
0 &\precsim \limsup_{n\to\infty} \left\| Sx_n - \tilde{S}\omega_n \right\| \\
&\precsim \limsup_{n\to\infty} \left\{ \frac{8\epsilon}{(1-\lambda)} \right\} \\
&= \frac{8\epsilon}{(1-\lambda)}
\end{aligned}
$$

We know from Theorem 2.1 that $Sx_n \to p$ and using hypotesis, we obtain

$$\|p - \tilde{p}\| \precsim \frac{8\epsilon}{1-\lambda}.$$

□

EXAMPLE 4.2. Let $E = [0,1]$ and define $\|\cdot\| : E \to \mathbb{C}$ by using norm (2) given by Example 1.4. If we take operators T and S as $Tx = xcosx$ and $Sx = x + sinx$, respectively. It is clear that $T(E) \subseteq S(E)$. For $\lambda \in [0.5, 1)$ and $\lambda + \mu < 1$, it can be seen from the following figure that these operators satisfy the condition (13) with a unique common fixed point $p = 0$:

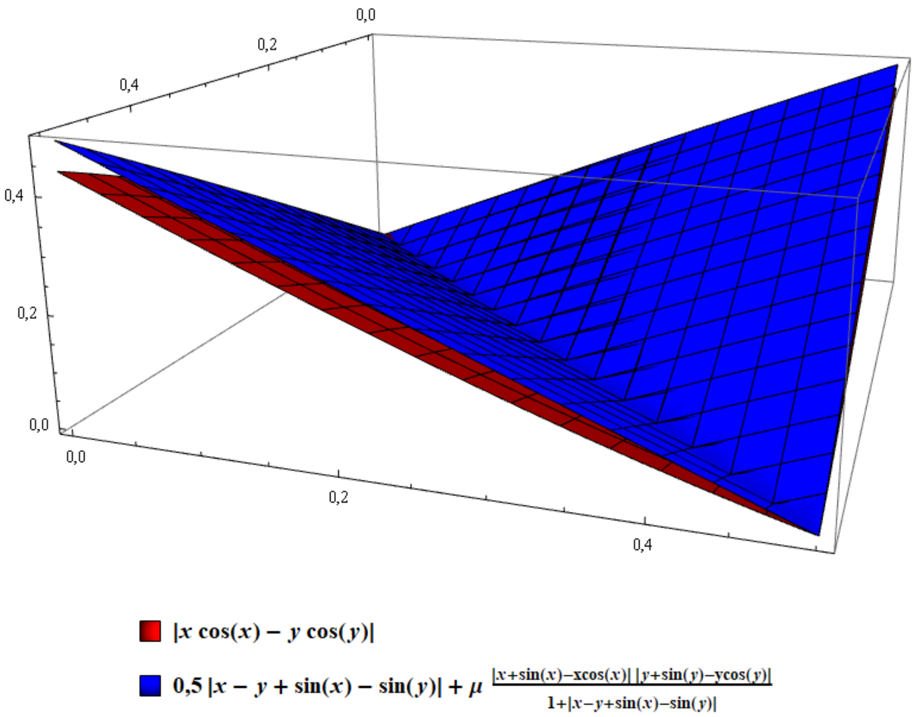

FIGURE 1. Graphical demonstration of T and S operators

Define operators $\tilde{T}$ and $\tilde{S}$

$$
\begin{aligned}
\tilde{T}x &= x + 0.036 - \frac{x^3}{2} + \frac{x^5}{15} - \frac{x^7}{42} \\
\tilde{S}x &= 2x - \frac{(x-0.6)^3}{6} + \frac{x^5}{30} - \frac{x^7}{84}
\end{aligned}
\tag{52}
$$

By utilizing Wolfram Mathematica 9 Software Package, we get $max_{x\in E}|T-\tilde{T}| = 0.04$. Hence for all $x \in E$ and for a fixed $\epsilon_1 > 0$, we have $|T - \tilde{T}| \leq 0.04$. Similarly, $max_{x\in E}|S - \tilde{S}| = 0.1693$ and hence, for all $x \in E$ and for a fixed $\epsilon_2 > 0$, we get $|S - \tilde{S}| \leq 0.1693$. Thus, $\tilde{T}$ and $\tilde{S}$ are approximate operators of T and S, respectively in the sense of Definition 1.13. From (52), $\tilde{p} = \tilde{T}0 = \tilde{S}0 = 0.036$. Hence the distance between two fixed points $|p - \tilde{p}| = 0.036$. If we take the initial point $x_0 = 0.5$ and we put $\alpha_n = \beta_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$, and $k = 40$ in the iteration method (43), then we get the following table:

Table 3. Behaviour of the iteration method (11) for the operators T, S, $\tilde{T}$ and $\tilde{S}$ given by the Example 4.2

Iteration Steps	x_{n+1}	Tx_n	Sx_n	$\tilde{x}_{n+1}$	$\tilde{T}x_n$	$\tilde{S}x_n$
1	0.50000000000000	0.43879128094518	0.97942553860420	0.50000000000000	0.47539732142857	1.00111532738095
2	0.02812766031015	0.02811653422968	0.05625161182900	0.03228222175568	0.06826540276515	0.09506068020702
3	0.00374753718996	0.00374751087471	0.00749506560816	0.00307576568855	0.03907575113968	0.04160072680391
4	0.00026916097544	0.00026916096568	0.00053832194762	0.00029655209478	0.03629655208174	0.03653975119109
5	0.00003632327199	0.00003632327196	0.00007264654397	0.00002871100999	0.03602871100997	0.03605225428547
6	0.00000245840286	0.00000245840285	0.00000491680571	0.00000278657400	0.03600278657399	0.03600278657399
7	0.00000016674963	0.00000016674962	0.00000033349925	0.00000027092306	0.03600027092306	0.03600049307999
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Then, we have the following estimate:

$$0.036 = |p - \tilde{p}| \precsim \frac{8(0.2093)}{1 - 0.5}$$

5. Application to a Functional-Integral Equation

The theory of functional-integral equations is one of the active fields of study of non-linear analysis, and fixed point theory is a useful method in showing the existence or uniqueness of the solutions of the equations in question. The basic idea in fixed point theory for the integral or differential equation to be solved is to construct algorithms called iterations by including the equation in an operator class under certain conditions and to determine the appropriate conditions for the convergence of the sequence obtained from this iteration. Therefore, fixed-point iteration algorithms have been studied by many researchers to solve integral or differential equations(see: [4, 13, 31]) and reference therein.

In this section, we show that the Jungck type iteration method (11) converges to the solution of following functional-integral equation, which is given in [29]:

$$x(t) = \int_a^b K(t, s, x(s), x(h(s)))\, ds + g(t) \tag{53}$$

where X is Banach sapace and α, β, a, b real numbers such that $\alpha \leq a \leq b \leq \beta$, and $K \in (C[\alpha, \beta]^2 \times X^2, X), g \in C([\alpha, \beta], X)$ and $h \in C([\alpha, \beta], [\alpha, \beta])$.
The authors in [29], gave the following theorem, which guarantees the uniqueness of the solution of the integral equation (53):

Theorem 5.1. *Let us consider the equation (53) under the above assumptions on K, g, h. Suppose that:*

i. there exist $L_1, L_2 > 0$ *such that*
$\|K(t,s,x_1,y_1) - K(t,s,x_2,y_2)\| \leq L_1 \|x_1 - x_2\| + L_2 \|y_1 - y_2\|$

for each $t, s \in [\alpha, \beta]$ *and* $x_1, y_1, x_2, y_2 \in X$.

ii. $(L_1 + L_2)(b - a) < 1$.

Then, we have the following conclusions:

1. *the equation (53) has in* $C([\alpha, \beta], X)$ *a unique solution* x*;*
2. *for all* $x_0 \in C([\alpha, \beta], X)$ *the sequence* $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1}(t) = \int_a^b K(t,s,x_n(s),x_n(h(s)))\, ds + g(t)$$

converges uniformly on $[\alpha, \beta]$ *to* x.

Theorem 5.2. *Let* $(X, \|\cdot\|)$ *be a complex Banach space, with the conditions given in Theorem 5.1. Suppose that* $\{Sx_n\}_{n=0}^{\infty}$ *be iterative sequence (11) with* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then, the equation (53) has a unique solution* $p \in C([\alpha, \beta], X)$*, and iterative sequence (11) converges to* p *with the following estimate:*

$$\begin{aligned} \|Sx_{n+1} - p\| &\precsim [(L_1 + L_2)(b-a)]^{2(n+1)} \\ &\quad e^{-\frac{(1-(L_1+L_2)(b-a))\sum_{i=0}^{n}\alpha_i}{k}} \|Sx_0 - p\| \end{aligned} \tag{54}$$

Proof. We consider the complex Banach space $(X, \|\cdot\|)$, where $\|\cdot\|$ is the complex Chebyshev's norm on X defined in Example 1.5.
Let $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence generated by iteration method (11) for the operator $A_1 : C([\alpha, \beta], X) \to C([\alpha, \beta], X)$ defined by

$$A_1(x_n(t)) = A_2(x_n(t)) = \int_a^b K(t,s,x_n(s),x_n(h(s)))\, ds + g(t) \tag{55}$$

By using iteration method (11) and the equation (55), we have

$$
\begin{aligned}
\|Sz_n - p\| &= \left\| \frac{(1-\beta_n)}{k} A_2 x_n + \left(1 - \frac{(1-\beta_n)}{k}\right) A_1 x_n - p \right\| \\
&\precsim \frac{(1-\beta_n)}{k} \|A_2 x_n - p\| + \left(1 - \frac{(1-\beta_n)}{k}\right) \|A_1 x_n - p\| \\
&\precsim \frac{(1-\beta_n)}{k} \left\| \int_a^b K(t,s,x_n(s),x_n(h(s)))\,ds \right. \\
&\quad \left. - \int_a^b K(t,s,p(s),p(h(s)))\,ds \right\| \\
&\quad + \left(1 - \frac{(1-\beta_n)}{k}\right) \left\| \int_a^b K(t,s,x_n(s),x_n(h(s)))\,ds \right. \\
&\quad \left. - \int_a^b K(t,s,p(s),p(h(s)))\,ds \right\| \\
&\precsim \frac{(1-\beta_n)}{k} \int_a^b \|K(t,s,x_n(s),x_n(h(s))) \\
&\quad - K(t,s,p(s),p(h(s)))\|\,ds \\
&\quad + \left(1 - \frac{(1-\beta_n)}{k}\right) \int_a^b \|K(t,s,x_n(s),x_n(h(s))) \\
&\quad - K(t,s,p(s),p(h(s)))\|\,ds \\
&\precsim \frac{(1-\beta_n)}{k} (L_1+L_2)(b-a)\|Sx_n - p\| \\
&\quad + \left(1 - \frac{(1-\beta_n)}{k}\right)(L_1+L_2)(b-a)\|Sx_n - p\| \\
&= (L_1+L_2)(b-a)\|Sx_n - p\|
\end{aligned} \tag{56}
$$

and

$$
\begin{aligned}
\|Sy_n - p\| &= \|A_1 z_n - A_1 p\| \\
&= \left\| \int_a^b K(t,s,z_n(s),z_n(h(s)))\,ds \right. \\
&\quad \left. - \int_a^b K(t,s,p(s),p(h(s)))ds \right\| \\
&\precsim \int_a^b \|K(t,s,z_n(s),z_n(h(s))) \\
&\quad - K(t,s,p(s),p(h(s)))\|\,ds \\
&\precsim (L_1+L_2)(b-a)\|Sz_n - p\|
\end{aligned} \tag{57}
$$

By doing calculations similar to the inequalities (56) and (57), we obtain

$$
\begin{aligned}
\|Su_n - p\| &\precsim \frac{(1-\alpha_n)}{k}\|A_1x_n - p\| + \left(1 - \frac{(1-\alpha_n)}{k}\right)\|A_1y_n - p\| \\
&= \frac{(1-\alpha_n)}{k}\left\|\int_a^b K(t,s,x_n(s),x_n(h(s)))ds \right. \\
&\quad \left. - \int_a^b K(t,s,p(s),p(h(s)))\,ds\right\| \\
&\quad + \left(1 - \frac{(1-\alpha_n)}{k}\right)\left\|\int_a^b K(t,s,y_n(s),y_n(h(s)))\,ds \right. \\
&\quad \left. - \int_a^b K(t,s,p(s),p(h(s)))\,ds\right\| \\
&\precsim (L_1+L_2)(b-a) \\
&\quad \left(1 - \frac{\alpha_n(1-(L_1+L_2)(b-a))}{k}\right)\|Sx_n - p\|
\end{aligned} \tag{58}
$$

and

$$
\begin{aligned}
\|Sx_{n+1} - p\| &= \|A_1u_n - A_1p\| \\
&= \left\|\int_a^b K(t,s,u_n(s),u_n(h(s)))ds + g(t) \right. \\
&\quad \left. - \int_a^b K(t,s,p(s),p(h(s)))\,ds - g(t)\right\| \\
&\precsim \int_a^b \|K(t,s,u_n(s),u_n(h(s))) \\
&\quad - K(t,s,p(s),p(h(s)))\|\ ds \\
&\precsim \int_a^b (L_1+L_2)\|u_n - p\|\ ds \\
&\precsim (L_1+L_2)(b-a)\|Su_n - p\|
\end{aligned} \tag{59}
$$

Substituting (58) into (59), we have

$$
\|Sx_{n+1} - p\| \precsim [(L_1+L_2)(b-a)]^2\left(1 - \frac{\alpha_n(1-(L_1+L_2)(b-a))}{k}\right)\|Sx_n - p\|
$$

by induction, we obtain

$$
\|Sx_n - p\| \precsim [(L_1+L_2)(b-a)]^2\left(1 - \frac{\alpha_n(1-(L_1+L_2)(b-a))}{k}\right)\|Sx_{n-1} - p\|
$$

$$
\|Sx_{n-1} - p\| \precsim [(L_1+L_2)(b-a)]^2\left(1 - \frac{\alpha_n(1-(L_1+L_2)(b-a))}{k}\right)\|Sx_{n-2} - p\|
$$

and hence,

$$\begin{aligned} \|Sx_{n+1}-p\| &\precsim [(L_1+L_2)(b-a)]^{2(n+1)} \prod_{i=0}^{n}\left(1-\frac{\alpha_i(1-(L_1+L_2)(b-a))}{k}\right) \\ &\quad \cdot \|Sx_0-p\| \\ &\precsim [(L_1+L_2)(b-a)]^{2(n+1)} e^{-\frac{(1-(L_1+L_2)(b-a))\sum_{i=0}^{n}\alpha_i}{k}} \|Sx_0-p\| \end{aligned} \tag{60}$$

Taking the limit on both sides of (60) and using $[(L_1+L_2)(b-a)]^2<1$, we obtain

$$\lim_{n\to\infty}\|Sx_n-p\|=0.$$

□

The following theorem indicates that the convergence result can be obtained without the $\sum_{n=0}^{\infty}\alpha_n=\infty$ condition for the sequence of $\{\alpha_n\}_{n=0}^{\infty}\subset[0,1]$:

THEOREM 5.3. *Let $(X,\|\cdot\|)$ be a complex Banach space, with the conditions given in Theorem 5.1. Suppose that $\{Sx_n\}_{n=0}^{\infty}$ be iterative sequence (11) with $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}\subset[0,1]$. Then, the equation (53) has a unique solution $p\in C([\alpha,\beta],X)$, and iterative sequence (11) converges to p with the following estimate:*

$$\|Sx_{n+1}-p\| \precsim [(L_1+L_2)(b-a)]^{2(n+1)}\|Sx_0-p\| \tag{61}$$

Proof. The proof is similar to that of Theorem 5.2. Consider the following inequality

$$\|Sx_{n+1}-p\| \quad \precsim \quad [(L_1+L_2)(b-a)]^2\left(1-\frac{\alpha_n(1-(L_1+L_2)(b-a))}{k}\right)\|Sx_n-p\|$$

Since $(L_1+L_2)(b-a)<1$ and $\{\alpha_n\}_{n=0}^{\infty}\subset[0,1]$ for all $n\in\mathbb{N}$, we have obtained in the following

$$\left(1-\frac{\alpha_n(1-(L_1+L_2)(b-a))}{k}\right)<1$$

Hence, we get

$$\|Sx_{n+1}-p\| \quad \precsim \quad [(L_1+L_2)(b-a)]^2\|Sx_n-p\|$$

By induction from the above ineqality, we obtain

$$\|Sx_{n+1}-p\| \precsim [(L_1+L_2)(b-a)]^{2(n+1)}\|Sx_0-p\| \tag{62}$$

Taking the limit the inequality (62), it can be seen that $\lim_{n\to\infty}\|Sx_n-p\|=0$. Hence, the condition $\sum_{n=0}^{\infty}\alpha_n=\infty$ is unnecessary. □

6. Conclusion

In this work, we obtain some strong convergence theorems by using the newly defined and classic Jungck-type iteration methods for certain mapping in complex-valued Banach spaces. We also show that the convergence speed of the new iteration method is better than the other iteration methods which are mentioned in this work. In addition, we obtain data dependence and stability results for this new iteration method. We also give some nontrivial examples to support these results. Finally, we analyze the convergence of the new iteration method to the solution of the functional integral equation in complex-valued Banach spaces.

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