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UNIQUENESS OF MEROMORPHIC SOLUTIONS OF A CERTAIN TYPE OF DIFFERENCE EQUATIONS

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Abstract. In this paper, we study the uniqueness of two finite order transcendental meromorphic solutions $f(z)$ and $g(z)$ of the following complex difference equation

$$A_1(z)f(z+1) + A_0(z)f(z) = F(z)e^{\alpha(z)}$$

when they share $0, \infty$ CM, where $A_1(z)$, $A_0(z)$, $F(z)$ are non-zero polynomials, $\alpha(z)$ is a polynomial. Our result generalizes and complements some known results given recently by Cui and Chen, Li and Chen. Examples for the precision of our result are also supplied.

1. Introduction

Let $\mathbb{C}$ denote the complex plane. In this paper, we assume that the reader is familiar with Nevanlinna theory and standard notations (see [9,11]), such as

$$T(r,f),\ N(r,f),\ m(r,f),\ \overline{N}(r,f),\ \overline{N}(r,\frac{1}{f}),\ldots.$$

We denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o(T(r,f)), \quad r \to \infty,$$

outside possibly an exceptional set of finite logarithmic measure.

Let $f(z)$ and $g(z)$ be meromorphic functions. We say that $f(z)$ and $g(z)$ share a CM, provided that $f(z)-a$ and $g(z)-a$ have the same zeros counting multiplicities. And we say that $f(z)$ and $g(z)$ share ∞ CM, provided that they have the same poles counting multiplicities.

The uniqueness theory of meromorphic functions is an important part of Nevanlinna theory. The famous Nevanlinna 4 CM theorem will be given in detail below.

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Theorem 1 (see [15]). *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let a_1, a_2, a_3, a_4 be four distinct values in the extended complex plane. Furthermore, suppose that $f(z)$ and $g(z)$ share a_1, a_2, a_3, a_4 CM. If $f(z) \not\equiv g(z)$, then $f(z) = T(g(z))$, where T is a Möbius transformation.*

Later, Yang and Yi [19] supplemented and perfected the uniqueness theory of meromorphic functions. One major problem in the study of complex difference equations has so far been the lack of efficient tools. Fortunately, till the 1970s and 1980s, Bank and Kaufman [1], Shimomura [17] and Yanagihara [18] initially obtained the existence of meromorphic solutions of certain difference equations in the complex plane. Recently, many scholars combined the uniqueness with the differences and the solutions of difference equations, and obtained lots of results [2–7, 10, 12–14, 16]. Specially, in 2013, Chen and Shon [3] studied the following non-homogeneous linear difference equation

$$A_n(z)f(z+n) + \cdots + A_1(z)f(z+1) + A_0(z)f(z) = F(z), \tag{1.1}$$

and proved the following theorem.

Theorem 2 (see [3]). *Let $A_0(z), \ldots, A_n(z), F(z)$ be polynomials such that*

$$F(z)A_0(z)A_n(z) \not\equiv 0, \quad A_0 + \cdots + A_n \not\equiv 0.$$

Then every finite order transcendental meromorphic solution $f(z)$ of the equation (1.1) satisfies $\lambda(f(z)) = \sigma(f(z)) \geq 1$. Here, the order $\sigma(f(z))$ of a meromorphic solution $f(z)$ is defined to be

$$\sigma(f(z)) = \limsup_{r\to\infty} \frac{\log^+ T(r, f(z))}{\log r},$$

and its exponent of convergence of zeros is defined by

$$\lambda(f(z)) = \limsup_{r\to\infty} \frac{\log^+ N\left(r, \frac{1}{f(z)}\right)}{\log r}.$$

In 2016, Cui and Chen [6] considered the uniqueness of meromorphic solutions sharing three values with a meromorphic function to the homogeneous linear difference equation

$$A_1(z)f(z+1) + A_0(z)f(z) = 0 \tag{1.2}$$

and proved the following theorem.

Theorem 3 (see [6]). *Let $f(z)$ be a transcendental meromorphic solution of (1.2) and assume that the order of $f(z)$ is finite. Suppose that $A_1(z)$ and $A_0(z)$ are non-zero polynomials such that $A_1(z) + A_0(z) \not\equiv 0$. If a meromorphic function $g(z)$ and $f(z)$ share $0, 1, \infty$ CM, then either $f(z) \equiv g(z)$ or $f(z)g(z) \equiv 1$.*

In 2017, Cui and Chen [7] extended the complex difference equation in Theorem 3 as

$$A_1(z)f(z+1)+A_0(z)f(z)=F(z) \tag{1.3}$$

and proved the following theorem.

Theorem 4 (see [7]). *Let $f(z)$ be a transcendental meromorphic solution of (1.3) and assume that the order of $f(z)$ is finite. Suppose that $A_1(z)$, $A_0(z)$, $F(z)$ are non-zero polynomials such that $A_1(z)+A_0(z)\not\equiv 0$. If a meromorphic function $g(z)$ and $f(z)$ share $0, 1, \infty$ CM, then one of the following cases holds*:

(i) $f(z)\equiv g(z)$; (ii) $f(z)+g(z)=f(z)g(z)$;

(iii) *there exist a polynomial $\beta(z)=b_1z+b_0$ and a constant a_0 satisfying $e^{a_0}\neq e^{b_0}$ such that*

$$f(z)=\frac{1-e^{\beta(z)}}{e^{\beta(z)}(e^{a_0-b_0}-1)},\quad g(z)=\frac{1-e^{\beta(z)}}{1-e^{b_0-a_0}},$$

where $b_1\neq 0$ and b_0 are constants.

In 2019, Li and Chen [12] considered the uniqueness of two meromorphic solutions of non-homogeneous linear difference equation

$$B_1(z)f(z+1)+B_2(z)f(z)=B_3(z) \tag{1.4}$$

when they share two values, and proved the following theorem.

Theorem 5 (see [12]). *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of* (1.4). *Furthermore, suppose that $B_1(z)\not\equiv 0$, $B_3(z)\not\equiv 0$, $B_2(z)$ are rational functions. If $f(z)$ and $g(z)$ share $0, \infty$ CM, then either $f(z)\equiv g(z)$ or*

$$f(z)=\frac{B_3(z)}{2B_2(z)}(e^{b_1z+b_0}+1)$$

and

$$g(z)=\frac{B_3(z)}{2B_2(z)}(e^{-b_1z-b_0}+1),$$

where b_1 and b_0 are constants such that $e^{-b_1}=e^{b_1}=-1$, and the coefficients of (1.4) *satisfy*

$$B_1(z)B_3(z+1)\equiv B_3(z)B_2(z+1).$$

Here in this paper, if we generalize the equation (1.3) to the following more general form

$$A_1(z)f(z+1)+A_0(z)f(z)=F(z)e^{\alpha(z)}, \tag{1.5}$$

then how can we guarantee the meromorphic solutions of the equation (1.5) are uniquely determined by their zeros and poles? Next we study this problem and prove the following main result.

Theorem 6. *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of the equation* (1.5). *Furthermore, suppose that $A_1(z)$, $A_0(z)$, $F(z)$ are non-zero polynomials, $\alpha(z)$ is a polynomial. If $f(z)$ and $g(z)$ share 0, ∞ CM, then one of the following cases holds*:

(i) $f(z) \equiv g(z)$;

(ii)

$$f(z) = \frac{c_1 F(z)}{2A_0(z)}(e^{b_1 z + b_0} + 1)$$

and

$$g(z) = \frac{c_1 F(z)}{2A_0(z)}(e^{-b_1 z - b_0} + 1),$$

where $e^{\alpha(z)} = c_1$ $(\neq 0)$, $e^{-b_1} = e^{b_1} = -1$, $b_1 = (2k+1)\pi i$, k is an integer and the coefficients of (1.5) *satisfy*

$$A_1(z)F(z+1) \equiv F(z)A_0(z+1);$$

(iii)

$$f(z) = \frac{F(z)e^{a_1 z + a_0}(e^{b_1 z + b_0} - e^{-b_1})}{A_0(z)(1 - e^{-b_1})}$$

and

$$g(z) = \frac{F(z)e^{a_1 z + a_0}(1 - e^{-b_1 z - b_1 - b_0})}{A_0(z)(1 - e^{-b_1})},$$

where $(1 - e^{-b_1})^{-1} \neq 0$, $a_1 \neq 0$, $b_1 \neq 0$, a_0, b_0 are constants and the coefficients of (1.5) *satisfy*

$$A_1(z)F(z+1) \equiv \frac{e^{-b_1}(e^{-b_1 z - b_0} - 1)}{e^{a_1}(1 - e^{-b_1 z - 2b_1 - b_0})} F(z)A_0(z+1).$$

By Theorem 6, we can also give the following corollaries.

Corollary 1. *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of the equation* (1.5), *where $A_1(z)$, $A_0(z)$, $F(z)$ are non-zero polynomials such that $A_1(z)F(z+1) \not\equiv qF(z)A_0(z+1)$, where $q = 1$ or $q = \frac{e^{-b_1}(e^{-b_1 z - b_0} - 1)}{e^{a_1}(1 - e^{-b_1 z - 2b_1 - b_0})}$, a_1 and b_1 are non-zero constants, b_0 is a constant. If $f(z)$ and $g(z)$ share 0, ∞ CM, then $f(z) \equiv g(z)$.*

Corollary 2. *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of the equation* (1.5), *where*

$$F(z)e^{\alpha(z)} \not\equiv A_1(z) + A_0(z),$$
$$A_1(z)[c_1 F(z+1) - A_1(z+1)] \not\equiv [c_1 F(z) - A_0(z)]A_0(z+1)$$

and

$$\frac{A_1(z)[F(z+1)e^{a_1 z + a_1 + a_0} - A_1(z+1) - A_0(z+1)]}{A_0(z+1)[F(z)e^{a_1 z + a_0} - A_1(z) - A_0(z)]} \not\equiv \frac{e^{-b_1 z - b_1 - b_0} - e^{-b_1}}{1 - e^{-b_1 z - 2b_1 - b_0}},$$

where a_1, b_1, c_1 are non-zero constants, a_0 and b_0 are constants. If $f(z)$ and $g(z)$ share 1, ∞ CM, then $f(z) \equiv g(z)$.

Remark 1. Corollary 1 and Corollary 2 follow from Theorem 6 immediately.

Example 1. The function $f(z)=\frac{ze^{\pi iz}+z}{2}$ and another function $g(z)=\frac{ze^{-\pi iz}+z}{2}$ solve the equation

$$zf(z+1)+(z+1)f(z)=z^2+z.$$

Here $f(z)$ and $g(z)$ share $0, \infty$ CM, $e^{-\pi i}=e^{\pi i}=-1$. Without loss of generality, we may suppose that $e^{\alpha(z)}\equiv 1$, $F(z)=z^2+z$. Then the coefficients of (1.5) satisfy $A_1(z)=z$, $A_0(z)=z+1$ and $A_1(z)F(z+1)\equiv F(z)A_0(z+1)=z^3+3z^2+2z$. The above statement shows that Case (ii) in Theorem 6 certainly exists.

Example 2. The function $f(z)=\frac{e^{3\pi iz}-e^{\pi i(2z-1)}}{2}$ and another function $g(z)=\frac{e^{2\pi iz}-e^{\pi i(z-1)}}{2}$ solve the equation

$$f(z+1)+f(z)=e^{2\pi iz}.$$

Here $f(z)$ and $g(z)$ share 0, ∞ CM, $(1-e^{-b_1})^{-1}=\frac{1}{2}\neq 0$, and the coefficients of (1.5) satisfy $A_1(z)\equiv A_0(z+1)\equiv F(z)\equiv 1$, $A_1(z)F(z+1)\equiv 1\equiv\frac{e^{-b_1}(e^{-b_1z-b_0}-1)}{e^{a_1}(1-e^{-b_1z-2b_1-b_0})}F(z)A_0(z+1)$, where $a_1=2\pi i$, $b_1=\pi i$, $a_0=b_0=0$. The above statement shows that Case (iii) in Theorem 6 certainly exists.

Example 3. The function $f(z)=-e^{3\pi iz}$ and another function $g(z)=-e^{-3\pi iz}$ solve the equation

$$f(z+1)+f(z)=0.$$

Here $f(z)$ and $g(z)$ share 1, ∞ CM, $e^{-3\pi i}=e^{3\pi i}=-1$, and the coefficients of (1.5) satisfy $A_0(z)\equiv A_1(z)\equiv 1$; $A_0(z)+A_1(z)\equiv 2\not\equiv F(z)e^{\alpha(z)}\equiv 0$. But

$$A_1(z)[c_1F(z+1)-A_1(z+1)]\equiv -1\equiv [c_1F(z)-A_0(z)]A_0(z+1)$$

and

$$\frac{A_1(z)[F(z+1)e^{a_1z+a_1+a_0}-A_1(z+1)-A_0(z+1)]}{A_0(z+1)[F(z)e^{a_1z+a_0}-A_1(z)-A_0(z)]}\equiv 1\equiv\frac{e^{-b_1z-b_1-b_0}-e^{-b_1}}{1-e^{-b_1z-2b_1-b_0}},$$

where $b_1=3\pi i$, $b_0=0$, $c_1\neq 0$, $a_1\neq 0$, a_0 are constants.

Remark 2. From Example 3, we find that the conditions

$$A_1(z)[c_1F(z+1)-A_1(z+1)]\equiv [c_1F(z)-A_0(z)]A_0(z+1)$$

and

$$\frac{A_1(z)[F(z+1)e^{a_1z+a_1+a_0}-A_1(z+1)-A_0(z+1)]}{A_0(z+1)[F(z)e^{a_1z+a_0}-A_1(z)-A_0(z)]}\equiv\frac{e^{-b_1z-b_1-b_0}-e^{-b_1}}{1-e^{-b_1z-2b_1-b_0}}$$

in Corollary 2 cannot be deleted.

Example 4. The functions $f(z)=\frac{e^z}{z}$ and $g(z)=\frac{e^z}{z}$ solve the equation

$$f(z+1)+ef(z)=\frac{2z+1}{z(z+1)}e^{z+1}.$$

Here $f(z)$ and $g(z)$ share 0, ∞ CM. The above statement shows that Case (i) in Theorem 6 certainly exists.

Example 5. The function $f(z) = \frac{z(e^{3\pi iz}+e^{2\pi iz})}{2}$ and another function $g(z) = \frac{z(e^{2\pi iz}+e^{\pi iz})}{2}$ solve the equation

$$\frac{z}{z+1}f(z+1) + f(z) = ze^{2\pi iz}.$$

Here $f(z)$ and $g(z)$ share 0, ∞ CM, $(1-e^{-b_1})^{-1} = \frac{1}{2} \neq 0$, and the coefficients of (1.5) satisfy $A_1(z) \equiv \frac{z}{z+1}$, $A_0(z) \equiv 1$, $F(z) \equiv z$, $A_1(z)F(z+1) \equiv z \equiv \frac{e^{-b_1}(e^{-b_1 z-b_0}-1)}{e^{a_1}(1-e^{-b_1 z-2b_1-b_0})}F(z)A_0(z+1)$, where $a_1 = 2\pi i$, $b_1 = \pi i$, $a_0 = b_0 = 0$. The above statement shows that when $F(z)$ and $\alpha(z)$ are non-constant polynomials, Case (iii) in Theorem 6 certainly exists.

2. Some lemmas

Lemma 1 (see [5,8])**.** *Let ε be a positive constant, ξ_1 and ξ_2 be two distinct complex constants, and let $f(z)$ be a meromorphic function of finite order $\sigma = \sigma(f(z))$. Then we have*

$$m\left(r, \frac{f(z+\xi_1)}{f(z+\xi_2)}\right) = O(r^{\sigma-1+\varepsilon}) = o(T(r,f)). \tag{2.1}$$

Lemma 2 (see [5])**.** *Let $f(z)$ be a meromorphic function such that the order $\sigma = \sigma(f(z)) < +\infty$, and let η be a nonzero complex number. If $\varepsilon > 0$, then*

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r). \tag{2.2}$$

Lemma 3 (see [19])**.** *If meromorphic functions $f_j(z)$ $(j = 1, 2, \ldots, n)$ $(n \geq 2)$ and entire functions $g_j(z)$ $(j = 1, 2, \ldots, n)$ $(n \geq 2)$ satisfy the following conditions*:

(1) $\sum_{j=1}^{n} f_j e^{g_j} \equiv 0$,
(2) *when* $1 \leq j < l \leq n$, $g_j - g_l$ *are not constants,*
(3) *when* $1 \leq j \leq n$, $1 \leq h < l \leq n$, $T(r, f_j) = o(T(r, e^{g_h - g_l}))$ $(r \to \infty$, $r \notin E)$,

then we have $f_j \equiv 0$ $(j = 1, 2, \ldots, n)$.

3. Proof of Theorem 6

Since $f(z)$ and $g(z)$ are finite order transcendental meromorphic functions that share 0, ∞ CM, we get

$$\frac{f(z)}{g(z)} = e^{\beta(z)}, \tag{3.1}$$

where $\beta(z)$ is a polynomial such that $\deg \beta(z) \leq \max\{\sigma(f), \sigma(g)\}$.

From the equations (1.5) and (3.1), we have

$$A_1(z)g(z+1) + A_0(z)g(z) = F(z)e^{\alpha(z)} \tag{3.2}$$

and

$$A_1(z)g(z+1)e^{\beta(z+1)} + A_0(z)g(z)e^{\beta(z)} = F(z)e^{\alpha(z)}. \tag{3.3}$$

Combining (3.2) with (3.3), we have

$$g(z)A_0(z)[1 - e^{\beta(z)-\beta(z+1)}] = F(z)e^{\alpha(z)}[1 - e^{-\beta(z+1)}]. \tag{3.4}$$

If $1 - e^{\beta(z)-\beta(z+1)} \equiv 0$, then from (3.4), we have $1 - e^{-\beta(z+1)} \equiv 0$, which means that $f(z) \equiv g(z)$.

If $1 - e^{\beta(z)-\beta(z+1)} \not\equiv 0$, then $\beta(z)$ must be a non-constant polynomial. Thus from (3.4) we may solve out $g(z)$ of the form

$$g(z) = \frac{F(z)e^{\alpha(z)}(1 - e^{-\beta(z+1)})}{A_0(z)[1 - e^{\beta(z)-\beta(z+1)}]}. \tag{3.5}$$

By (3.2) and (3.5), we see that

$$\frac{A_1(z)F(z+1)e^{\alpha(z+1)}[1-e^{-\beta(z+2)}]}{A_0(z+1)[1-e^{\beta(z+1)-\beta(z+2)}]} + \frac{A_0(z)F(z)e^{\alpha(z)}[1-e^{-\beta(z+1)}]}{A_0(z)[1-e^{\beta(z)-\beta(z+1)}]} = F(z)e^{\alpha(z)}.$$

Equally,

$$A_1(z)q(z+1)[1 - e^{-\beta(z+2)}] + A_0(z)q(z)[1 - e^{-\beta(z+1)}] = F(z)e^{\alpha(z)}, \tag{3.6}$$

where

$$q(z) = \frac{F(z)e^{\alpha(z)}}{A_0(z)[1 - e^{\beta(z)-\beta(z+1)}]}.$$

Now we set

$$\begin{aligned} \alpha(z) &= a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0, \\ \beta(z) &= b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0, \end{aligned} \tag{3.7}$$

where $a_m \neq 0$, a_{m-1}, $\ldots$, a_0, $b_n \neq 0$, b_{n-1}, $\ldots$, b_0 are constants, m and n are integers.

Note that $\beta(z)$ is a non-constant polynomial and $\alpha(z)$ is a polynomial. Next, we divide our argument into two cases respectively: Case 1, $\alpha(z)$ is a constant, $\beta(z)$ is not a constant; Case 2, $\alpha(z)$ and $\beta(z)$ are not constants.

Case 1. If $\alpha(z)$ is a constant, $\beta(z)$ is not a constant, then it is natural to have $\deg\beta(z) \geq 1$. Now we will prove that $\deg\beta(z) = 1$. Suppose on the contrary that $\deg\beta(z) = n \geq 2$. Then, it is obvious that

$$\deg[\beta(z+1) - \beta(z)] = \deg[\beta(z+2) - \beta(z+1)] = n - 1, \tag{3.8}$$

and $\sigma(e^{\beta(z+2)-\beta(z+1)}) = \sigma(e^{\beta(z+1)-\beta(z)}) = n - 1 \geq 1$.

Without loss of generality, we assume that $e^{\alpha(z)} = c_1$ $(\neq 0)$. By the first main theorem of Nevanlinna theory, we have

$$\begin{aligned} T(r, q(z)) &= T\left(r, \frac{1}{q(z)}\right) + O(1) = T\left(r, \frac{A_0(z) - A_0(z)e^{\beta(z)-\beta(z+1)}}{c_1F(z)}\right) + O(1) \\ &= T\left(r, e^{\beta(z)-\beta(z+1)}\right) + O(\log r). \end{aligned}$$

Therefore, $\sigma(q(z)) = n-1$. From Lemma 1, we get

$$m\left(r, \frac{q(z+1)}{q(z)}\right) = O(r^{\sigma(q)-1+\varepsilon}) = O(r^{n-2+\varepsilon}) = o(r^{n-1}) \tag{3.9}$$

for each $\varepsilon \in (0,1)$.

The equation (3.6) can be changed to

$$\begin{aligned} &- A_1(z)q(z+1) - A_0(z)q(z)e^{\beta(z+2)-\beta(z+1)} \\ &= [c_1F(z) - A_1(z)q(z+1) - A_0(z)q(z)]e^{\beta(z+2)}. \end{aligned} \tag{3.10}$$

If $c_1F(z) - A_1(z)q(z+1) - A_0(z)q(z) \not\equiv 0$, then by (3.8), (3.10) and the fact $\sigma(q(z)) = n-1$, we have

$$\begin{aligned} n &= \sigma\left((c_1F(z) - A_1(z)q(z+1) - A_0(z)q(z))e^{\beta(z+2)}\right) \\ &= \sigma\left(-A_1(z)q(z+1) - A_0(z)q(z)e^{\beta(z+2)-\beta(z+1)}\right) \le n-1. \end{aligned}$$

This is absurd. Thus $c_1F(z) - A_1(z)q(z+1) - A_0(z)q(z) \equiv 0$.

From the above discussion and (3.10), we see that

$$-A_1(z)q(z+1) - A_0(z)q(z)e^{\beta(z+2)-\beta(z+1)} = 0. \tag{3.11}$$

Since $A_1(z)$ and $A_0(z)$ are non-zero polynomials, by (3.9) and (3.11), we obtain

$$\begin{aligned} T\left(r, e^{\beta(z+2)-\beta(z+1)}\right) &= m\left(r, e^{\beta(z+2)-\beta(z+1)}\right) \\ &= m\left(r, \frac{-A_1(z)q(z+1)}{A_0(z)q(z)}\right) \le o(r^{n-1}) + O(\log r), \end{aligned}$$

which contradicts $\sigma(e^{\beta(z+2)-\beta(z+1)}) = n-1 \ge 1$. We thus have proved that $\deg \beta(z) = 1$. Next we derive from (3.7) that there exist two constants b_1 ($\neq 0$), b_0 such that $\beta(z) = b_1z + b_0$.

Substituting $e^{\alpha(z)} = c_1$, $\beta(z) = b_1z + b_0$ into (3.5), we obtain

$$g(z) = \frac{c_1F(z)(1 - e^{-b_1z-b_1-b_0})}{A_0(z)(1 - e^{-b_1})}, \tag{3.12}$$

where $(1 - e^{-b_1})^{-1} \neq 0$.

Then, combining (3.2) with (3.12), we have

$$\begin{aligned} &\left[\frac{-c_1A_1(z)F(z+1)}{A_0(z+1)(1-e^{-b_1})}e^{-b_1} - \frac{c_1F(z)}{(1-e^{-b_1})}\right] e^{-b_1z-b_1-b_0} \\ &= c_1F(z) - \frac{c_1A_1(z)F(z+1)}{A_0(z+1)(1-e^{-b_1})} - \frac{c_1F(z)}{(1-e^{-b_1})}. \end{aligned}$$

From Lemma 3, we get

$$\frac{-c_1A_1(z)F(z+1)}{A_0(z+1)(1-e^{-b_1})}e^{-b_1} - \frac{c_1F(z)}{(1-e^{-b_1})} \equiv 0 \tag{3.13}$$

and

$$c_1F(z) - \frac{c_1A_1(z)F(z+1)}{A_0(z+1)(1-e^{-b_1})} - \frac{c_1F(z)}{(1-e^{-b_1})} \equiv 0. \tag{3.14}$$

By using the above equations, we have

$$e^{-b_1} = \frac{-A_0(z+1)F(z)}{A_1(z)F(z+1)} = e^{b_1},$$

where $e^{-b_1} = e^{b_1} = -1$.

Further, from (3.1) and (3.12), we obtain

$$f(z) = \frac{c_1F(z)}{2A_0(z)}(1 + e^{b_1z+b_0})$$

and

$$g(z) = \frac{c_1F(z)}{2A_0(z)}(1 + e^{-b_1z-b_0}),$$

where $e^{-b_1} = e^{b_1} = -1$, $b_1 = (2k+1)\pi i$, k is an integer. Finally, we obtain from (3.13) or (3.14) that

$$A_1(z)F(z+1) \equiv F(z)A_0(z+1)$$

holds for this case.

Case 2. If $\alpha(z)$ and $\beta(z)$ are not constants with $\deg\alpha(z) = m$, $\deg\beta(z) = n$, then (3.6) can be expressed as

$$\begin{aligned} &A_1(z)F(z+1)e^{\alpha(z+1)} - A_1(z)F(z+1)e^{\alpha(z+1)-\beta(z+2)} - A_0(z+1) \\ &F(z)e^{\alpha(z)-\beta(z+1)} + A_0(z+1)F(z)e^{\alpha(z)-\beta(z+2)} - A_1(z)F(z+1) \\ &e^{\beta(z)-\beta(z+1)+\alpha(z+1)} + A_0(z+1)F(z)e^{\alpha(z)+\beta(z)-\beta(z+1)} \\ &- A_0(z+1)F(z)e^{\alpha(z)+\beta(z)-\beta(z+2)} + A_1(z)F(z+1) \\ &e^{\alpha(z+1)+\beta(z)-\beta(z+1)-\beta(z+2)} = 0. \end{aligned} \tag{3.15}$$

We will discuss three subcases.

Subcase 2.1: If $\deg\alpha(z) > \deg\beta(z) \geq 1$, then (3.15) can be expressed as

$$J_{11}(z)e^{\alpha(z+1)} = 0,$$

so that

$$J_{11}(z) = 0, \tag{3.16}$$

where

$$\begin{aligned} J_{11}(z) = {} & A_1(z)F(z+1) - A_1(z)F(z+1)e^{-\beta(z+2)} \\ & - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)-\beta(z+1)} \\ & + A_0(z+1)F(z)e^{\alpha(z)-\beta(z+2)-\alpha(z+1)} - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} \\ & + A_0(z+1)F(z)e^{\alpha(z)+\beta(z)-\beta(z+1)-\alpha(z+1)} \\ & - A_0(z+1)F(z)e^{\alpha(z)+\beta(z)-\beta(z+2)-\alpha(z+1)} \end{aligned}$$

$$+ A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)-\beta(z+2)}.$$

Now we distinguish $m-1 > n \geq 1$, $m-1 = n \geq 1$ two subcases to derive the contradictions.

Subcase 2.1.1: If $m-1 > n \geq 1$, then (3.16) can be expressed as

$$\begin{aligned} &J_{24}(z)e^{-\beta(z+2)} + J_{23}(z)e^{\alpha(z)-\alpha(z+1)-\beta(z+1)} + J_{22}(z) \\ &e^{\beta(z)-\beta(z+1)+\alpha(z)-\alpha(z+1)} + J_{21}(z)e^{h_0(z)} = 0, \end{aligned} \tag{3.17}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} J_{24}(z) = -A_1(z)F(z+1) + A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)}, \\ J_{23}(z) = -A_0(z+1)F(z) + A_0(z+1)F(z)e^{\beta(z+1)-\beta(z+2)}, \\ J_{22}(z) = A_0(z+1)F(z) - A_0(z+1)F(z)e^{\beta(z+1)-\beta(z+2)}, \\ J_{21}(z) = A_1(z)F(z+1) - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)}. \end{cases}$$

Now by $m-1 > n \geq 1$ it follows that $\deg(\beta(z+i) - \beta(z+j)) = \deg\beta(z) - 1 = n-1 \geq 0$ $(i = 0, 1;\ j = 1, 2)$, $\deg(\alpha(z+1)-\alpha(z)+\beta(z+1)-\beta(z+2)) = m-1$, $\deg(\alpha(z+1) - \alpha(z) + \beta(z+1) - \beta(z+2) - \beta(z)) = m-1$, $\deg(\alpha(z) - \alpha(z+1) - \beta(z+1)) = m-1$, $\deg(\beta(z) - \beta(z+1) + \alpha(z) - \alpha(z+1)) = m-1$. From Lemma 2, for $j = 1, 2, 3, 4$, we get

$$\begin{cases} T(r, J_{2j}(z)) = o\{T(r, e^{\beta(z)})\}, \\ T(r, J_{2j}(z)) = o\{T(r, e^{\alpha(z)-\alpha(z+1)-\beta(z+1)})\}, \\ T(r, J_{2j}(z)) = o\{T(r, e^{\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)})\}, \\ T(r, J_{2j}(z)) = o\{T(r, e^{\alpha(z+1)-\alpha(z)+\beta(z+1)-\beta(z+2)})\}, \\ T(r, J_{2j}(z)) = o\{T(r, e^{\alpha(z+1)-\alpha(z)+\beta(z+1)-\beta(z+2)-\beta(z)})\}. \end{cases}$$

Applying Lemma 3 to (3.17), we have $J_{2j}(z) \equiv 0$ $(j = 1, 2, 3, 4)$. Then, by $J_{24}(z) \equiv 0$ and the assumption that $A_1(z)$ and $F(z)$ are non-zero polynomials, we have $e^{\beta(z)-\beta(z+1)} \equiv 1$, which contradicts that $e^{\beta(z)-\beta(z+1)} \not\equiv 1$.

Subcase 2.1.2: If $m-1 = n \geq 1$, then from (3.7), we see that $\alpha(z)-\alpha(z+1) = -ma_m z^{m-1} - (C_m^2 a_m + C_{m-1}^1 a_{m-1})z^{m-2} - \cdots - (a_m + a_{m-1} + \cdots + a_1)$ is non-constant polynomial with $\deg(\alpha(z) - \alpha(z+1)) = m-1 \geq 1$. Similarly, we can get $\deg(\beta(z) - \beta(z+1)) = n-1 \geq 0$. The following discussion is divided into three subcases according to whether $b_n = \pm ma_m$.

Subcase 2.1.2.1: If $b_n \neq ma_m, b_n \neq -ma_m$, then (3.16) can be rewritten as (3.17). Therefore, using the same method as in the proof of Subcase 2.1.1, we can get a contradiction.

Subcase 2.1.2.2: If $b_n = -ma_m$, then (3.16) can be expressed as

$$J_{33}(z)e^{-\beta(z+2)} + J_{32}(z)e^{\beta(z)-\beta(z+1)+\alpha(z)-\alpha(z+1)} + J_{31}(z)e^{h_0(z)} = 0, \tag{3.18}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} J_{33}(z) = -A_1(z)F(z+1) + A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)}, \\ J_{32}(z) = A_0(z+1)F(z) - A_0(z+1)F(z)e^{\beta(z+1)-\beta(z+2)}, \\ J_{31}(z) = A_1(z)F(z+1) - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} \\ \qquad\qquad + A_0(z+1)F(z)e^{\alpha(z)-\beta(z+2)-\alpha(z+1)} \\ \qquad\qquad - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)-\beta(z+1)}. \end{cases}$$

By $b_n = -ma_m$, $m-1 = n \geq 1$, then $\deg(-\beta(z+2) - \beta(z) + \beta(z+1) - \alpha(z) + \alpha(z+1)) = \deg(\beta(z) - \beta(z+1) + \alpha(z) - \alpha(z+1)) = n$, $\deg(\beta(z) - \beta(z+1)) = \deg(\beta(z+1) - \beta(z+2)) = n-1$, $\deg(\alpha(z) - \alpha(z+1) - \beta(z+1)) = \deg(\alpha(z) - \alpha(z+1) - \beta(z+2)) \leq n-1$. From Lemma 2, for $j = 1, 2, 3$, we get

$$\begin{cases} T(r, J_{3j}(z)) = o\{T(r, e^{-\beta(z+2)})\}, \\ T(r, J_{3j}(z)) = o\{T(r, e^{\beta(z)-\beta(z+1)+\alpha(z)-\alpha(z+1)})\}, \\ T(r, J_{3j}(z)) = o\{T(r, e^{-\beta(z+2)-\beta(z)+\beta(z+1)-\alpha(z)+\alpha(z+1)})\}. \end{cases}$$

Applying Lemma 3 to (3.18), we have $J_{3j}(z) \equiv 0$ $(j = 1, 2, 3)$. Then, by $J_{33}(z) \equiv 0$ and the assumption that $A_1(z)$ and $F(z)$ are non-zero polynomials, we obtain a contradiction again.

Subcase 2.1.2.3: If $b_n = ma_m$, then (3.16) can be expressed as

$$J_{43}(z)e^{-\beta(z+2)} + J_{42}(z)e^{\alpha(z)-\alpha(z+1)-\beta(z+1)} + J_{41}(z)e^{h_0(z)} = 0, \tag{3.19}$$

where $h_0(z) \equiv 0$ and

$$\begin{cases} J_{43}(z) = -A_1(z)F(z+1) + A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} \\ \qquad\qquad - A_0(z+1)F(z)e^{\beta(z)+\alpha(z)-\alpha(z+1)} \\ \qquad\qquad + A_0(z+1)F(z)e^{\beta(z)-\beta(z+1)+\beta(z+2)+\alpha(z)-\alpha(z+1)}, \\ J_{42}(z) = -A_0(z+1)F(z) + A_0(z+1)F(z)e^{\beta(z+1)-\beta(z+2)}, \\ J_{41}(z) = A_1(z)F(z+1) - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)}. \end{cases}$$

By $b_n = ma_m$, $m-1 = n \geq 1$, then $\deg(-\beta(z+2) - \alpha(z) + \alpha(z+1) + \beta(z+1)) = \deg(\alpha(z) - \alpha(z+1) - \beta(z+1)) = n$, $\deg(\beta(z) - \beta(z+1)) = \deg(\beta(z+1) - \beta(z+2)) = n-1$, $\deg(\beta(z) - \beta(z+1) + \beta(z+2) + \alpha(z) - \alpha(z+1)) = \deg(\beta(z) + \alpha(z) - \alpha(z+1)) \leq n-1$. From Lemma 2, for $j = 1, 2, 3$, we get

$$\begin{cases} T(r, J_{4j}(z)) = o\{T(r, e^{-\beta(z+2)})\}, \\ T(r, J_{4j}(z)) = o\{T(r, e^{\alpha(z)-\alpha(z+1)-\beta(z+1)})\}, \\ T(r, J_{4j}(z)) = o\{T(r, e^{-\beta(z+2)-\alpha(z)+\alpha(z+1)+\beta(z+1)})\}. \end{cases}$$

Applying Lemma 3 to (3.19), we have $J_{4j}(z) \equiv 0$ $(j = 1, 2, 3)$. Then by $J_{41}(z) \equiv 0$, we can get the same contradiction as in Subcase 2.1.2.2.

Subcase 2.2: If $\deg \beta(z) > \deg \alpha(z) \geq 1$, then (3.15) can be expressed as

$$J_{52}(z)e^{\alpha(z+1)} + J_{51}(z)e^{\alpha(z+1)-\beta(z+2)} = 0, \tag{3.20}$$

where

$$\begin{cases} J_{52}(z) = A_1(z)F(z+1) - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} \\ \qquad\qquad + A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)} \\ \qquad\qquad - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+2)}, \\ J_{51}(z) = -A_1(z)F(z+1) - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)-\beta(z+1)+\beta(z+2)} \\ \qquad\qquad + A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)} + A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)}. \end{cases}$$

By $\deg\beta(z) > \deg\alpha(z) \geq 1$, then $\deg(\beta(z+2)) = n$, $\deg(\beta(z)-\beta(z+1)) = n-1$, $\deg(\alpha(z)-\alpha(z+1)) = m-1 < n$, $\deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)) = n-1$, $\deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+2)) = n-1$, $\deg(\alpha(z)-\alpha(z+1)-\beta(z+1)+\beta(z+2)) = n-1$. From Lemma 2, for $j = 1, 2$, we get $T(r, J_{5j}(z)) = o\{T(r, e^{\beta(z+2)})\}$. Applying Lemma 3 to (3.20), we have $J_{5j}(z) \equiv 0$ $(j = 1, 2)$. Then by $J_{52}(z) \equiv 0$, we have

$$\begin{aligned} &A_1(z)F(z+1) - A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} + A_0(z+1)F(z) \\ &e^{\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)} - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+2)} \equiv 0. \end{aligned} \tag{3.21}$$

We will discuss the degree of $\alpha(z)$ and obtain the contradiction.

Firstly, if $\deg\alpha(z) = m \geq 2$, by $\deg\beta(z) > \deg\alpha(z) \geq 1$, then we obtain $\deg\beta(z) = n \geq 3$, thus $\deg(\beta(z)-\beta(z+1)) = n-1 \geq 2$, $\deg(\beta(z+2)-\beta(z+1)) = n-1 \geq 2$, $\deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)) = \deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+2)) = \deg(\alpha(z+1)-\alpha(z)+\beta(z+2)-\beta(z+1)) = n-1 \geq 2$, $\deg(\alpha(z+1)-\alpha(z)) = m-1 \geq 1$. Applying Lemma 3 to (3.21), we have $A_1(z)F(z+1) \equiv A_0(z+1)F(z) \equiv 0$. This contradicts the assumption that $A_1(z)$, $A_0(z)$, $F(z)$ are non-zero polynomials. Hence, $\deg\alpha(z) = 1$. Setting $\alpha(z) = a_1 z + a_0$, where $a_1 \neq 0$ and a_0 are constants. Then (3.21) can be expressed as

$$\begin{aligned} &A_1(z)F(z+1) + [A_0(z+1)F(z)e^{-a_1} - A_1(z)F(z+1)]e^{\beta(z)-\beta(z+1)} \\ &- A_0(z+1)F(z)e^{-a_1+\beta(z)-\beta(z+2)} \equiv 0. \end{aligned} \tag{3.22}$$

By $\deg\alpha(z) = 1$, then $\deg\beta(z) \geq 2$, thus $\deg(\beta(z+2)-\beta(z+1)) \geq 1$, $\deg(\beta(z)-\beta(z+i)) \geq 1$ $(i = 1,\ 2)$. Applying Lemma 3 to (3.22), we have $A_1(z)F(z+1) \equiv A_0(z+1)F(z) \equiv 0$, which is a contradiction.

Subcase 2.3: If $\deg\beta(z) = \deg\alpha(z) = n \geq 1$, then (3.15) can be expressed as (3.20). By $\deg\beta(z) = \deg\alpha(z) = n \geq 1$, then $\deg(\beta(z)-\beta(z+1)) = \deg\beta(z) - 1 = \deg(\alpha(z)-\alpha(z+1)) = n-1$, $\deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+1)) \leq n-1$, $\deg(\alpha(z)-\alpha(z+1)+\beta(z)-\beta(z+2)) \leq n-1$, $\deg(\alpha(z)-\alpha(z+1)-\beta(z+1)+\beta(z+2)) \leq n-1$. From Lemma 2, for $j = 1,\ 2$, we get $T(r, J_{5j}(z)) = o\{T(r, e^{\beta(z+2)})\}$. Applying Lemma 3 to (3.20), we have $J_{5j}(z) \equiv 0$ $(j = 1,\ 2)$. Then by $J_{51}(z) \equiv 0$, we have

$$\begin{aligned} &- A_1(z)F(z+1) - A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)+\beta(z+2)-\beta(z+1)} \\ &+ A_0(z+1)F(z)e^{\alpha(z)-\alpha(z+1)} + A_1(z)F(z+1)e^{\beta(z)-\beta(z+1)} \equiv 0. \end{aligned} \tag{3.23}$$

Next we will prove that $\deg\alpha(z)=\deg\beta(z)=1$.

Subcase 2.3.1: If $\deg\alpha(z)=\deg\beta(z)=n\geq 2$, $a_m\neq b_n$, then (3.23) can be expressed as

$$\begin{aligned}&J_{64}(z)e^{\alpha(z)-\alpha(z+1)}+J_{63}(z)e^{\beta(z)-\beta(z+1)}+J_{62}(z)\\&e^{\alpha(z)-\alpha(z+1)+\beta(z+2)-\beta(z+1)}+J_{61}(z)e^{h_0(z)}\equiv 0,\end{aligned}\tag{3.24}$$

where $h_0(z)\equiv 0$ and

$$\begin{cases}J_{64}(z)=A_0(z+1),\\ J_{63}(z)=A_1(z)\frac{F(z+1)}{F(z)},\\ J_{62}(z)=-A_0(z+1),\\ J_{61}(z)=-A_1(z)\frac{F(z+1)}{F(z)}.\end{cases}$$

By $\deg\alpha(z)=\deg\beta(z)=n\geq 2$, $a_m\neq b_n$, then $\deg(\alpha(z)-\alpha(z+1)-\beta(z)+\beta(z+1))=n-1\geq 1$, $\deg(\beta(z+1)-\beta(z+2))=n-1\geq 1$, $\deg(\alpha(z)-\alpha(z+1))=n-1\geq 1$, $\deg(\beta(z)-\alpha(z)+\alpha(z+1)-\beta(z+2))=n-1\geq 1$, $\deg(\beta(z)-\beta(z+1))=n-1\geq 1$, $\deg(\alpha(z)-\alpha(z+1)-\beta(z+1)+\beta(z+2))=n-1\geq 1$. From Lemma 2, for $j=1,\ 2,\ 3,\ 4$, we get

$$\begin{cases}T(r,J_{6j}(z))=o\{T(r,e^{\alpha(z)-\alpha(z+1)})\},\\ T(r,J_{6j}(z))=o\{T(r,e^{\beta(z)-\beta(z+1)})\},\\ T(r,J_{6j}(z))=o\{T(r,e^{\beta(z+1)-\beta(z+2)})\},\\ T(r,J_{6j}(z))=o\{T(r,e^{\beta(z)-\alpha(z)+\alpha(z+1)-\beta(z+2)})\},\\ T(r,J_{6j}(z))=o\{T(r,e^{\alpha(z)-\alpha(z+1)-\beta(z)+\beta(z+1)})\},\\ T(r,J_{6j}(z))=o\{T(r,e^{\alpha(z)-\alpha(z+1)-\beta(z+1)+\beta(z+2)})\}.\end{cases}$$

Applying Lemma 3 to (3.24), we have $J_{6j}(z)\equiv 0$ $(j=1,\ 2)$. Then by $J_{64}(z)\equiv 0$, we have $A_0(z+1)\equiv 0$. This contradicts the assumption that $A_0(z)$ is a non-zero polynomial.

Subcase 2.3.2: If $\deg\alpha(z)=\deg\beta(z)=n\geq 2,\ a_m=b_n$, then (3.23) can be expressed as

$$J_{72}(z)e^{\alpha(z)-\alpha(z+1)}+J_{71}(z)e^{\alpha(z)-\alpha(z+1)+\beta(z+2)-\beta(z+1)}\equiv 0,\tag{3.25}$$

where

$$\begin{cases}J_{72}(z)=A_0(z+1),\\ J_{71}(z)=\ -A_0(z+1)-A_1(z)\dfrac{F(z+1)}{F(z)}e^{\alpha(z+1)-\alpha(z)+\beta(z+1)-\beta(z+2)}\\ \qquad\qquad +A_1(z)\dfrac{F(z+1)}{F(z)}e^{\alpha(z+1)-\alpha(z)+\beta(z)-\beta(z+2)}.\end{cases}$$

By $\deg\alpha(z)=\deg\beta(z)=n\geq 2$, $a_m=b_n$, then $\deg(\beta(z+1)-\beta(z+2))=n-1\geq 1$, $\deg(\alpha(z+1)-\alpha(z)+\beta(z)-\beta(z+2))=\deg(\alpha(z+1)-\alpha(z)-\beta(z+2)+\beta(z+1))\leq n-2$. From Lemma 2, for $j=1$, 2, we get $T(r,J_{7j}(z))=o\{T(r,e^{\beta(z+1)-\beta(z+2)})\}$. Applying Lemma 3 to (3.25), we have $J_{7j}(z)\equiv 0$ $(j=$

1, 2). Then by $J_{72}(z) \equiv 0$, we have $A_0(z+1) \equiv 0$, a contradiction. Hence, $\deg \alpha(z) = \deg \beta(z) = 1$.

Now we set $\alpha(z) = a_1 z + a_0$, $\beta(z) = b_1 z + b_0$. Then, substituting it into (3.1) and (3.5), we obtain

$$f(z) = \frac{F(z) e^{a_1 z + a_0}(e^{b_1 z + b_0} - e^{-b_1})}{A_0(z)(1 - e^{-b_1})}$$

and

$$g(z) = \frac{F(z) e^{a_1 z + a_0}(1 - e^{-b_1 z - b_1 - b_0})}{A_0(z)(1 - e^{-b_1})},$$

where $(1 - e^{-b_1})^{-1} \neq 0$, $a_1 \neq 0$, $b_1 \neq 0$, a_0, b_0 are constants.

What is more, from (3.6), we see that the coefficients of (1.5) satisfy

$$A_1(z) F(z+1) \equiv \frac{e^{-b_1}(e^{-b_1 z - b_0} - 1)}{e^{a_1}(1 - e^{-b_1 z - 2b_1 - b_0})} F(z) A_0(z+1).$$

This completes the proof of Theorem 6.

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References

[1] S. B. Bank and R. P. Kaufman, *An extension of Hölder's theorem concerning the gamma function*, Funkcial. Ekvac. **19** (1976), no. 1, 53–63.

[2] Z. X. Chen, *Complex Differences and Difference Equations*, Science Press, Beijing, 2014.

[3] Z.-X. Chen and K. H. Shon, *On growth of meromorphic solutions for linear difference equations*, Abstr. Appl. Anal. **2013** (2013), Art. ID 619296, 6 pp. https://doi.org/10.1155/2013/619296

[4] Z.-X. Chen and H.-X. Yi, *On sharing values of meromorphic functions and their differences*, Results Math. **63** (2013), no. 1-2, 557–565. https://doi.org/10.1007/s00025-011-0217-7

[5] Y.-M. Chiang and S.-J. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J. **16** (2008), no. 1, 105–129. https://doi.org/10.1007/s11139-007-9101-1

[6] N. Cui and Z. X. Chen, *Unicity for meromorphic solutions of some difference equations sharing three values of any meromorphic function*, J. South China Normal Univ. Natur. Sci. Ed. **48** (2016), no. 4, 83–87.

[7] N. Cui and Z. X. Chen, *Uniqueness for meromorphic solutions sharing three values with a meromorphic function to some linear difference equations*, Chinese J. Contemp. Math. **38** (2017), no. 1, 13–22; translated from Chinese Ann. Math. Ser. A **38** (2017), no. 1, 13–22.

[8] R. G. Halburd and R. J. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. **314** (2006), no. 2, 477–487. https://doi.org/10.1016/j.jmaa.2005.04.010

[9] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[10] J. Heittokangas, R. Korhonen, I. Laine, and J. Rieppo, *Uniqueness of meromorphic functions sharing values with their shifts*, Complex Var. Elliptic Equ. **56** (2011), no. 1-4, 81–92. https://doi.org/10.1080/17476930903394770

[11] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, De Gruyter Studies in Mathematics, 15, Walter de Gruyter & Co., Berlin, 1993. https://doi.org/10.1515/9783110863147

[12] S. Li and B. Chen, *Uniqueness of meromorphic solutions of the difference equation* $R_1(z)f(z+1)+R_2(z)f(z)=R_3(z)$, Adv. Difference Equ. **2019** (2019), Paper No. 250, 11 pp. https://doi.org/10.1186/s13662-019-2194-1

[13] X.-M. Li and H.-X. Yi, *Meromorphic functions sharing four values with their difference operators or shifts*, Bull. Korean Math. Soc. **53** (2016), no. 4, 1213–1235. https://doi.org/10.4134/BKMS.b150609

[14] F. Lü, Q. Han, and W. Lü, *On unicity of meromorphic solutions to difference equations of Malmquist type*, Bull. Aust. Math. Soc. **93** (2016), no. 1, 92–98. https://doi.org/10.1017/S0004972715000787

[15] R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Chelsea Publishing Co., New York, 1974.

[16] X. Qi, N. Li, and L. Yang, *Uniqueness of meromorphic functions concerning their differences and solutions of difference Painlevé equations*, Comput. Methods Funct. Theory **18** (2018), no. 4, 567–582. https://doi.org/10.1007/s40315-018-0241-7

[17] S. Shimomura, *Entire solutions of a polynomial difference equation*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 2, 253–266.

[18] N. Yanagihara, *Meromorphic solutions of some difference equations*, Funkcial. Ekvac. **23** (1980), no. 3, 309–326.

[19] C.-C. Yang and H.-X. Yi, *Uniqueness Theory of Meromorphic Functions*, Mathematics and its Applications, 557, Kluwer Academic Publishers Group, Dordrecht, 2003.

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