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COMPLETE CHARACTERIZATION OF ODD FACTORS VIA THE SIZE, SPECTRAL RADIUS OR DISTANCE SPECTRAL RADIUS OF GRAPHS

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Abstract. Given a graph G, a $\{1,3,\dots,2n-1\}$-factor of G is a spanning subgraph of G, in which each degree of vertices is one of $\{1,3,\dots,2n-1\}$, where n is a positive integer. In this paper, we first establish a lower bound on the size (resp. the spectral radius) of G to guarantee that G contains a $\{1,3,\dots,2n-1\}$-factor. Then we determine an upper bound on the distance spectral radius of G to ensure that G has a $\{1,3,\dots,2n-1\}$-factor. Furthermore, we construct some extremal graphs to show all the bounds obtained in this contribution are best possible.

1. Introduction

In this paper, we only deal with finite and undirected graphs without loops or multiple edges. For graph theoretic notation and terminology not defined here, we refer to [5, 14].

Let G be a graph with vertex set $V(G) = \{v_1,\dots,v_\nu\}$ and edge set $E(G)$. The *order* of G is the number $\nu := |V(G)|$ of its vertices and its *size* is the number $\varepsilon := |E(G)|$ of its edges. A graph G is called *trivial* if $\nu = 1$. Let $V_1 \subseteq V(G)$ and $E_1 \subseteq E(G)$. Then $G - V_1, G - E_1$ are the graphs formed from G by deleting the vertices in V_1 and their incident edges, the edges in E_1, respectively. For convenience, denote $G - \{v\}$ and $G - \{uv\}$ by $G - v$ and $G - uv$, respectively. For a given subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. As usual, let P_n and K_n denote the path and complete graph on n vertices, respectively.

For a vertex $v \in V(G)$, let $N_G(v)$ be the set of all neighbours of v in G. Then $d_G(v) = |N_G(v)|$ is the *degree* of v in G. A vertex v of G is called a *pendant vertex* if $d_G(v) = 1$. A *quasi-pendant vertex* is a vertex being adjacent to some pendant vertex. A graph is *r-regular* if each vertex has the same degree r. The

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complement of a graph G is a graph $\overline{G}$ with the same vertex set as G, in which any two distinct vertices are adjacent if and only if they are non-adjacent in G.

For two graphs G_1 and G_2, we define $G_1 \cup G_2$ to be their *disjoint union*. The *join* $G_1 \vee G_2$ is obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2 by an edge. The graph formed by joining $\overline{K}_q$ with a vertex of the complete graph K_p (i.e., $K_1 \vee (K_{p-1} \cup \overline{K}_q)$) is called a *pineapple graph*, written by P_p^q.

Given a graph G of order ν, the *adjacency matrix* $A(G) = (a_{ij})_{\nu\times\nu}$ of G is a 0-1 matrix in which the entry $a_{ij} = 1$ if and only if v_i and v_j are adjacent. The eigenvalues of the adjacency matrix $A(G)$ are also called *eigenvalues* of G. Note that $A(G)$ is a real non-negative symmetric matrix. Hence, its eigenvalues are real, which can be arranged in nonincreasing order as $\lambda_1(G) \geqslant \cdots \geqslant \lambda_\nu(G)$. Note that the adjacency spectral radius (or spectral radius, for short) of G is equal to $\lambda_1(G)$, written as $\rho(G)$.

Let G be a connected graph. The *distance* between v_i and v_j in G, denoted by d_{ij}, is the length of a shortest path from v_i to v_j. The *Winer index* of G is defined as $W(G) = \sum_{i<j} d_{ij}$. The *distance matrix* of G, denoted by $D(G)$, is a $\nu \times \nu$ real symmetric matrix whose (i, j)-entry is d_{ij}. Then we can order the eigenvalues of $D(G)$ as

$$\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_\nu(G).$$

By the Perron-Frobenius theorem, $\mu_1(G)$ is always positive (unless G is trivial) and $\mu_1(G) \geqslant |\mu_i(G)|$ for $i = 2, 3, \ldots, \nu$, and we call $\mu_1(G)$ the *distance spectral radius*.

A subset M of $E(G)$ is called a *matching* if any two members of M do not have a common vertex in G. A matching with the maximum size in G is called the *maximum matching*. The *matching number* $\alpha'(G)$ is the size of a maximum matching in G. We call M a *perfect matching* if each vertex of G is incident with an edge in M.

Let I be a set of non-negative integers. A graph G is called an *I-graph* if $d_G(v) \in I$ for all $v \in V(G)$. In particular, a $\{r\}$-graph is a r-regular graph. An *I-factor* is a spanning I-subgraph of G. Let n be a positive integer. Then, a *$\{1, 3, \ldots, 2n-1\}$-factor* of graph G is a spanning subgraph, in which each degree of vertices is one of $\{1, 3, \ldots, 2n-1\}$. The $\{1, 3, \ldots, 2n-1\}$-factor is also called an *odd factor* of G. Note that a perfect matching of graph G is indeed a $\{1\}$-factor of G.

In mathematical literature, it is interesting to study I-factor. For example, Tutte [12] gave a necessary and sufficient condition for the existence of a $\{1\}$-factor in a graph, which is well known as Tutte's 1-Factor Theorem. Bondy [3] gave a necessary and sufficient condition of a tree containing a $\{1\}$-factor. Las Vergnas [13] obtained a necessary and sufficient condition for graphs with a $\{1, 2, \ldots, n\}$-factor, while Amahashi [1] characterized a necessary and sufficient condition for graphs with a $\{1, 3, \ldots, 2n-1\}$-factor.

In 2020, O [11] showed that there is a close relationship between the spectral radius and $\{1\}$-factor by Tutte's 1-Factor Theorem. He established a sharp upper bound on the number of edges (resp. spectral radius) of a graph without a $\{1\}$-factor. Cui and Kano [16] gave a sufficient condition for the existence of a $\{1, 3, \ldots, 2n-1\}$-factor in a graph by using neighborhoods, and gave an extension of Amahashi's Theorem. For more advances on this topic, one may be referred to [4, 7] and the references cited in.

Motivated by [1, 11] directly, it is natural and interesting to give some sufficient conditions to ensure that a graph contains a $\{1, 3, \ldots, 2n-1\}$-factor. Here, we focus on the sufficient conditions including the size, the spectral radius or the distance spectral radius of graphs.

Our first main result gives a sufficient condition to ensure that a graph G contains a $\{1, 3, \ldots, 2n-1\}$-factor according to the size of G. Note that if G has odd-factors, then the order of G is even (based on handshaking lemma). We call G a *ν-vertex graph* if the graph G has ν vertices.

Theorem 1.1. *Let G be a ν-vertex connected graph, where ν is an even integer, and let $n \leqslant \frac{\nu}{2} - 1$ be a positive integer.*

(i) *For $\nu = 4$ or $\nu \geqslant 10$, if $|E(G)| > 2n + \binom{\nu-2n}{2}$, then G contains a $\{1, 3, \ldots, 2n-1\}$-factor.*

(ii) *For $\nu = 6$, if $|E(G)| > 9$, then G contains a $\{1\}$-factor; if $|E(G)| > 5$, then G contains a $\{1, 3\}$-factor.*

(iii) *For $\nu = 8$, if $|E(G)| > 18$, then G contains a $\{1\}$-factor; if $|E(G)| > 10$, then G contains a $\{1, 3\}$-factor; if $|E(G)| > 7$, then G contains a $\{1, 3, 5\}$-factor.*

Our second main result gives a sufficient condition to ensure that a graph G contains a $\{1, 3, \ldots, 2n-1\}$-factor according to the adjacency spectral radius of G.

Theorem 1.2. *Let G be a ν-vertex connected graph, where ν is an even integer, and let $n \leqslant \frac{\nu}{2} - 1$ be a positive integer. Assume the largest root of $x^3 + (2n - \nu + 2)x^2 - (\nu - 1)x - 2n(2n - \nu + 2) = 0$ is $\theta(\nu)$.*

(i) *For $\nu = 4$ or $\nu \geqslant 8$, if $\rho(G) > \theta(\nu)$, then G contains a $\{1, 3, \ldots, 2n-1\}$-factor.*

(ii) *For $\nu = 6$, if $\rho(G) > \frac{1+\sqrt{33}}{2}$, then G contains a $\{1\}$-factor; if $\rho(G) > \theta(6)$, then G contains a $\{1, 3\}$-factor.*

Our last main result gives a sufficient condition to ensure that a graph G contains a $\{1, 3, \ldots, 2n-1\}$-factor in regard to the distance spectral radius of G.

Theorem 1.3. *Let G be a ν-vertex connected graph, where ν is an even integer, and let $n \leqslant \frac{\nu}{2} - 1$ be a positive integer.*

(i) *For $\nu = 4$ or $\nu \geqslant 10$, if $\mu_1(G) < \mu_1(P^{2n}_{\nu-2n})$, then G contains a $\{1, 3, \ldots, 2n-1\}$-factor.*

(ii) *For* $\nu = 6$, *if* $\mu_1(G) < \mu_1(K_2 \vee 4K_1)$, *then* G *contains a* $\{1\}$*-factor; if* $\mu_1(G) < \mu_1(P_2^4)$, *then* G *contains a* $\{1,3\}$*-factor.*

(iii) *For* $\nu = 8$, *if* $\mu_1(G) < \mu_1(K_3 \vee 5K_1)$, *then* G *contains a* $\{1\}$*-factor; if* $\mu_1(G) < \mu_1(P_4^4)$, *then* G *contains a* $\{1,3\}$*-factor; if* $\mu_1(G) < \mu_1(P_2^6)$, *then* G *contains a* $\{1,3,5\}$*-factor.*

The proof techniques for our main results follow the idea of O [11]. Together with some new idea we make the proofs work. Our paper is organized as follows. In Section 2, we give some preliminary results. In Section 3, we give the proof of Theorem 1.1. In Section 4, we give the proof of Theorem 1.2 and in Section 5, we give the proof of Theorem 1.3.

2. Some preliminaries

In this section, we present some necessary preliminary results, which will be used to prove our main results. The first one follows directly from [2, Theorem 6.8].

Lemma 2.1 ([2])**.** *Let* G *be a connected graph and let* H *be a proper subgraph of* G*. Then* $\rho(G) > \rho(H)$*.*

Let M be an $n \times n$ irreducible and nonnegative matrix, by the Perron-Frobenius theorem [6], we know that there exists a unit positive eigenvector, say $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, of M corresponding to spectral radius of M. As usual, we call $\mathbf{x}$ the *Perron vector* of M.

The following lemma is a direct consequence of [10, Proposition 16].

Lemma 2.2 ([10])**.** *Let* G *be a* ν*-vertex connected graph and let* $\mathbf{x} = (x_1, x_2, \ldots, x_\nu)^T$ *be the Perron vector of* $A(G)$ *corresponding to* $\rho(G)$*. If* $v_i, v_j \in V(G)$ *satisfy* $N_G(v_i) \setminus \{v_j\} = N_G(v_j) \setminus \{v_i\}$*, then* $x_i = x_j$*.*

Let M be a real matrix whose rows and columns are indexed by $V = \{1, \ldots, n\}$. Assume that M, with respect to to the partition $\pi: V = V_1 \cup \cdots \cup V_s$, can be written as

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1s} \\ \vdots & \ddots & \vdots \\ M_{s1} & \cdots & M_{ss} \end{pmatrix},$$

where M_{ij} denotes the submatrix (block) of M formed by rows in V_i and columns in V_j. Let q_{ij} denote the average row sum of M_{ij}. Then matrix $M_\pi = (q_{ij})$ is called the *quotient matrix* of M. If the row sum of each block M_{ij} is a constant, then the partition is *equitable*.

Lemma 2.3 ([15])**.** *Let* M *be a square matrix with an equitable partition* π *and let* M_π *be the corresponding quotient matrix. Then every eigenvalue of* M_π *is an eigenvalue of* M*. Furthermore, if* M *is nonnegative, then the largest eigenvalues of* M *and* M_π *are equal.*

Lemma 2.4 ([9]). *Let G be a connected graph with two nonadjacent vertices $u, v \in V(G)$. Then $\mu_1(G+uv) < \mu_1(G)$, where $G+uv$ denotes the graph obtained from G by adding an edge to connect u and v.*

Lemma 2.5 ([8]). *Let G be a ν-vertex connected graph and let $\mathbf{y} = (y_1, y_2, \ldots, y_\nu)^T$ be the Perron vector of $D(G)$ corresponding to $\mu_1(G)$. If v_i, v_j are in $V(G)$ satisfying*

$$N_G(v_i) \setminus \{v_j\} = N_G(v_j) \setminus \{v_i\},$$

then $y_i = y_j$.

The following lemma can be easily derived by the Rayleigh quotient [6].

Lemma 2.6. *Let G be a connected graph with order ν. Then*

$$\mu_1(G) = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T D(G)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \geqslant \frac{\mathbf{1}^T D(G)\mathbf{1}}{\mathbf{1}^T\mathbf{1}} = \frac{2W(G)}{\nu},$$

where $\mathbf{1} = (1, 1, \ldots, 1)^T$.

Lemma 2.7 ([9]). *Let M be a Hermitian matrix of order s, and let N be a principal submatrix of M with order t. If $\hat{\lambda}_1 \geqslant \hat{\lambda}_2 \geqslant \cdots \geqslant \hat{\lambda}_s$ are the eigenvalues of M and $\hat{\mu}_1 \geqslant \hat{\mu}_2 \geqslant \cdots \geqslant \hat{\mu}_t$ are the eigenvalues of N, then $\hat{\lambda}_i \geqslant \hat{\mu}_i \geqslant \hat{\lambda}_{s-t+i}$ for $1 \leqslant i \leqslant t$.*

Let $o(G)$ be the number of odd components (components with odd order) of G. The following lemma gives a sufficient and necessary condition for a graph containing a $\{1, 3, \ldots, 2n-1\}$-factor.

Lemma 2.8 ([1]). *Let G be a graph and n be a positive integer. Then G has a $\{1, 3, \ldots, 2n-1\}$-factor if and only if $o(G-S) \leqslant (2n-1)|S|$ for every set $S \subseteq V(G)$.*

Lemma 2.9. *Let G be a connected graph and let n be a positive integer. If G does not have a $\{1, 3, \ldots, 2n-1\}$-factor, then there exists a non-empty subset $S \subseteq V(G)$ such that $o(G-S) > (2n-1)|S|$. Furthermore, if $|V(G)|$ is even, then*

$$o(G-S) \equiv (2n-1)|S| \pmod 2 \tag{1}$$

and $o(G-S) \geqslant (2n-1)|S| + 2$.

Proof. By Lemma 2.8, it is easy to see that there exists a subset $S \subseteq V(G)$ satisfying $o(G-S) > (2n-1)|S|$. Assume $|V(G)|$ is even. Then $o(G-S)$ is odd (resp. even) if and only if $|S|$ is odd (resp. even), which is equivalent to say that $(2n-1)|S|$ is odd (resp. even). Hence, (1) holds, and so $o(G-S) \geqslant (2n-1)|S| + 2$. □

By Lemma 2.8, the following corollary can be shown easily.

Corollary 2.10. *Let ν be an even number. If G is a ν-vertex connected graph, then G has a $\{1, 3, \ldots, \nu-1\}$-factor.*

Since $\{1, 3, \ldots, 2n-1\} \subset \{1, 3, \ldots, 2n+1\}$, we have that G must contain a $\{1, 3, \ldots, 2n+1\}$-factor if G contains a $\{1, 3, \ldots, 2n-1\}$-factor. By Corollary 2.10, it suffices to consider $2n-1 < |V(G)|-1$ (i.e., $n \leqslant \frac{\nu}{2}-1$) in the whole context.

3. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1, which gives a sufficient condition via the size of a connected graph to ensure that the graph contains a $\{1, 3, \ldots, 2n-1\}$-factor. We also show that the bounds obtained in Theorem 1.1 are the best possible.

Proof of Theorem 1.1. Suppose to the contrary that G has no $\{1, 3, \ldots, 2n-1\}$-factor. By Lemma 2.9, there exists a non-empty subset $S \subseteq V(G)$ satisfying $o(G-S) \geqslant (2n-1)|S|+2$. Choose such a connected graph G of order ν so that its size is as large as possible.

According to the choice of G, the induced subgraph $G[S]$ and each connected component of $G-S$ are complete graphs, respectively. Furthermore, all components of $G-S$ are odd and G is just the graph $G[S] \vee (G-S)$.

For convenience, let $o(G-S) = q$ and $|S| = s$. Assume that $G_1, G_2, \ldots, G_q$ are all the components of $G-S$ with $n_i = |V(G_i)|$ and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_q$. Then, $G = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})$. We proceed by showing $n_2 = \cdots = n_q = 1$.

In fact, if there exists some $i \in \{2, \ldots, q\}$ such that $n_i \geqslant 3$, then we consider a new graph $H_1 = K_s \vee (K_{n_1+2} \cup K_{n_2} \cup \cdots \cup K_{n_i-2} \cup \cdots \cup K_{n_q})$. It is easy to see that

$$o(H_1 - S) = o(G-S) \geqslant (2n-1)|S|+2.$$

On the other hand,

$$\begin{aligned} |E(H_1)| &= |E(G)| - 2(n_i-2) + 2n_1 \\ &= |E(G)| + 2(n_1-n_i) + 4 \\ &> |E(G)|. \end{aligned}$$

This contradicts the choice of G. Therefore, $n_i = 1$ for all $i \in \{2, 3, \ldots, q\}$ and so $n_1 = \nu - s - q + 1$. Then, $G = K_s \vee (K_{n_1} \cup (q-1)K_1)$.

Notice that $q \geqslant (2n-1)s+2$. We are to show $q = (2n-1)s+2$. In fact, if $q \geqslant (2n-1)s+4$, then let $H_2 = K_s \vee (K_{n_1+2} \cup (q-3)K_1)$. Clearly, G is a proper subgraph of H_2. Hence $|E(H_2)| > |E(G)|$. Bear in mind that

$$o(H_2 - S) = o(G-S) - 2 \geqslant (2n-1)|S|+2.$$

Hence, we obtain a contradiction to the choice of G. Therefore, $q < (2n-1)s+4$. Together with (1), one has $q = (2n-1)s+2$ and so $G = K_s \vee (K_{n_1} \cup (2ns - s+1)K_1)$.

It is straightforward to check that $\nu = n_1 + 2ns + 1 \geqslant 2ns+2$ and $|E(G)| = s(2ns-s+1) + \binom{\nu-2ns+s-1}{2} =: f(n, s)$. In what follows, we are to prove

$f(n,s) \leqslant 2n+\binom{\nu-2n}{2}$ for $n \geqslant 2$. In fact,

$$\begin{aligned} f(n,s) - 2n - \binom{\nu-2n}{2} &= s(2ns-s+1) + \binom{\nu-2ns+s-1}{2} \\ &\quad - 2n - \binom{\nu-2n}{2} \\ &= \frac{(s-1)[4n^2s+4n^2-(4n-2)\nu+6n-s-2]}{2}. \end{aligned}$$

Obviously, $f(n,s) = 2n+\binom{\nu-2n}{2}$ if $s=1$. So it suffices to prove $4n^2s+4n^2-(4n-2)\nu+6n-s-2 \leqslant 0$ for $s \geqslant 2$. By $\nu \geqslant 2ns+2$, we have

$$\begin{aligned} 4n^2s+4n^2-(4n-2)\nu+6n-s-2 &\leqslant 4n^2s+4n^2-(4n-2)(2ns+2) \\ &\quad + 6n-s-2 \\ &= (-4n^2+4n-1)s+4n^2-2n+2 \\ &\leqslant 2(-4n^2+4n-1)+4n^2-2n+2 \\ &= -4n^2+6n < 0. \end{aligned}$$

Therefore, when $n \geqslant 2$, we have

$$f(n,s) \leqslant 2n + \binom{\nu-2n}{2}. \tag{2}$$

Recall that $\nu \geqslant 2ns+2$ and $n \leqslant \frac{\nu}{2}-1$. If $\nu=4$, then $n=1$ and $s=1$. Note that $|E(G)| = f(1,1) = 2+\binom{4-2}{2}$, a contradiction to the assumption that $|E(G)| > 2n+\binom{\nu-2n}{2}$.

If $\nu=6$, then $n \leqslant 2$. For $n=1$, one has $s \leqslant 2$. By a direct calculation, one has $f(1,1)=8$ and $f(1,2)=9$, each of which deduces a contradiction. For $n=2$, by (2), we have $f(2,s) \leqslant 5$, a contradiction.

If $\nu=8$, then $n \leqslant 3$. For $n=1$, we have $s \leqslant 3$. By a direct calculation, one has $f(1,1)=17$, $f(1,2)=16$ and $f(1,3)=18$, each of which induces a contradiction. For $n \geqslant 2$, by (2), we get $f(2,s) \leqslant 10$ and $f(3,s) \leqslant 7$, in which each contradicts the condition.

At last we consider the case $\nu \geqslant 10$. By (2), we have $f(n,s) \leqslant 2n+\binom{\nu-2n}{2}$ for $n \geqslant 2$. Thus, it is sufficient to show $f(n,s) \leqslant 2n+\binom{\nu-2n}{2}$ for $n=1$. Note that

$$\begin{aligned} f(1,s) - 2n - \binom{\nu-2n}{2} &= s(s+1) + \binom{\nu-s-1}{2} - 2 - \binom{\nu-2}{2} \\ &= \frac{(s-1)(3s-2\nu+8)}{2}. \end{aligned}$$

Recall that $\nu = n_1+2s+1 \geqslant 10$. Hence, $2n_1+s \geqslant 6$. So we have

$$\begin{aligned} f(1,s) - 2n - \binom{\nu-2n}{2} &\leqslant \frac{(s-1)[3s-2(n_1+2s+1)+8]}{2} \\ &= \frac{(s-1)(-2n_1-s+6)}{2} \leqslant 0. \end{aligned}$$

By the argument as above, we obtain $|E(G)| \leqslant 2n+\binom{\nu-2n}{2}$, a contradiction to the condition for $\nu \geqslant 10$. $\square$

Bearing in mind that a perfect matching of graph G is a $\{1\}$-factor. So the next result follows immediately.

Corollary 3.1 ([11]). *Let G be a ν-vertex connected graph. If $\nu = 4$ or $\nu \geqslant 10$, and $|E(G)| > 2+\binom{\nu-2}{2}$, then G has a perfect matching. If $\nu = 6$ and $|E(G)| > 9$, or $\nu = 8$ and $|E(G)| > 18$, then G has a perfect matching.*

At last we show the bounds obtained in Theorem 1.1 are the best possible. It is straightforward to check that $|E(P^{2n}_{\nu-2n})| = 2n+\binom{\nu-2n}{2}$, $|E(K_2 \vee 4K_1)| = 9$, $|E(P^4_2)| = 5$, $|E(K_3 \vee 5K_1)| = 18$, $|E(P^4_4)| = 10$ and $|E(P^6_2)| = 7$.

Theorem 3.2. *Let G be a ν-vertex connected graph, where ν is an even integer, and let $n \leqslant \frac{\nu}{2} - 1$ be a positive integer.*

(i) *For $\nu = 4$ or $\nu \geqslant 10$, $P^{2n}_{\nu-2n}$ has no $\{1, 3, \dots, 2n-1\}$-factor.*
(ii) *For $\nu = 6$, $K_2 \vee 4K_1$ has no $\{1\}$-factor, and P^4_2 has no $\{1, 3\}$-factor.*
(iii) *For $\nu = 8$, $K_3 \vee 5K_1$ has no $\{1\}$-factor, P^4_4 has no $\{1, 3\}$-factor and P^6_2 has no $\{1, 3, 5\}$-factor.*

Proof. Let v be the vertex with the maximum degree of $P^{2n}_{\nu-2n}$. Put $S = \{v\}$ (resp. $V(K_2)$ and $V(K_3)$), then $o(P^{2n}_{\nu-2n} - S) = 2n + 1 > 2n - 1$ (resp. $o(K_2 \vee 4K_1 - S) = 4$ and $o(K_3 \vee 5K_1 - S) = 5$). By Lemma 2.8, we get that $P^{2n}_{\nu-2n}$ has no $\{1, 3, \dots, 2n-1\}$-factor, $K_2 \vee 4K_1$ and $K_3 \vee 5K_1$ have no $\{1\}$-factor. $\square$

4. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2, which presents a sufficient spectral condition to ensure a graph containing a $\{1, 3, \dots, 2n-1\}$-factor. Based on the idea in the proof of Theorem 1.1, we prove Theorem 1.2 by comparing the spectral radius rather than the number of edges. We also show that the bounds obtained in Theorem 1.2 are the best possible.

Proof of Theorem 1.2. Suppose to the contrary that G has no $\{1, 3, \dots, 2n-1\}$-factor. Then by Lemma 2.9, one has that there exists a non-empty subset $S \subseteq V(G)$ satisfying $o(G-S) \geqslant (2n-1)|S|+2$. Choose such a connected graph G of order ν so that its spectral radius is as large as possible.

Together with Lemma 2.1 and the choice of G, we obtain that all the components of $G - S$ are odd, and the induced subgraph $G[S]$ (resp. each connected component of $G - S$) is a complete subgraph. Thus, $G = G[S] \vee (G - S)$.

For convenience, let $o(G - S) = q$ and $|S| = s$. Assume $G_1, G_2, \dots, G_q$ are all the components of $G - S$ and let $n_i = |V(G_i)|$ with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_q$. Then, $G = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_q})$.

In what follows, we show that $n_2 = n_3 = \cdots = n_q = 1$. If $n_2 \geqslant 3$, then let $H = K_s \vee (K_{n_1+2} \cup K_{n_2-2} \cup K_{n_3} \cup \cdots \cup K_{n_q})$. Note that $o(H-S) = o(G-S) \geqslant (2n-1)|S| + 2$.

Assume $\mathbf{x} = (x_1, \dots, x_\nu)^T$ is the Perron vector of $A(G)$, and let x_i denote the entry of $\mathbf{x}$ corresponding to the vertex $v_i \in V(G)$. By Lemma 2.2, one has $x_r = x_s$ for all v_r, v_s in S (resp. $V(G_i)$, $i \in \{1, 2, \dots, q\}$). For convenience, let $x_0 = x_r$ for all $v_r \in S$, and $x_i = x_r$ for all $v_r \in V(G_i)$, $i \in \{1, 2, \dots, q\}$. Then

$$\begin{cases} \rho(G)x_1 = sx_0 + (n_1 - 1)x_1, \\ \rho(G)x_2 = sx_0 + (n_2 - 1)x_2. \end{cases}$$

Thus, $x_1 \geqslant x_2$. By the Rayleigh quotient, we have

$$\begin{aligned} \rho(H) - \rho(G) &\geqslant \mathbf{x}^T(A(H) - A(G))\mathbf{x} \\ &= 4n_1x_1x_2 - 4(n_2 - 2)x_2^2 \\ &\geqslant 4x_2^2(n_1 - n_2 + 2) \\ &> 0. \end{aligned}$$

Hence, H is a ν-vertex connected graph with $\rho(H) > \rho(G)$, a contradiction to the choice of G. Therefore, $n_2 = n_3 = \cdots = n_q = 1$. This gives us $n_1 = \nu - s - q + 1$. Thus, $G = K_s \vee (K_{n_1} \cup (q-1)K_1)$.

Next we show $q = (2n-1)s+2$. Note that $q \geqslant (2n-1)s+2$ and $o(G-S)$ and $(2n-1)|S|$ have the same parity based on Lemma 2.9. Hence, if $q \geqslant (2n-1)s+4$, then let $\hat{H} = K_s \vee (K_{n_1+2} \cup (q-3)K_1)$. Note that $o(\hat{H} - S) = o(G-S) - 2 \geqslant (2n-1)|S| + 2$ and G is a proper subgraph of $\hat{H}$. Hence, by Lemma 2.1, $\rho(\hat{H}) > \rho(G)$, a contradiction to the choice of G. Therefore, $q < (2n-1)s + 4$. Thus, $q = (2n-1)s + 2$ and so $G = K_s \vee (K_{n_1} \cup (2ns - s + 1)K_1)$.

Let $f(x) = x^3 + (2n - \nu + 2)x^2 - (\nu - 1)x - 2n(2n - \nu + 2)$ be a real function in x, and let $\theta(\nu)$ be the largest root of $f(x) = 0$. Note that $\nu = n_1 + 2ns + 1$ and n_1 is odd. In what follows, we proceed by considering the two possible cases.

Case 1. $n_1 = 1$.

In this case, $\nu = 2ns + 2$ and $G = K_s \vee (2ns - s + 2)K_1$. Consider the partition π_1: $V(G) = S \cup V((2ns - s + 2)K_1)$. Then the quotient matrix of $A(G)$ corresponding to the partition π_1 equals

$$B_1 = \begin{pmatrix} s-1 & 2ns - s + 2 \\ s & 0 \end{pmatrix}.$$

Thus the characteristic polynomial of B_1 is

$$\Phi_{B_1}(x) = x^2 - (s-1)x - s(2ns - s + 2).$$

Since the partition $\pi_1 : V(G) = S \cup V((2ns - s + 2)K_1)$ is equitable, by Lemma 2.3, the largest root, say ρ_1, of $\Phi_{B_1}(x) = 0$ satisfies $\rho_1 = \rho(G)$. By a simple calculation, one has

$$\rho(G) = \rho_1 = \frac{s - 1 + \sqrt{(8n-3)s^2 + 6s + 1}}{2}.$$

This gives us

$$\begin{aligned}f(\rho_1) =&(s-1)[(-4n^2+4n-1)s^2+(-3n+2)s\\&+n\sqrt{(8n-3)s^2+6s+1}+4n^2-n].\end{aligned}$$

We proceed by considering the following two subcases.

Subcase 1.1. $n = 1$. In this case $\nu = 2s + 2$. If $\nu = 4$, then $s = 1$. Note that $f(\rho_1) = 0$ when $s = 1$. Hence, $\rho(G) \leqslant \theta(4)$, which is a contradiction to the assumption. If $\nu = 6$, then $s = 2$. By a direct calculation, one has $\rho(G) = \frac{1+\sqrt{33}}{2}$, a contradiction to the assumption. So we consider $\nu \geqslant 8$ in what follows. Together with $\nu = 2s + 2 \geqslant 8$, one has $s \geqslant 3$.

Note that

$$\begin{aligned}f(\rho_1) &= (s-1)(-s^2-s+3+\sqrt{5s^2+6s+1})\\&= (s-1)(-s^2-s+3+\sqrt{5(s+\frac{3}{5})^2-\frac{4}{5}})\\&< (s-1)[-s^2-s+3+\frac{5}{2}(s+\frac{3}{5})]\\&= (s-1)(-s^2+\frac{3}{2}s+\frac{9}{2})\\&= -(s-1)(s+\frac{3}{2})(s-3)\\&\leqslant 0.\end{aligned}$$

Thus, $f(\rho_1) < 0$. Then $\rho(G) = \rho_1 \leqslant \theta(\nu)$, a contradiction to the assumption.

Subcase 1.2. $n \geqslant 2$. In this case $\nu = 2ns + 2 \geqslant 4s + 2 \geqslant 6$.

Obviously, $f(\rho_1) = 0$ when $s = 1$. Then $\rho(G) = \rho_1 \leqslant \theta(\nu)$ for $s = 1$, a contradiction to the assumption. Next our purpose is to show, for $s \geqslant 2$, $f(\rho_1) < 0$, which is equivalent to show

$$\begin{aligned}g_1 := &(-4n^2+4n-1)s^2+(-3n+2)s\\&+n\sqrt{(8n-3)s^2+6s+1}+4n^2-n<0\end{aligned}$$

for $s \geqslant 2$. Note that

$$\begin{aligned}\sqrt{(8n-3)s^2+6s+1} &< \sqrt{8ns^2+6s+1}\\&= \sqrt{8n(s+\frac{3}{8n})^2+\frac{8n-9}{8n}} < 2\sqrt{2}n(s+\frac{3}{8n})+1.\end{aligned}$$

This gives us

$$\begin{aligned}g_1 &< (-4n^2+4n-1)s^2+(-3n+2)s+n[2\sqrt{2}n(s+\frac{3}{8n})+1]+4n^2-n\\&= (-4n^2+4n-1)s^2+(2\sqrt{2}n^2-3n+2)s+\frac{3\sqrt{2}n}{4}+4n^2\\&\leqslant s[2(-4n^2+4n-1)+2\sqrt{2}n^2-3n+2]+\frac{3\sqrt{2}n}{4}+4n^2\end{aligned}$$

$$= s(-8n^2 + 2\sqrt{2}n^2 + 5n) + \frac{3\sqrt{2}n}{4} + 4n^2.$$

Together with $n \geqslant 2$ and $-8n^2 + 2\sqrt{2}n^2 + 5n < 0$, one has

$$\begin{aligned} g_1 &< 2(-8n^2 + 2\sqrt{2}n^2 + 5n) + \frac{3\sqrt{2}n}{4} + 4n^2 \\ &= (-12 + 4\sqrt{2})n^2 + (10 + \frac{3\sqrt{2}}{4})n < 0, \end{aligned}$$

i.e., $g_1 < 0$ for $n \geqslant 2$, and so $\rho(G) = \rho_1 < \theta(\nu)$, a contradiction to the assumption.

Case 2. $n_1 \geqslant 3$.

In this case, one has $\nu \geqslant 2ns + 4 \geqslant 6$. Consider the partition π_2: $V(G) = V(K_{n_1}) \cup V(K_s) \cup V((2ns - s + 1)K_1)$. Then the quotient matrix of $A(G)$ corresponding to the partition π_2 equals

$$B_2 = \begin{pmatrix} \nu - 2ns - 2 & s & 0 \\ \nu - 2ns - 1 & s - 1 & 2ns - s + 1 \\ 0 & s & 0 \end{pmatrix}.$$

Thus the characteristic polynomial of B_2 is

$$\begin{aligned} \Phi_{B_2}(x) = {} & x^3 + (2ns - s - \nu + 3)x^2 - (2ns^2 - 2ns - s^2 + 2s + \nu - 2)x \\ & + s(2ns - s + 1)(\nu - 2ns - 2). \end{aligned}$$

Since the partition π_2: $V(G) = V(K_{n_1}) \cup V(K_s) \cup V((2ns - s + 1)K_1)$ is equitable, by Lemma 2.3, the largest root, say ρ_2, of $\Phi_{B_2}(x) = 0$ satisfies $\rho_2 = \rho(G)$.

In what follows, we are to show $\rho(G) \leqslant \theta(\nu)$, i.e., $\rho_2 \leqslant \theta(\nu)$. By a direct calculation, we have

$$\begin{aligned} f(x) - \Phi_{B_2}(x) = {} & (-2ns + 2n + s - 1)x^2 + (2ns^2 - 2ns - s^2 + 2s - 1)x \\ & + 4n^2s^3 - 2ns^3 - 2ns^2\nu + 6ns^2 + s^2\nu - 4n^2 \\ & + 2n\nu - 2s^2 - s\nu - 4n + 2s \\ = {} & -(s-1)[(2n-1)x^2 + (s - 1 - 2ns)x \\ & + (\nu - 2n - 2)(2ns + 2n - s) - 2ns(2ns - s + 1)]. \end{aligned}$$

Obviously, $f(x) = \Phi_{B_2}(x)$ and $\rho_2 \leqslant \theta(\nu)$ when $s = 1$. Let $g_2(x) = (2n-1)x^2 + (s - 1 - 2ns)x + (\nu - 2n - 2)(2ns + 2n - s) - 2ns(2ns - s + 1)$ be a real function in x. Then it suffices to show $g_2(\rho_2) > 0$ for $s \geqslant 2$. We proceed by considering the following two possible subcases.

Subcase 2.1. $n = 1$. In this case, $\nu \geqslant 2s + 4 \geqslant 6$, and $g_2(x) = x^2 - (1 + s)x + (\nu - 4)(s + 2) - 2s(s + 1)$. Note that K_{s+3} is a proper subgraph of G. By Lemma 2.1, one has $\rho_2 > s + 2$.

It is routine to check that the derivative function of $g_2(x)$ is

$$g_2'(x) = 2x - (1 + s).$$

Hence, $\frac{1+s}{2}$ is the unique solution of $g_2'(x) = 0$. As $\frac{1+s}{2} < s+2$, one has $g_2(x)$ is increasing for $x \in [s+2, +\infty)$. Then

$$\begin{aligned}g_2(\rho_2) &> g_2(s+2)\\ &= s+2+(\nu-4)(s+2)-2s(s+1)\\ &\geqslant s+2+2s(s+2)-2s(s+1)\\ &= 3s+2\\ &> 0.\end{aligned}$$

Hence, $\rho_2 \leqslant \theta(\nu)$ if $n = 1$. Then we can get a contradiction for $\nu \geqslant 8$. As $\theta(6) < \frac{1+\sqrt{33}}{2}$, one also obtains a contradiction for $\nu = 6$.

Subcase 2.2. $n \geqslant 2$. In this case, $\nu \geqslant 4s+4 \geqslant 8$, and $K_s \vee \overline{K}_{\nu-s}$ is a proper subgraph of G. By Lemma 2.1, one has $\rho_2 > \rho(K_s \vee \overline{K}_{\nu-s})$. Consider a equitable partition $V(K_s \vee \overline{K}_{\nu-s}) = V(K_s) \cup V(\overline{K}_{\nu-s})$. One may obtain the quotient matrix of $A(K_s \vee \overline{K}_{\nu-s})$ corresponding to the partition as

$$B_3 = \begin{pmatrix} s-1 & \nu-s \\ s & 0 \end{pmatrix}.$$

Its characteristic polynomial is $\Phi_{B_3}(x) = x^2+(1-s)x+s(s-\nu)$. By Lemma 2.3, the largest root, say ρ_3, of $\Phi_{B_3}(x) = 0$ is equal to $\rho(K_s \vee \overline{K}_{\nu-s})$. Hence, $\rho_2 > \rho_3$.

Note that

$$\begin{aligned}(1-s)^2-4s(s-\nu) &= -3s^2+4s\nu-2s+1\\ &\geqslant -3s^2+4s(2ns+4)-2s+1\\ &= (8n-3)s^2+14s+1\\ &> 9s^2\\ &> 0.\end{aligned}$$

Hence,

$$\rho_3 = \frac{s-1+\sqrt{-3s^2+4s\nu-2s+1}}{2} > 2s-\frac{1}{2}.$$

Recall that $g_2(x) = (2n-1)x^2+(s-1-2ns)x+(\nu-2n-2)(2ns+2n-s)-2ns(2ns-s+1)$. It is routine to check that the derivative function of $g_2(x)$ is

$$g_2'(x) = 2(2n-1)x+s-1-2ns.$$

Hence $\frac{s}{2}+\frac{1}{4n-2}$ is the unique solution of $g_2'(x) = 0$. As

$$\frac{s}{2}+\frac{1}{4n-2} < 2s-\frac{1}{2} < \rho_3,$$

one obtains that $g_2(x)$ is increasing for $x \in [\rho_3, +\infty)$. Together with $\rho_2 > \rho_3$, one has $g_2(\rho_2) > g_2(\rho_3)$, where

$$\begin{aligned}g_2(\rho_3) = {}& (4ns+2n-2s)\nu-4n^2s^2-4n^2s-n\sqrt{-3s^2+4s\nu-2s+1}\\ &-4n^2-5ns+s^2-3n+2s.\end{aligned}$$

Note that

$$ns\nu > n\sqrt{4s\nu} > n\sqrt{-3s^2+4s\nu-2s+1}.$$

Hence,

$$g_2(\rho_2) > (3ns+2n-2s)\nu - 4n^2s^2 - 4n^2s - 4n^2 - 5ns + s^2 - 3n + 2s.$$

Recall that $\nu \geqslant 2ns+4$. Then

$$\begin{aligned} g_2(\rho_2) &> (3ns+2n-2s)(2ns+4) - 4n^2s^2 - 4n^2s - 4n^2 - 5ns + s^2 - 3n + 2s \\ &= (2n^2-4n+1)s^2 + (7n-6)s - 4n^2 + 5n. \end{aligned}$$

Together with $s \geqslant 2$ and $n \geqslant 2$, one has

$$g_2(\rho_2) > 4(2n^2-4n+1) + 2(7n-6) - 4n^2 + 5n = 4n^2 + 3n - 8 > 0.$$

Hence, $g_2(\rho_2) > 0$. Then $\rho_2 \leqslant \theta(\nu)$ for $n \geqslant 2$, a contradiction to the condition.

Together with Cases 1 and 2, we complete the proof. □

Bearing in mind a perfect matching of graph G is a $\{1\}$-factor. So we have the following corollary directly.

Corollary 4.1 ([11])**.** *Let $\nu = 4$ or $\nu \geqslant 8$ be an even integer. If G is a ν-vertex connected graph with $\rho(G) > \theta(\nu)$, where $\theta(\nu)$ is the largest root of $x^3 + (4-\nu)x^2 - (\nu-1)x - 2(4-\nu) = 0$, then G has a perfect matching. For $\nu = 6$, if G is a ν-vertex connected graph with $\rho(G) > \frac{1+\sqrt{33}}{2}$, then G has a perfect matching.*

We close this section by showing the bounds in Theorem 1.2 are best possible.

Theorem 4.2. *Let G be a ν-vertex connected graph, where ν is even, and let $n \leqslant \frac{\nu}{2} - 1$ be a positive integer. Assume the largest root of $x^3 + (2n-\nu+2)x^2 - (\nu-1)x - 2n(2n-\nu+2) = 0$ is $\theta(\nu)$.*

(i) *For $\nu = 4$ or $\nu \geqslant 8$, we have that $\rho(P^{2n}_{\nu-2n}) = \theta(\nu)$ and $P^{2n}_{\nu-2n}$ has no $\{1,3,\ldots,2n-1\}$-factor.*

(ii) *For $\nu = 6$, we have that $\rho(K_2 \vee 4K_1) = \frac{1+\sqrt{33}}{2}$ and $\rho(P^4_2) = \theta(6)$, $K_2 \vee 4K_1$ has no $\{1\}$-factor and P^4_2 has no $\{1,3\}$-factor.*

Proof. Here we only prove (i). By a similar discussion, we can also show (ii), which is omitted here.

Let v be the maximum degree vertex of $P^{2n}_{\nu-2n}$ and let $S = \{v\}$, then $o(P^{2n}_{\nu-2n} - S) = 2n+1 > 2n-1$. By Lemma 2.8, we get $P^{2n}_{\nu-2n}$ has no $\{1,3,\ldots,2n-1\}$-factor. Consider the partition $V(G) = V(K_{\nu-2n-1}) \cup V(K_1) \cup V(\overline{K}_{2n})$, the quotient matrix of $A(G)$ corresponding to the above partition equals

$$B = \begin{pmatrix} \nu-2n-2 & 1 & 0 \\ \nu-2n-1 & 0 & 2n \\ 0 & 1 & 0 \end{pmatrix}.$$

Then we obtain

$$\Phi_B(x) = x^3 + (2n-\nu+2)x^2 - (\nu-1)x - 2n(2n-\nu+2).$$

By Lemma 2.3, one has $\rho(P^{2n}_{\nu-2n}) = \theta(\nu)$. This completes the proof. $\square$

5. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3, which presents a sufficient condition via distance spectral radius to ensure a graph containing a $\{1, 3, \dots, 2n-1\}$-factor.

Proof of Theorem 1.3. Suppose to the contrary that G has no $\{1, 3, \dots, 2n-1\}$-factor. By Lemma 2.9, there exists a subset $S \subseteq V(G)$ satisfying $o(G-S) \equiv (2n-1)|S| \pmod 2$ and

$$o(G-S) \geqslant (2n-1)|S| + 2. \tag{3}$$

Choose such a connected graph G of order ν so that its distance spectral radius is as small as possible.

Together with Lemma 2.4 and the choice of G, we know that all the components are odd, and the induced subgraph $G[S]$ (resp. each connected component of $G-S$) is a complete subgraph. Thus G is just the graph $G[S] \vee (G-S)$.

For convenience, let $o(G-S) = q$ and $|S| = s$. Suppose that $G_1, G_2, \dots, G_q$ are all the components of $G-S$ and let $n_i = |V(G_i)|$ with $n_1 \geqslant n_2 \geqslant \dots \geqslant n_q$. Then,

$$G = K_s \vee (K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_q}). \tag{4}$$

In order to characterize the structure of G, we need the following facts.

Fact 1. In (4), one has $n_2 = 1$, and so $n_3 = \dots = n_q = 1$.

Proof of Fact 1. If it is not true, then $n_2 \geqslant 3$. Consider a graph $\tilde{G} := K_s \vee ((q-1)K_1 \cup K_{\nu-s-q+1})$. Note that $o(\tilde{G}-S) = o(G-S)$, one obtains that $\tilde{G}$ satisfies (3). Suppose $\mathbf{z} = (z_1, \dots, z_\nu)^T$ is the Perron vector of $D(\tilde{G})$, and let z_i denote the entry of $\mathbf{z}$ corresponding to the vertex $v_i \in V(\tilde{G})$. By Lemma 2.5, one has $z_r = z_s$ for all v_r, v_s in $V(K_s)$ (resp. $V((q-1)K_1)$ and $V(K_{\nu-s-q+1})$). For convenience, let $z_r = a$ for all $v_r \in V(K_s)$, $z_r = b$ for all $v_r \in V((q-1)K_1)$ and $z_r = c$ for all $v_r \in V(K_{\nu-s-q+1})$. Then

$$\begin{cases} \mu_1(\tilde{G})b = sa + 2(\nu-s-q+1)c + 2(q-2)b, \\ \mu_1(\tilde{G})c = sa + (\nu-s-q)c + 2(q-1)b. \end{cases}$$

Thus, $b = (1 + \frac{\nu-q-s}{\mu_1(\tilde{G})+2})c$. By the Rayleigh quotient, we have

$$\begin{aligned} \mu_1(G) - \mu_1(\tilde{G}) &\geqslant \mathbf{z}^T(D(G) - D(\tilde{G}))\mathbf{z} \\ &= n_1 \sum_{k=2}^{q} (n_k - 1)c^2 + (n_2 - 1)[(\nu - s - n_2 - (q-2))c^2 - 2bc] \\ &\quad + (n_3 - 1)[(\nu - s - n_3 - (q-2))c^2 - 2bc] \\ &\quad + (n_4 - 1)[(\nu - s - n_4 - (q-2))c^2 - 2bc] + \cdots \end{aligned}$$

$$+ (n_q - 1)[(\nu - s - n_q - (q-2))c^2 - 2bc]. \tag{5}$$

We proceed by considering the following two possible cases.

(1) $n_3 = 1$. In this case, we may consider either (i) $n_1 = 3$ or (ii) $n_1 \geqslant 5$.

(2) $n_3 \geqslant 3$.

For item (1)(i), one has $n_2 = 3$. Then

$$\begin{aligned}\mu_1(G) - \mu_1(\tilde{G}) &\geqslant \mathbf{z}^T(D(G) - D(\tilde{G}))\mathbf{z} \\ &= n_1(n_2-1)c^2 + (n_2-1)(n_1c^2 - 2bc) \\ &= 4c(3c - b) \\ &= 4c^2(2 - \frac{\nu - s - q}{\mu_1(\tilde{G}) + 2}) \\ &= 4c^2(2 - \frac{4}{\mu_1(\tilde{G}) + 2}) > 0.\end{aligned}$$

Thus, $\mu_1(G) > \mu_1(\tilde{G})$, a contradiction to the choice of G.

For item (1)(ii) and item (2), we are to show each term in (5) is positive. In view of (5), it suffices to show that $(\nu - s - n_2 - (q-2))c^2 - 2bc > 0$.

Note that $K_{\nu-q+1}$ is a subgraph of $\tilde{G}$. By Lemma 2.7, one has

$$\mu_1(\tilde{G}) \geqslant \mu_1(K_{\nu-q+1}) = \nu - q.$$

Thus,

$$\begin{aligned}(\nu - s - n_2 - (q-2))c^2 - 2bc &= c^2(\nu - s - n_2 - q - \frac{2\nu - 2s - 2q}{\mu_1(\tilde{G}) + 2}) \\ &\geqslant c^2(\nu - s - n_2 - q - \frac{2\nu - 2s - 2q}{\nu - q + 2}) \\ &= c^2(\nu - s - n_2 - q - 2 + \frac{2s + 4}{\nu - q + 2}) \\ &> c^2(\nu - s - n_2 - q - 2) \\ &= c^2[\sum_{i=1, i\neq 2}^{q} (n_i - 1) - 3]. \qquad (6)\end{aligned}$$

If $n_1 \geqslant 5$, $n_2 \geqslant 3$ and $n_3 = n_4 = \cdots = n_q = 1$, then $\sum_{i=1,i\neq 2}^{q}(n_i - 1) - 3 = n_1 - 4 > 0$. If $n_3 \geqslant 3$, then $n_1 \geqslant n_2 \geqslant n_3 \geqslant 3$. Thus, $\sum_{i=1,i\neq 2}^{q}(n_i - 1) - 3 \geqslant 2(3-1) - 3 > 0$. Together with (5) and (6), we obtain $\mu_1(G) > \mu_1(\tilde{G})$, a contradiction to the choice of G. Therefore, we obtain $n_2 = 1$, and so $n_3 = \cdots = n_q = 1$. $\square$

By Fact 1 and the choice of G, one has $n_1 = \nu - s - q + 1$. So we obtain that

$$G = K_s \vee ((q-1)K_1 \cup K_{\nu-s-q+1}). \tag{7}$$

Fact 2. In (7), one has $q = (2n-1)s + 2$.

Proof of Fact 2. If it is not true, then in view of (3) we may suppose $q \geqslant (2n-1)s+4$. Consider a graph $\hat{G} := K_s \vee ((2ns-s+1)K_1 \cup K_{\nu-2ns-1})$. Thus, $o(\hat{G}-S) = (2n-1)s+2$. Note that G is a proper subgraph of $\hat{G}$. By Lemma 2.4, one has $\mu_1(G) > \mu_1(\hat{G})$, a contradiction. Hence, one has $q < (2n-1)s+4$. Together with (3) and $o(G-S) \equiv (2n-1)|S| \pmod 2$, we get $q = (2n-1)s+2$. □

By Fact 2, one has $G = K_s \vee ((2ns-s+1)K_1 \cup K_{\nu-2ns-1})$. As $\nu-2ns-1 \geqslant 1$, one has $\nu \geqslant 2ns+2$. Furthermore, in order to complete the proof of (i), we show the following claim to deduce a contradiction.

Claim 1. $\mu_1(G) \geqslant \mu_1(P^{2n}_{\nu-2n})$ if $\nu = 4$ or $\nu \geqslant 10$.

Proof of Claim 1. Consider the equitable partition $V(P^{2n}_{\nu-2n}) = V(K_{\nu-2n-1}) \cup V(K_1) \cup V(\overline{K}_{2n})$. Then one may obtain the quotient matrix of $D(P^{2n}_{\nu-2n})$ corresponding to the partition as

$$\begin{pmatrix} \nu-2n-2 & 1 & 4n \\ \nu-2n-1 & 0 & 2n \\ 2(\nu-2n-1) & 1 & 2(2n-1) \end{pmatrix}.$$

So we obtain that its characteristic polynomial is $\Phi(x) = x^3 - (2n+\nu-4)x^2 + (8n^2 - 4n\nu + 4n - 3\nu + 5)x + 4n^2 - 2n\nu + 4n - 2\nu + 2$. By Lemma 2.3, the largest root, say η, of $\Phi(x) = 0$ satisfies $\eta = \mu_1(P^{2n}_{\nu-2n})$.

By Lemma 2.6, one has

$$\eta = \mu_1(P^{2n}_{\nu-2n}) \geqslant \frac{2W(P^{2n}_{\nu-2n})}{\nu} = \frac{\nu^2 + (4n-1)\nu - 2n(3+2n)}{\nu}. \tag{8}$$

Note that $n \leqslant \frac{\nu}{2} - 1$. Then

$$\eta \geqslant \nu + (4n-1) - \frac{(\nu-2)(3+2n)}{\nu} > \nu + 2n - 4. \tag{9}$$

Clearly, ν is even. Hence, we proceed by consider the following two possible cases.

Case 1. $\nu = 2ns+2$. In this case, we proceed by considering the following three possible subcases.

Subcase 1.1. $n = 1$. In this subcase, $\nu = 2s+2$. If $\nu = 4$, then $s = 1$. It is easy to see $G = P^2_2$, and so $\mu_1(G) = \mu_1(P^2_2)$, a contradiction. If $\nu \geqslant 10$, then $s \geqslant 4$. Observe that $G = K_s \vee (s+2)K_1$. Consider the partition $V(G) = V(K_s) \cup V((s+2)K_1)$. One has the quotient matrix of $D(G)$ corresponding to the partition as

$$M_1 = \begin{pmatrix} s-1 & s+2 \\ s & 2s+2 \end{pmatrix}.$$

Then the characteristic polynomial of M_1 is

$$\Phi_{M_1}(x) = x^2 - (3s+1)x + s^2 - 2s - 2.$$

Since the partition $V(G) = V(K_s) \cup V((s+2)K_1)$ is equitable, by Lemma 2.3, the largest root, say ξ_1, of $\Phi_{M_1}(x) = 0$ satisfies $\xi_1 = \mu_1(G)$. In what follows, we are to prove $\xi_1 > \eta$. For $s = 4$ (i.e., $\nu = 10$), we have $\xi_1 \approx 12.5208 > 12.4504 \approx \eta$. So we consider $s \geqslant 5$ in what follows. Note that

$$\Phi(x) = x^3 - 2sx^2 - (14s-3)x - 8s + 2.$$

Let $l_1(x) = x\Phi_{M_1}(x) - \Phi(x)$ be a real function in x. By a direct calculation, we have

$$l_1(x) = -(s+1)x^2 + (s^2 + 12s - 5)x + 8s - 2.$$

It is sufficient to prove $l_1(\eta) < 0$. In fact,

$$l_1'(x) = -2(s+1)x + s^2 + 12s - 5.$$

Hence $\frac{s^2+12s-5}{2(s+1)}$ is the unique solution of $l_1'(x) = 0$. As $\frac{s^2+12s-5}{2(s+1)} < 2s+4$, $l_1(x)$ is decreasing in the interval $[2s+4, +\infty)$. By (8), we see that

$$\eta \geqslant 2s + 5 - \frac{10}{2s+2} \geqslant 2s + 4.$$

Thus

$$l_1(\eta) \leqslant l_1(2s+4) = -2s^3 + 8s^2 + 14s - 38.$$

Let $l_2(x) = -2x^3 + 8x^2 + 14x - 38$ be a real function in x for $x \in [5, +\infty)$. It is routine to check that the derivative function of $l_2(x)$ is

$$l_2'(x) = -6x^2 + 16x + 14 = -6(x - \frac{4+\sqrt{37}}{3})(x - \frac{4-\sqrt{37}}{3}).$$

Note that

$$\frac{4-\sqrt{37}}{3} < \frac{4+\sqrt{37}}{3} < 5.$$

Hence $l_2(x)$ is a monotonically decreasing function for $x \geqslant 5$, and $l_2(x) \leqslant l_2(5) = -18 < 0$ when $x \geqslant 5$. Thus, $l_1(\eta) < 0$, and so $\xi_1 > \eta$ (i.e., $\mu_1(G) > \mu_1(P^{2n}_{\nu-2n})$).

Subcase 1.2. $n = 2$. In this subcase, by $\nu = 4s + 2 \geqslant 10$, we have $s \geqslant 2$. Observe that $G = K_s \vee (3s+2)K_1$. Consider the partition $V(G) = V(K_s) \cup V((3s+2)K_1)$. Then we obtain the quotient matrix of $D(G)$ corresponding to the partition as

$$M_2 = \begin{pmatrix} s-1 & 3s+2 \\ s & 6s+2 \end{pmatrix}.$$

Thus, $\Phi_{M_2}(x) = x^2 - (7s+1)x + 3s^2 - 6s - 2$. Since the partition $V(G) = V(K_s) \cup V((3s+2)K_1)$ is equitable, by Lemma 2.3, the largest root, say ξ_2, of $\Phi_{M_2}(x) = 0$ satisfies $\xi_2 = \mu_1(G)$. In what follows, we are to prove $\xi_2 > \eta$. Recall that

$$\Phi(x) = x^3 - (4s+2)x^2 - (44s-23)x - 24s + 14.$$

Let $l_3(x) = x\Phi_{M_2}(x) - \Phi(x)$ be a real function in x, i.e.,

$$l_3(x) = (1-3s)x^2 + (3s^2 + 38s - 25)x + 24s - 14.$$

It is sufficient to prove $l_3(\eta) < 0$. In fact, the derivative function of $l_3(x)$ is

$$l_3'(x) = 2(1-3s)x + 3s^2 + 38s - 25.$$

Hence $\frac{3s^2+38s-25}{6s-2}$ is the unique solution of $l_3'(x) = 0$. Together with

$$\frac{3s^2+38s-25}{6s-2} < 4s+6,$$

one may obtain $l_3(x)$ is a monotonically decreasing function for $x \geqslant 4s+6$. By (8), we have

$$\eta \geqslant 4s+9-\frac{28}{4s+2} > 4s+6.$$

Thus

$$l_3(\eta) < l_3(4s+6) = -36s^3 + 42s^2 + 92s - 128.$$

Let $l_4(x) = -36x^3 + 42x^2 + 92x - 128$ be a real function in x for $x \in [2, +\infty)$. It is routine to check that the derivative function of $l_4(x)$ is

$$l_4'(x) = -108x^2 + 84x + 92 = -108(x - \frac{7+5\sqrt{13}}{18})(x - \frac{7-5\sqrt{13}}{18}).$$

Clearly,

$$\frac{7-5\sqrt{13}}{18} < \frac{7+5\sqrt{13}}{18} < 2.$$

Hence, $l_4(x)$ is a monotonically decreasing function for $x \geqslant 2$. Hence, $l_3(\eta) < l_4(x) \leqslant l_4(2) = -64 < 0$ when $x \geqslant 2$. Thus, $\xi_2 > \eta$, i.e., $\mu_1(G) > \mu_1(P^{2n}_{\nu-2n})$.

Subcase 1.3. $n \geqslant 3$. In this subcase, $G = K_s \vee ((2ns-s+1)K_1 \cup K_1)$. Consider the partition $V(G) = V(K_1) \cup V(K_s) \cup V((2ns-s+1)K_1)$. Then the quotient matrix of $D(G)$ corresponding to the partition is given as

$$M_3 = \begin{pmatrix} 0 & s & 2(2ns-s+1) \\ 1 & s-1 & 2ns-s+1 \\ 2 & s & 2(2ns-s) \end{pmatrix}.$$

Thus, $\Phi_{M_3}(x) = x^3-(4ns-s-1)x^2-(-2ns^2+12ns+s^2-4s+4)x+4ns^2-8ns-2s^2+4s-4$. Since the partition $V(G) = V(K_1) \cup V(K_s) \cup V((2ns-s+1)K_1)$ is equitable, by Lemma 2.3, the largest root, say ξ_3, of $\Phi_{M_3}(x) = 0$ satisfies $\xi_3 = \mu_1(G)$. Next we are to prove $\xi_3 \geqslant \eta$. In fact,

$$\Phi(x) = x^3+(-2ns-2n+2)x^2+(-8n^2s+8n^2-6ns-4n-1)x-4n^2s+4n^2-4ns-2.$$

By a direct calculation, one has

$$\Phi_{M_3}(x)-\Phi(x) = (s-1)[(1-2n)x^2+(8n^2+2ns-4n-s+3)x+4n^2+4ns-2s+2].$$

Obviously, $\Phi_{M_3}(\eta) = \Phi(\eta) = 0$ when $s=1$, and so $\xi_3 \geqslant \eta$. In what follows, we show $\xi_3 > \eta$ for $s \geqslant 2$.

Let

$$l_5(x) = (1-2n)x^2 + (8n^2+2ns-4n-s+3)x + 4n^2 + 4ns - 2s + 2$$

be a real function in x. It is routine to check that the derivative function of $l_5(x)$ is

$$l_5'(x) = 2(1-2n)x + 8n^2 + 2ns - 4n - s + 3.$$

Hence $-\frac{8n^2+2ns-4n-s+3}{2(1-2n)}$ is the unique solution of $l_5'(x) = 0$. Together with

$$-\frac{8n^2+2ns-4n-s+3}{2(1-2n)} < 2ns + 2n - 2,$$

we obtain that $l_5(x)$ is a decreasing function in the interval $[2ns+2n-2, +\infty)$. By (9), we have

$$\eta > \nu + 2n - 4 = 2ns + 2n - 2.$$

Then

$$l_5(\eta) < l_5(2ns+2n-2) = (-8n^3+8n^2-2n)s^2 + (20n^2-4n)s + 8n^3 - 2n.$$

Let $l_6(x) = (-8n^3+8n^2-2n)x^2 + (20n^2-4n)x + 8n^3 - 2n$ be a real function in x for $x \in [2, +\infty)$. It is routine to check that the derivative function of $l_6(x)$ is

$$l_6'(x) = 2(-8n^3+8n^2-2n)x + 20n^2 - 4n.$$

Hence $-\frac{20n^2-4n}{2(-8n^3+8n^2-2n)}$ is the unique solution of $l_6'(x) = 0$. It is straightforward to check that

$$-\frac{20n^2-4n}{2(-8n^3+8n^2-2n)} < 2.$$

Hence, $l_6(x)$ is a decreasing function in the interval $[2, +\infty)$. Thus,

$$l_6(x) \leqslant l_6(2) = -24n^3 + 72n^2 - 18n = -24n(n - \frac{3-\sqrt{6}}{2})(n - \frac{3+\sqrt{6}}{2}).$$

Note that $n \geqslant 3$, and

$$\frac{3-\sqrt{6}}{2} < \frac{3+\sqrt{6}}{2} < 3.$$

Therefore, $l_5(\eta) < l_6(x) < 0$, and so $\xi_3 > \eta$, i.e., $\mu_1(G) > \mu_1(P^{2n}_{\nu-2n})$.

Case 2. $\nu \geqslant 2ns + 4$.

If $s = 1$, then it is easy to see that $G \cong P^{2n}_{\nu-2n}$ and $\mu_1(G) = \mu_1(P^{2n}_{\nu-2n})$. So in what follows, we consider $s \geqslant 2$.

According to the partition $V(G) = V(K_{\nu-2ns-1}) \cup V(K_s) \cup V((2ns-s+1)K_1)$, we may obtain the quotient matrix of $D(G)$ corresponding to the partition as

$$M_4 = \begin{pmatrix} \nu - 2ns - 2 & s & 2(2ns-s+1) \\ \nu - 2ns - 1 & s - 1 & 2ns - s + 1 \\ 2(\nu - 2ns - 1) & s & 2(2ns - s) \end{pmatrix}.$$

Then the characteristic polynomial of M_4 is

$$\begin{aligned}\Phi_{M_4}(x) = {} & x^3 - (2ns + \nu - s - 3)x^2 - (-8n^2s^2 + 2ns^2 + 4ns\nu - 6ns + s^2 \\ & - 2s\nu + 5\nu - 6)x - 4n^2s^3 + 8n^2s^2 + 2ns^3 + 2ns^2\nu - 6ns^2 \\ & - 4ns\nu - s^2\nu + 8ns + 3s\nu - 2s - 4\nu + 4. \qquad (10)\end{aligned}$$

Since the partition $V(G) = V(K_{\nu-2ns-1}) \cup V(K_s) \cup V((2ns-s+1)K_1)$ is equitable, by Lemma 2.3, the largest root, say ξ_4, of $\Phi_{M_4}(x) = 0$ satisfies $\xi_4 = \mu_1(G)$.

Recall that

$$\Phi(x) = x^3-(2n+\nu-4)x^2+(8n^2-4n\nu+4n-3\nu+5)x+4n^2-2n\nu+4n-2\nu+2.$$

Together with (10), we have

$$\begin{aligned}\Phi_{M_4}(x) - \Phi(x) = (s-1)[&(1-2n)x^2 + (8n^2s - 4n\nu + 8n^2 - 2ns + 2\nu + 4n \\ &- s - 1)x - 4n^2s^2 + 2ns\nu + 4n^2s + 2ns^2 - 2n\nu \\ &- s\nu + 4n^2 - 4ns + 2\nu + 4n - 2].\end{aligned}$$

Let

$$\begin{aligned}h_1(x) = (1-2n)x^2 &+ (8n^2s - 4n\nu + 8n^2 - 2ns + 2\nu + 4n - s - 1)x \\ &- 4n^2s^2 + 2ns\nu + 4n^2s + 2ns^2 - 2n\nu - s\nu + 4n^2 \\ &- 4ns + 2\nu + 4n - 2 \end{aligned} \tag{11}$$

be a real function in x. We are to show $h_1(\eta) < 0$ by considering the following two possible subcases.

Subcase 2.1. $n = 1$. In this case

$$h_1(x) = -x^2 - (2\nu - 5s - 11)x - 2s^2 + s\nu + 6.$$

It is routine to check that the derivative function of $h_1(x)$ is

$$h_1'(x) = -2x - (2\nu - 5s - 11).$$

Hence $-\nu + \frac{5}{2}s + \frac{11}{2}$ is the unique solution of $h_1'(x) = 0$. Obviously, $-\nu + \frac{5}{2}s + \frac{11}{2} < \nu + 2$. Then $h_1(x)$ is decreasing in the interval $[\nu+2, +\infty)$. By (8), one has

$$\eta \geqslant \nu + 3 - \frac{10}{\nu} \geqslant \nu + 2.$$

So $h_1(\eta) \leqslant h_1(\nu+2) = -3\nu^2 + (6s+3)\nu - 2s^2 + 10s + 24$.

Let $h_2(x) = -3x^2 + (6s+3)x - 2s^2 + 10s + 24$ be a real function in x. It is routine to check that the derivative function of $h_2(x)$ is

$$h_2'(x) = -6x + 6s + 3.$$

Hence $s + \frac{3}{2}$ is the unique solution of $h_2'(x) = 0$. Obviously, $s + \frac{3}{2} < 2s + 4$. Therefore $h_2(x)$ is decreasing in the interval $[2s+4, +\infty)$. By $\nu \geqslant 2s+4$, one has

$$h_2(\nu) \leqslant h_2(2s+4) = -2s^2 - 8s - 12 < 0.$$

Hence, $h_1(\eta) < 0$. Therefore, $\mu_1(G) > \mu_1(P^{2n}_{\nu-2n})$.

Subcase 2.2. $n \geqslant 2$. In view of (11), we have

$$h_1'(x) = 2(1-2n)x + 8n^2s - 4n\nu + 8n^2 - 2ns + 2\nu + 4n - s - 1.$$

Hence, $-\nu + \frac{4n+1}{2}s + \frac{8n^2+4n-1}{2(2n-1)}$ is the unique solution of $h_1'(x) = 0$. Obviously,

$$-\nu + \frac{4n+1}{2}s + \frac{8n^2+4n-1}{2(2n-1)} < \nu + 2n - 4.$$

Thus, $h_1(x)$ is decreasing in the interval $[\nu + 2n - 4, +\infty)$. By (9), we have $\eta \geqslant \nu + 2n - 4$. Thus, $h_1(\eta) \leqslant h_1(\nu + 2n - 4)$, where

$$\begin{aligned} h_1(\nu + 2n - 4) = {} & (-6n + 3)\nu^2 + (8n^2 s - 8n^2 + 42n - 2s - 15)\nu + 16n^3 s \\ & - 4n^2 s^2 + 8n^3 - 32n^2 s + 2ns^2 + 16n^2 \\ & + 2ns - 62n + 4s + 18. \end{aligned}$$

Let $h_3(x) = (-6n+3)x^2 + (8n^2 s - 8n^2 + 42n - 2s - 15)x + 16n^3 s - 4n^2 s^2 + 8n^3 - 32n^2 s + 2ns^2 + 16n^2 + 2ns - 62n + 4s + 18$ be a real function in x. It is routine to check that the derivative function of $h_3(x)$ is

$$h_3'(x) = 2(-6n + 3)x + 8n^2 s - 8n^2 + 42n - 2s - 15.$$

Hence $-\frac{8n^2 s - 8n^2 + 42n - 2s - 15}{2(-6n+3)}$ is the unique solution of $h_3'(x) = 0$. Clearly,

$$-\frac{8n^2 s - 8n^2 + 42n - 2s - 15}{2(-6n + 3)} < 2ns + 4.$$

Then $h_3(x)$ is decreasing in the interval $[2ns + 4, +\infty)$. Thus,

$$\begin{aligned} h_3(x) \leqslant h_3(2ns + 4) = {} & (-8n^3 + 8n^2 - 2n)s^2 + (-12n^2 + 20n - 4)s \\ & + 8n^3 - 16n^2 + 10n + 6. \end{aligned}$$

Let $h_4(x) = (-8n^3 + 8n^2 - 2n)x^2 + (-12n^2 + 20n - 4)x + 8n^3 - 16n^2 + 10n + 6$ be a real function in x. Hence, the derivative function of $h_4(x)$ is

$$h_4'(x) = 2(-8n^3 + 8n^2 - 2n)x - 12n^2 + 20n - 4.$$

Hence $-\frac{-12n^2+20n-4}{2(-8n^3+8n^2-2n)}$ is the unique solution of $h_4'(x) = 0$. Obviously,

$$-\frac{-12n^2 + 20n - 4}{2(-8n^3 + 8n^2 - 2n)} < 2.$$

Therefore, $h_4(x)$ is decreasing in the interval $[2, +\infty)$. Hence,

$$h_4(x) \leqslant h_4(2) = -24n^3 - 8n^2 + 42n - 2.$$

Let $h_5(x) = -24x^3 - 8x^2 + 42x - 2$ be a real function in x. Thus,

$$h_5'(x) = -72x^2 - 16x + 42 = -72\left(x + \frac{2 + \sqrt{193}}{18}\right)\left(x - \frac{-2 + \sqrt{193}}{18}\right).$$

Note that

$$-\frac{2 + \sqrt{193}}{18} < \frac{-2 + \sqrt{193}}{18} < 2.$$

Thus, $h_5(x)$ is decreasing in the interval $[2, +\infty)$. Therefore, $h_5(x) \leqslant h_5(2) = -142 < 0$. Thus, $h_1(\eta) < 0$ and so $\mu_1(G) > \mu_1(P^{2n}_{\nu-2n})$. $\square$

By Claim 1, we get $\mu_1(G) > \mu_1(P^{2n}_{\nu-2n})$ for $\nu = 4$ or $\nu \geqslant 10$. Thus, we get a contradiction to the condition of (i). So we need to consider the rest case for $\nu = 6, 8$. Recall that $G = K_s \vee ((2ns - s + 1)K_1 \cup K_{\nu-2ns-1})$.

For $\nu = 6$, by $n \leqslant \frac{\nu}{2} - 1$, we have $n \leqslant 2$. If $n = 1$, then $s \leqslant 2$ by $\nu \geqslant 2ns + 2$. Hence $G \in \{P^2_4, K_2 \vee 4K_1\}$. By a direct calculation, $\mu_1(P^2_4) \approx 7.5546$, $\mu_1(K_2 \vee 4K_1) \approx 7.2744$. Then $\mu_1(G) \geqslant \mu_1(K_2 \vee 4K_1)$. If $n = 2$, then $s = 1$ by $\nu \geqslant 2ns + 2$. Hence $G = P^4_2$, and so $\mu_1(G) = \mu_1(P^4_2)$. So we obtain each of the subcases deduces a contradiction to the condition of (ii).

For $\nu = 8$, by $n \leqslant \frac{\nu}{2} - 1$, we have $n \leqslant 3$. If $n = 1$, then $s \leqslant 3$ by $\nu \geqslant 2ns + 2$. Hence $G \in \{P^2_6, K_2 \vee (3K_1 \cup K_3), K_3 \vee 5K_1\}$. By a direct calculation, $\mu_1(P^2_6) \approx 10.0839$, $\mu_1(K_3 \vee 5K_1) \approx 9.8990$, and $\mu_1(K_2 \vee (3K_1 \cup K_3)) \approx 10.3573$, Thus, $\mu_1(G) \geqslant \mu_1(K_3 \vee 5K_1)$. If $n = 2$, then $s = 1$ by $\nu \geqslant 2ns + 2$. Hence, $G = P^4_4$, and so $\mu_1(G) = \mu_1(P^4_4)$. If $n = 3$, then $s = 1$ by $\nu \geqslant 2ns + 2$. Hence $G = P^6_2$, and so $\mu_1(G) = \mu_1(P^6_2)$. Therefore, we deduce a contradiction to the condition of (iii) for $n = 1, 2, 3$. $\square$

Bearing in mind that a perfect matching of graph G is a $\{1\}$-factor. So we have the following corollary immediately.

Corollary 5.1. *Let G be a ν-vertex connected graph, where ν is an even integer. If $\nu = 4$ or $\nu \geqslant 10$ and $\mu_1(G) < \mu_1(P^2_{\nu-2})$, then G has a perfect matching. If $\nu = 6$ and $\mu_1(G) < \mu_1(K_2 \vee 4K_1)$, or $\nu = 8$ and $\mu_1(G) < \mu_1(K_3 \vee 5K_1)$, then G has a perfect matching.*

By the proof of Theorems 3.2 and 4.2, we know $P^{2n}_{\nu-2n}$ has no $\{1, 3, \ldots, 2n-1\}$-factor, $K_2 \vee 4K_1$ and $K_3 \vee 5K_1$ have no $\{1\}$-factor. Thus the condition in Theorem 1.3 can not be replaced by the condition that $\mu_1(G) \leqslant \mu_1(P^{2n}_{\nu-2n})$, $\mu_1(G) \leqslant \mu_1(K_2 \vee 4K_1)$ and $\mu_1(G) \leqslant \mu_1(K_3 \vee 5K_1)$, which implies the bounds established in Theorem 1.3 are the best possible.

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