Bull. Korean Math. Soc. **59** (2022), No. 4, pp. 961–977
https://doi.org/10.4134/BKMS.b210567
pISSN: 1015-8634 / eISSN: 2234-3016

POSITIVE SOLUTION AND GROUND STATE SOLUTION FOR A KIRCHHOFF TYPE EQUATION WITH CRITICAL GROWTH

Caixia Chen and Aixia Qian

Abstract. In this paper, we consider the following Kirchhoff type equation on the whole space

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\triangle u = u^5+\lambda k(x)g(u), \ x\in\mathbb{R}^3, \\ u\in\mathcal{D}^{1,2}(\mathbb{R}^3), \end{cases}$$

where $\lambda > 0$ is a real number and k, g satisfy some conditions. We mainly investigate the existence of ground state solution via variational method and concentration-compactness principle.

1. Introduction

In this paper, we are concerned with the following Kirchhoff type problem

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\triangle u = u^5+\lambda k(x)g(u), \ x\in\mathbb{R}^3, \\ u\in\mathcal{D}^{1,2}(\mathbb{R}^3), \end{cases} \tag{1.1}$$

where the nonlinear growth u^5 reaches the Sobolev critical exponent since the critical exponent $2^* = 6$ in three spatial dimensions. In recent years, because of the strong physical meaning in mechanics models, the following Kirchhoff type problem involving the critical Sobolev exponent has attracted a lot of attention,

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\triangle u = f(x,u), \ \text{in } \Omega, \\ u=0, \qquad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth domain in $\mathbb{R}^3$. Therefore many results have been obtained, including the existence of positive solutions, multiple solutions, sign-changing

Received August 2, 2021; Revised October 10, 2021; Accepted November 8, 2021.
2020 *Mathematics Subject Classification.* Primary 35J20, 35J60, 58E05.
Key words and phrases. Kirchhoff type equation, critical nonlinearity, concentration-compactness principle, ground state solution.
This work was financially supported by SSF ZR2021MA096, ZR2020MA005.

solutions and so on, see for example [2,5,7,8,16] and the reference therein on bounded domain, [3,9,11,19] and the reference therein on unbounded domains.

The existence of positive ground state solution for Kirchhoff equation has also been widely investigated in recent years. Wang et al. in [19] showed the existence of positive ground state solutions by using the variational method, where the subcritical nonlinear function $f(u)$ satisfies some stronger conditions. By a monotonicity trick and a new version of global compactness lemma, Li and Ye in [10] considered the following Kirchhoff type problem with pure power nonlinearities

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\triangle u+V(x)u=|u|^{p-1},\ x\in\mathbb{R}^3,\\ u\in \mathrm{H}^1(\mathbb{R}^3),\ u>0,\end{cases}$$

and gave the existence of positive ground state solution for $2<p<5$. In [4], Hu and Lu obtained the multiplicity of positive solutions for the following Kirchhoff type problem with pure power nonlinearities

$$\begin{cases} -(\varepsilon^2a+\epsilon b\int_{\mathbb{R}^3}|\nabla u|^2dx)\triangle u+u=Q(x)|u|^{p-2}u,\ x\in\mathbb{R}^3,\\ u\in \mathrm{H}^1(\mathbb{R}^3),\ u>0\end{cases}$$

with $p\in(2,6)$. Recently, Lei et al. in [9] studied the following Kirchhoff type problem involving critical growth

$$\begin{cases} -(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx)\triangle u=u^5+\lambda k(x)u^{q-1},\ x\in\mathbb{R}^3,\\ u\in \mathcal{D}^{1,2}(\mathbb{R}^3)\end{cases}$$

with $2<q<6$.

In the present paper, we are concerned with a more general nonlinearity $\lambda k(x)g(u)$, where $g(u)$ is a nonlinear function with superquadratic growth both at zero and at infinity. Besides, we use a new cut-off technique together with critical point theorems to investigate the ground state solutions of (1.1). Nowadays, it has become a classical variational method due to its useful features. It is highlighted in the recent contribution [1,8,15] and so on.

Before stating our results, we assume that $k(x)$ satisfies the following conditions.

(k_1) $k(x)\in L^{\frac{6}{6-q}}(\mathbb{R}^3)\bigcap L^{\frac{6}{6-p}}(\mathbb{R}^3)$, $k(x)\geq 0$ for any $x\in\mathbb{R}^3$, $4<p,q<6$ and $k(x)\not\equiv 0$;

(k_2) There exist $x_0\in\mathbb{R}^3$ and $\delta_1,\rho_1>0$ such that $k(x)\geq\delta_1|x-x_0|^{-\beta}$ for $|x-x_0|<\rho_1$ with $3-\frac{p}{2}<\beta<3$.

Moreover, we assume the nonlinearity $g(u)\in C(\mathbb{R},\mathbb{R})$ satisfies the following hypotheses.

(G_1) There is a $q\in\mathbb{R}$ with $4<q<6$ such that $\lim_{|u|\to 0}\frac{g(u)}{|u|^{q-2}u}=1$;

(G_2) There is a $p\in\mathbb{R}$ with $4<p<6$ such that $\lim_{|u|\to\infty}\frac{g(u)}{|u|^{p-2}u}=1$;

(G_3) $g(u) > 0$ for all $u > 0$.

Since our aim is to find the positive solutions, it is only necessary to consider $u > 0$ for the equation (1.1). Then throughout the paper we assume, without loss of generality, that $g(u)$ is defined in $\mathbb{R}$ as an odd function.

Now the main result of the present paper reads as follows.

Theorem 1.1. *Assume* (k_1)-(k_2) *and* (G_1)-(G_3) *hold. Then there exists* $\lambda_* > 0$ *such that problem* (1.1) *has at least one positive ground state solution for any* $0 < \lambda < \lambda_*$.

Remark 1.2. From the hypotheses (G_1)-(G_3), we know the nonlinear term $g(u)$ is more general than the specific power-type nonlinearity u^{q-1} for $q \in (4, 6)$. Furthermore, under some suitable assumptions, we can still prove the existence of positive state ground solution. Therefore, we generalize some corresponding results in the relative references.

Hereafter we use the following notations.

- $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard norm $\|u\|^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx$;
- $L^p(\mathbb{R}^3)$ and $L^q(\mathbb{R}^3)$ ($1 \le q, p \le \infty$) denote Lebesgue spaces, the norm in $L^p(\mathbb{R}^3)$ and $L^q(\mathbb{R}^3)$ are denoted by $|\cdot|_p$ and $|\cdot|_q$, respectively;
- $C, C_0, \hat{C}, C_1, \hat{C}_1, C_2, \ldots$ denote various positive constants, which may vary from lines to lines;
- S_r (respectively, B_r) is the sphere (respectively, the closed ball) of center zero and radius r, i.e., $S_r = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \|u\| = r\}$, $B_r = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \|u\| \le r\}$;
- S is the best Sobolev constant for the embedding of $\mathcal{D}^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$, that is

$$S = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx\right)^{\frac{1}{3}}}; \tag{1.3}$$

- $\to$ ($\rightharpoonup$) means strong (weak) convergence.

This paper is organized as follows. After introducing some preliminary results in Section 2, we shall demonstrate the proof of Theorem 1.1 in Section 3.

2. Some preliminaries

The solution of (1.1) corresponds to critical points of the following energy functional

$$I_\lambda(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx - \lambda \int_{\mathbb{R}^3} k(x)G(u)dx$$

with $G(u) = \int_0^u g(t)dt$. The hypotheses (G_1)-(G_2) imply that

$$|g(t)| \le b_1|t|^{q-1} + b_2|t|^{p-1}, \quad |G(t)| \le b_1|t|^q + b_2|t|^p \quad \text{for all } t \in \mathbb{R}, \tag{2.1}$$

and

$$b_3|t|^q \le G(t), \quad b_3|t|^q \le g(t)t, \quad \text{if } |t| \le \delta_0, \tag{2.2}$$
$$b_4|t|^p \le G(t), \quad b_4|t|^p \le g(t)t, \quad \text{if } |t| \ge \delta_0,$$

for some $b_1, b_2, b_3, b_4, \delta_0 > 0$ (see e.g. [8]).

It follows from (G_2) that there exist $r \ge 4$ and $M > 0$ such that

$$|u| \ge M \Longrightarrow 0 < rG(u) \le ug(u). \tag{2.3}$$

Lemma 2.1 ([6]). *Under the assumption (k_1), the function $\mathcal{K} : \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ given by*

$$\mathcal{K}(u) = \int_{\mathbb{R}^3} k(x)g(u)udx$$

is weakly continuous.

Lemma 2.2. *Suppose that (k_1)-(k_2) hold. Then*
(i) *There are two constants $\alpha, \rho > 0$ such that $I_\lambda|_{S_\rho} \ge \alpha$;*
(ii) *There exists $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ with $\|u_0\| > \rho$ satisfying $I_\lambda(u_0) < 0$.*

Proof. (i) For any $u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \setminus \{0\}$, $4 < q, p < 6$, it follows from (1.3) that

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx - \lambda \int_{\mathbb{R}^3} k(x)G(u)dx \\ &\ge \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6S^3}\|u\|^6 - \lambda \int_{\mathbb{R}^3} k(x)(b_1|u|^q + b_2|u|^p)dx \\ &\ge \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6S^3}\|u\|^6 \\ &\quad - \lambda b_1 |k(x)|_{\frac{6}{6-q}} S^{-\frac{q}{2}}\|u\|^q - \lambda b_2 |k(x)|_{\frac{6}{6-p}} S^{-\frac{p}{2}}\|u\|^p. \end{aligned}$$

Let $\rho = \|u\|$ be sufficiently small such that

$$\frac{a}{2}\rho^2 + \frac{b}{4}\rho^4 - \frac{1}{6S^3}\rho^6 - \lambda b_1 |k(x)|_{\frac{6}{6-q}} S^{-\frac{q}{2}}\rho^q - \lambda b_2 |k(x)|_{\frac{6}{6-p}} S^{-\frac{p}{2}}\rho^p > \frac{a}{4}\rho^2.$$

Therefore we get

$$I_\lambda|_{S_\rho} \ge \frac{a}{4}\rho^2 = \alpha.$$

(ii) (k_1) and (k_2) imply that

$$I_\lambda(tu) = \frac{a}{2}t^2\|u\|^2 + \frac{b}{4}t^4\|u\|^4 - \frac{t^6}{6}\int_{\mathbb{R}^3} |u|^6 dx - \lambda \int_{\mathbb{R}^3} k(x)G(tu)dx \to -\infty$$

as $t \to +\infty$.

Therefore there exists $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ with $\|u_0\| > \rho$ such that $I_\lambda(u_0) < 0$. Then (ii) follows and the proof of Lemma 2.2 is complete. □

Denote $\mathcal{M}^+$ as a space of positive finite Radon measures on $\mathbb{R}^3$, and δ_x as the Dirac mass at point x.

Lemma 2.3 ([12, 13])**.** *Let* $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^3)$ *be a bounded sequence, due to Hardy-Sobolev inequality, passing to a subsequence if necessary, we may assume that* $u_n \rightharpoonup u$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^3)$, $|\nabla u_n|^2 \rightharpoonup \mu$ *in* $\mathcal{M}^+$, $|u_n|^6 \rightharpoonup \nu$ *in* $\mathcal{M}^+$. *Define*

$$\mu_\infty := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^3 \bigcap |x|>R} |\nabla u_n|^2 dx,$$

$$\nu_\infty := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\mathbb{R}^3 \bigcap |x|>R} |u_n|^6 dx.$$

Then for every j *in an at most countable set* J, *there hold*

(1) $\mu_\infty \geq S\nu_\infty^{\frac{1}{3}}$;
(2) $\nu = |u|^6 + \nu_0\delta_0 + \sum \delta_{x_j}\nu_j > 0,\ \mu \geq \|\nabla u\|^2 + \mu_0\delta_0 + \sum \delta_{x_j}\mu_j$;
(3) $\mu_j \geq S\nu_j^{\frac{1}{3}}$;
(4) $\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^6 dx = \int_{\mathbb{R}^3} |u|^6 dx + \|\nu\| + \nu_\infty$.

Next, we define

$$\Lambda = \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4 + 4aS)^{\frac{3}{2}}}{24}.$$

Lemma 2.4. *Suppose* $4 < p, q < 6$. *Let* $\{u_n\}$ *be a* $(PS)_c$ *sequence of* I_λ *with* $c < \Lambda - C_0\lambda$. *Then there exists* $u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ *such that* $|u_n|_6^6 \to |u|_6^6$ *as* $n \to \infty$.

Proof. For $4 < p, q < 6$, let $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^3)$ be a $(PS)_c$ sequence for I_λ at level $c < \Lambda - C_0\lambda$, i.e.,

$$I_\lambda(u_n) \to c, \quad I_\lambda'(u_n) \to 0 \ \text{ as } \ n \to \infty. \tag{2.4}$$

We firstly claim that $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. It follows from (2.1) and Hölder inequality that there exists C_0 such that

$$\int_{\{x:|u_n|\leq M\}} k(x)\left(\frac{1}{4}g(u_n)u_n - G(u_n)\right)dx > -C_0.$$

For n large enough, from (2.1) and (2.3), we conclude that

$$\begin{aligned}
&1 + o(\|u_n\|) + c \\
\geq\ & I_\lambda(u_n) - \frac{1}{4}\langle I_\lambda'(u_n), u_n\rangle \\
=\ & \frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6 dx - \lambda\int_{\mathbb{R}^3} k(x)G(u_n)dx \\
& - \frac{1}{4}\left\{a\|u_n\|^2 + b\|u_n\|^4 - \int_{\mathbb{R}^3} |u_n|^6 dx - \lambda\int_{\mathbb{R}^3} k(x)g(u_n)u_n dx\right\} \\
=\ & \frac{a}{4}\|u_n\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} |u_n|^6 dx + \lambda\int_{\mathbb{R}^3} k(x)\left(\frac{1}{4}g(u_n)u_n - G(u_n)\right)dx \\
=\ & \frac{a}{4}\|u_n\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} |u_n|^6 dx + \lambda\int_{\{x:|u_n|\leq M\}} k(x)\left(\frac{1}{4}g(u_n)u_n - G(u_n)\right)dx
\end{aligned}$$

$$
\begin{aligned}
&+\lambda \int_{\{x:|u_n|\geq M\}} k(x)\left(\frac{1}{4}g(u_n)u_n - G(u_n)\right)dx\\
&\geq \frac{a}{4}\|u_n\|^2 + \frac{1}{12}\int_{\mathbb{R}^3}|u_n|^6dx + \lambda\int_{\{x:|u_n|\leq M\}} k(x)\left(\frac{1}{4}g(u_n)u_n - G(u_n)\right)dx\\
&\geq \frac{a}{4}\|u_n\|^2 + \frac{1}{12}\int_{\mathbb{R}^3}|u_n|^6dx - C_0\lambda.
\end{aligned}
$$

Thus $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Since $g(u)$ is odd, we get $G(u)$ is even and $I_\lambda(u_n) = I_\lambda(|u_n|)$. Here, we suppose straight away that $u_n(x) \geq 0$ a.e. in $\mathbb{R}^3$ for all n. Then there exist $u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ and a subsequence (still denoted $\{u_n\}$) satisfying

$$
\begin{cases}
u_n \rightharpoonup u, \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3),\\
u_n(x) \to u(x), \quad \text{a.e.} \quad \text{in } \mathbb{R}^3,\\
|\nabla u_n| \rightharpoonup \mu, \quad \text{in } \mathcal{M}^+,\\
|u_n|^6 \rightharpoonup \nu, \ \text{in } \mathcal{M}^+.
\end{cases}
$$

Therefore $u(x) \geq 0$ a.e. in $\mathbb{R}^3$.

Let x_j be a singular point of the measure μ and ν. For any $\epsilon > 0$ small enough, we define a cut-off function $\phi_{\epsilon,j} \in \mathrm{C}_0^\infty(\mathbb{R}^3, [0,1])$ such that

$$
\begin{cases}
\phi_{\epsilon,j}(x) = 1, \quad \text{in } B(x_j,\epsilon),\\
\phi_{\epsilon,j}(x) = 0, \quad \text{in } \ \mathbb{R}^3 \setminus B(x_j, 2\epsilon),\\
|\nabla\phi_{\epsilon,j}(x)| \leq \dfrac{4}{\epsilon}, \quad \text{in } \mathbb{R}^3.
\end{cases}
$$

It is easy to show that $\{\phi_{\epsilon,j}u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Then by (2.4) we get that

$$\langle I_\lambda'(u_n), \phi_{\epsilon,j}u_n\rangle \to 0,$$

which implies

$$
\begin{aligned}
(a + b\|u_n\|^2)\int_{\mathbb{R}^3}(\nabla u_n, \nabla(\phi_{\epsilon,j}u_n))dx - \int_{\mathbb{R}^3}|u_n|^6\phi_{\epsilon,j}dx&\\
- \lambda\int_{\mathbb{R}^3} k(x)g(u_n)\phi_{\epsilon,j}u_ndx \to 0.&
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|u_n|^6\phi_{\epsilon,j}dx + \lambda\int_{\mathbb{R}^3}k(x)g(u_n)\phi_{\epsilon,j}u_ndx\\
&= (a + b\|u_n\|^2)\int_{\mathbb{R}^3}(\nabla u_n, \nabla\phi_{\epsilon,j})u_ndx \qquad (2.5)\\
&\quad + (a + b\|u_n\|^2)\int_{\mathbb{R}^3}|\nabla u_n|^2\phi_{\epsilon,j}dx + o(1).
\end{aligned}
$$

Next we claim that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}(a+b\|u_n\|^2)\int_{\mathbb{R}^3}(\nabla u_n,\nabla\phi_{\epsilon,j})u_n dx=0.$$

Actually, since $\{u_n\}$ is bounded, for any $\epsilon>0$, combining Hölder inequality with the definition of $\phi_{\epsilon,j}$, we have

$$\begin{aligned}\left|\limsup_{n\to\infty}\int_{\mathbb{R}^3}(\nabla u_n,\nabla\phi_{\epsilon,j})u_n dx\right| &\le \limsup_{n\to\infty}\left|\int_{B(x_j,2\epsilon)}(\nabla u_n,\nabla\phi_{\epsilon,j})u_n dx\right|\\ &\le \limsup_{n\to\infty}\frac{4}{\epsilon}\int_{B(x_j,2\epsilon)}|\nabla u_n|\cdot|u_n|\cdot 1dx\\ &\le \frac{4}{\epsilon}C_1\left(\int_{B(x_j,2\epsilon)}|u|^6dx\right)^{\frac{1}{6}}\left(\int_{B(x_j,2\epsilon)}1^3dx\right)^{\frac{1}{3}}\\ &\le 4C_2\left(\int_{B(x_j,2\epsilon)}|u|^6dx\right)^{\frac{1}{6}},\end{aligned}$$

with $C_1,C_2>0$, which implies that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}(\nabla u_n,\nabla\phi_{\epsilon,j})u_n dx=0.$$

Now by Lemma 2.3(2)-(3), we know that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}u_n^6\phi_{\epsilon,j}(x)dx=\lim_{\epsilon\to 0}\int_{\mathbb{R}^3}\phi_{\epsilon,j}(x)d\nu=\nu_j$$

and

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2\phi_{\epsilon,j}(x)dx &= \int_{\mathbb{R}^3}\phi_{\epsilon,j}(x)d\mu\\ &\ge \int_{\mathbb{R}^3}\phi_{\epsilon,j}(x)|\nabla u|^2dx+\sum\delta_j\phi_{\epsilon,j}(x)\mu_j\ge\mu_j.\end{aligned}$$

Thus

$$\begin{aligned}&\limsup_{n\to\infty}(a+b\|u_n\|^2)\int_{\mathbb{R}^3}|\nabla u_n|^2\phi_{\epsilon,j}(x)dx\\ \ge&\limsup_{n\to\infty}\left(a+b\int_{\mathbb{R}^3}|\nabla u_n|^2\phi_{\epsilon,j}(x)dx\right)\int_{\mathbb{R}^3}|\nabla u_n|^2\phi_{\epsilon,j}(x)dx\ge(a+b\mu_j)\mu_j.\end{aligned}$$

By (2.1), one has

$$\begin{aligned}&\left|\lim_{\epsilon\to 0}\limsup_{n\to\infty}\int_{\mathbb{R}^3}\phi_{\epsilon,j}(x)k(x)g(u_n)u_n dx\right|\\ =&\left|\lim_{\epsilon\to 0}\int_{\mathbb{R}^3}\phi_{\epsilon,j}(x)k(x)g(u)udx\right|\\ \le&\left|\lim_{\epsilon\to 0}\int_{\mathbb{R}^3}\phi_{\epsilon,j}(x)k(x)(b_1|u|^{q-1}u+b_2|u|^{p-1}u)dx\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \lim_{\epsilon\to 0}\int_{B(x_j,2\epsilon)} k(x)b_1|u|^q dx + \lim_{\epsilon\to 0}\int_{B(x_j,2\epsilon)} k(x)b_2|u|^p dx \\
&\leq b_1|k(x)|_{\frac{6}{6-q}} S^{-\frac{q}{2}} \lim_{\epsilon\to 0}\left(\int_{B(x_j,2\epsilon)} |\nabla u|^2 dx\right)^{\frac{q}{2}} \\
&\quad + b_2|k(x)|_{\frac{6}{6-p}} S^{-\frac{p}{2}} \lim_{\epsilon\to 0}\left(\int_{B(x_j,2\epsilon)} |\nabla u|^2 dx\right)^{\frac{p}{2}} \\
&= 0.
\end{aligned}
$$

Then by (2.5), we have

$$\nu_j \geq a\mu_j + b\mu_j^2.$$

Combining this with Lemma 2.3(3), a direct computation shows that either

$$\text{(i)}\ \ \mu_j = 0 \qquad \text{or} \qquad \text{(ii)}\ \ \mu_j \geq \frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2} := A.$$

Now, we define another cut-off function $\phi_R \in \mathrm{C}_0^\infty(\mathbb{R}^3, [0,1])$ such that

$$
\begin{cases}
\phi_R(x) = 0, & \text{on } |x| < R, \\
\phi_R(x) = 1, & \text{on } |x| > 2R, \\
|\nabla \phi_R(x)| \leq \dfrac{4}{R}, & \text{in } \mathbb{R}^3.
\end{cases}
$$

By the same argument as above, we can obtain

$$\nu_\infty \geq a\mu_\infty + b\mu_\infty^2.$$

Then by Lemma 2.3(1), one has that

$$\text{(iii)}\ \ \mu_\infty = 0 \qquad \text{or} \qquad \text{(iv)}\ \ \mu_\infty \geq \frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2} = A.$$

Next we claim that (ii) and (iv) cannot hold. Otherwise, we suppose that there exists $j \in J$ such that (iv) holds. By the weak lower semicontinuity of the norm, (2.3) and (2.4), we conclude that

$$
\begin{aligned}
c &= \lim_{n\to\infty}\left\{ I_\lambda(u_n) - \frac{1}{4}\langle I_\lambda'(u_n), u_n\rangle \right\} \\
&= \lim_{n\to\infty}\left\{ \frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6 dx - \lambda\int_{\mathbb{R}^3} k(x)G(u_n)dx \right. \\
&\quad \left. -\frac{1}{4}\left(a\|u_n\|^2 + b\|u_n\|^4 - \int_{\mathbb{R}^3}|u_n|^6 dx - \lambda\int_{\mathbb{R}^3} k(x)g(u_n)u_n dx \right) \right\} \\
&= \lim_{n\to\infty}\left\{ \left(\frac{1}{2} - \frac{1}{4}\right) a\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{4}\right) b\|u_n\|^4 + \frac{1}{12}\int_{\mathbb{R}^3}|u_n|^6 dx \right. \\
&\quad \left. +\lambda\int_{\mathbb{R}^3} k(x)\left(\frac{1}{4}g(u_n)u_n - G(u_n)\right)dx \right\}
\end{aligned}
\tag{2.6}
$$

$$\geq \left(\frac{1}{2}-\frac{1}{4}\right)a\mu_\infty + \left(\frac{1}{4}-\frac{1}{4}\right)b\mu_\infty^2 + \frac{1}{12}\nu_\infty + \frac{1}{12}\int_{\mathbb{R}^3}|u|^6dx - C_0\lambda$$
$$\geq \left(\frac{1}{2}-\frac{1}{4}\right)aA + \left(\frac{1}{4}-\frac{1}{4}\right)bA^2 + \frac{aA+bA^2}{12} - C_0\lambda.$$

Here we show that

$$\frac{aA}{2}+\frac{b}{4}A^2-\frac{aA+bA^2}{6}=\Lambda.$$

Indeed,

$$\frac{aA}{2}+\frac{b}{4}A^2-\frac{aA+bA^2}{6}=\frac{aA}{3}+\frac{b}{12}A^2$$
$$=\frac{abS^3+a\sqrt{b^2S^6+4aS^3}}{6}+\frac{b^3S^6+2abS^3+b^2S^3\sqrt{b^2S^6+4aS^3}}{24}$$
$$=\frac{abS^3}{4}+\frac{b^3S^6}{24}+\frac{(4a+b^2S^3)\sqrt{b^2S^6+4aS^3}}{24}$$
$$=\Lambda.$$

Consequently $\Lambda - C_0\lambda \leq c < \Lambda - C_0\lambda$, we deduce the contradiction.

Similarly (ii) cannot hold for any j. This implies that J is empty.

Up to now, we have proved that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^6dx=\int_{\mathbb{R}^3}|u|^6dx.$$

This completes the proof. □

Moreover, it is well known that S is attained by the function

$$U_{\epsilon,x_0}(x)=C\frac{\epsilon^{\frac{1}{4}}}{(\epsilon+|x-x_0|^2)^{\frac{1}{2}}},$$

where C is a normalizing constant and x_0 is defined in (k_2).

Next, let a cut-off function $\varphi\in C_0^\infty(\mathbb{R}^3)$ be such that $0\leq\varphi\leq 1, \varphi|_{B_r}\equiv 1$ and supp$\varphi\subset B_{2r}$ for some $r>0$. Set $u_\epsilon(x)=\varphi U_{\epsilon,x_0}(x)$, then $u_\epsilon\in\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $u_\epsilon\geq 0$ for all $x\in\mathbb{R}^3$. Let ϵ be small enough, it follows from [20] that

$$\begin{cases} |\nabla u_\epsilon|_2^2 = K_1 + O(\epsilon^{\frac{1}{2}}), \qquad |u_\epsilon|_6^2 = K_2 + O(\epsilon), \\ \int_{\mathbb{R}^3} u_\epsilon^2 dx = O(\epsilon^{\frac{1}{2}}), \end{cases} \tag{2.7}$$

where K_1, K_2 are positive constants. Moreover,

$$\frac{K_1}{K_2}=S. \tag{2.8}$$

Lemma 2.5. *Suppose that (k_1)-(k_2) hold. Then there exists $u_1\in\mathcal{D}^{1,2}(\mathbb{R}^3)$ such that*

$$\sup_{t\geq 0} I_\lambda(tu_1) < \Lambda - C_0\lambda. \tag{2.9}$$

Proof. We first claim that there exist $t_\epsilon > 0$ and positive constants t_0, T_1 independent of ϵ, λ, such that $\sup_{t\geq 0} I_\lambda(tu_\epsilon) = I_\lambda(t_\epsilon u_\epsilon)$ and

$$0 < t_0 \leq t_\epsilon \leq T_1 < \infty. \tag{2.10}$$

Indeed, since

$$\begin{aligned}
&\lim_{t\to+\infty} I_\lambda(tu_\epsilon)\\
=& \lim_{t\to+\infty}\left(\frac{a}{2}t^2\|u_\epsilon\|^2 + \frac{b}{4}t^4\|u_\epsilon\|^4 - \frac{1}{6}t^6\int_{\mathbb{R}^3}|u_\epsilon|^6dx - \lambda\int_{\mathbb{R}^3}k(x)G(tu_\epsilon)dx\right)\\
=& -\infty,
\end{aligned}$$

$I_\lambda(tu_\epsilon) = 0$ for $t = 0$ and $I_\lambda(tu_\epsilon) > 0$ for t small enough, there exists $t_\epsilon > 0$ such that

$$I_\lambda(t_\epsilon u_\epsilon) = \sup_{t\geq 0} I_\lambda(tu_\epsilon) \quad\text{and}\quad \left.\frac{dI_\lambda(tu_\epsilon)}{dt}\right|_{t=t_\epsilon} = 0. \tag{2.11}$$

It follows from (2.11) and (2.1) that

$$\begin{aligned}
0 \geq\ & t_\epsilon a\|u_\epsilon\|^2 + t_\epsilon^3 b\|u_\epsilon\|^4 - t_\epsilon^5\int_{\mathbb{R}^3}|u_\epsilon|^6dx\\
&- \lambda\int_{\mathbb{R}^3}k(x)(b_1|t_\epsilon u_\epsilon|^{q-1} + b_2|t_\epsilon u_\epsilon|^{p-1})u_\epsilon dx
\end{aligned} \tag{2.12}$$

or

$$\begin{aligned}
0 \leq\ & t_\epsilon a\|u_\epsilon\|^2 + t_\epsilon^3 b\|u_\epsilon\|^4 - t_\epsilon^5\int_{\mathbb{R}^3}|u_\epsilon|^6dx\\
&+ \lambda\int_{\mathbb{R}^3}k(x)(b_1|t_\epsilon u_\epsilon|^{q-1} + b_2|t_\epsilon u_\epsilon|^{p-1})u_\epsilon dx.
\end{aligned} \tag{2.13}$$

On one hand, as $4 < p, q < 6$, by (2.12) and

$$\begin{aligned}
&a\|u_\epsilon\|^2 + 3t_\epsilon^2 b\|u_\epsilon\|^4 - 5t_\epsilon^4\int_{\mathbb{R}^3}|u_\epsilon|^6dx\\
&- \lambda\int_{\mathbb{R}^3}k(x)\{b_1(q-1)t_\epsilon^{q-2}|u_\epsilon|^q dx + b_2(p-1)t_\epsilon^{p-2}|u_\epsilon|^p\}dx \leq 0,
\end{aligned}$$

we can obtain easily that t_ϵ is bounded from below. Therefore, there exists a positive constant t_0 independent of ϵ and λ, satisfying $0 < t_0 \leq t_\epsilon$.

On the other hand, by (2.13) and

$$\frac{a\|u_\epsilon\|^2}{t_\epsilon^2} + b\|u_\epsilon\|^4 - t_\epsilon^2\int_{\mathbb{R}^3}|u_\epsilon|^6dx + \lambda\int_{\mathbb{R}^3}k(x)(b_1 t_\epsilon^{q-4}|u_\epsilon|^q + b_2 t_\epsilon^{p-4}|u_\epsilon|^p)dx \geq 0,$$

we see that t_ϵ is bounded from above for all $\epsilon > 0$ small enough. Consequently, we can conclude that (2.10) holds.

Set

$$h(t_\epsilon u_\epsilon) = \frac{a}{2}t_\epsilon^2\|u_\epsilon\|^2 + \frac{b}{4}t_\epsilon^4\|u_\epsilon\|^4 - \frac{1}{6}t_\epsilon^6\int_{\mathbb{R}^3}u_\epsilon^6dx.$$

We claim that

$$h(t_\epsilon u_\epsilon) \le \Lambda + C_3 \epsilon^{\frac{1}{2}}, \tag{2.14}$$

where C_3 is independent of ϵ, λ.

Indeed define

$$g(t) = \frac{a}{2} t^2 \|u_\epsilon\|^2 + \frac{b}{4} t^4 \|u_\epsilon\|^4 - \frac{1}{6} t^6 \int_{\mathbb{R}^3} u_\epsilon^6 dx.$$

It follows from

$$\lim_{t\to\infty} g(t) = -\infty, \; g(0) = 0 \; \text{ and } \; \lim_{t\to 0^+} g(t) > 0$$

that $\sup_{t\ge 0} g(t)$ is achieved at $T_\epsilon > 0$, that is

$$g^{'}(t)\Big|_{T_\epsilon} = aT_\epsilon \|u_\epsilon\|^2 + bT_\epsilon^3 \|u_\epsilon\|^4 - T_\epsilon^5 \int_{\mathbb{R}^3} u_\epsilon^6 dx = 0.$$

A direct computation implies

$$T_\epsilon = \left(\frac{b\|u_\epsilon\|^4 + \sqrt{b^2\|u_\epsilon\|^8 + 4a\|u_\epsilon\|^2 \int_{\mathbb{R}^3} u_\epsilon^6 dx}}{2\int_{\mathbb{R}^3} u_\epsilon^6 dx} \right)^{\frac{1}{2}}.$$

Note that $g(t)$ is increasing in the interval $[0, T_\epsilon]$, then by (2.7) and (2.8), we obtain

$$\begin{aligned} h(t_\epsilon u_\epsilon) &\le g(T_\epsilon) \\ &= \frac{ab\|u_\epsilon\|^6}{4\int_{\mathbb{R}^3} u_\epsilon^6 dx} + \frac{b^3\|u_\epsilon\|^{12}}{24(\int_{\mathbb{R}^3} u_\epsilon^6 dx)^2} + \frac{(b^2\|u_\epsilon\|^8 + 4a\|u_\epsilon\|^2 \int_{\mathbb{R}^3} u_\epsilon^6 dx)^{\frac{3}{2}}}{24(\int_{\mathbb{R}^3} u_\epsilon^6 dx)^2} \\ &= \frac{abS^3}{4} + \frac{b^3 S^6}{24} + \frac{(b^2 S^4 + 4aS)^{\frac{3}{2}}}{24} + O(\epsilon^{\frac{1}{2}}) \\ &= \Lambda + O(\epsilon^{\frac{1}{2}}). \end{aligned}$$

Hence there exists $C_3 > 0$ (independent of ϵ, λ) such that (2.14) holds.

It is easy to show that $\{x|\,|x - x_0| < \rho_1\} \subset \{x|\,|t_\epsilon u_\epsilon(x)| \ge \delta_0\}$. Since $4 < p, q < 6$, together with (k_2) and (2.2), when $0 < \epsilon < \min\left\{ \left(\frac{t_0 \varphi(x_0)\cdot c}{\delta_0}\right)^4, \rho_1^2 \right\}$, we get

$$\begin{aligned} \int_{\mathbb{R}^3} k(x) G(t_\epsilon u_\epsilon) dx &= \int_{|t_\epsilon u_\epsilon(x_0) \le \delta_0|} k(x) G(t_\epsilon u_\epsilon) dx + \int_{|t_\epsilon u_\epsilon(x_0) \ge \delta_0|} k(x) G(t_\epsilon u_\epsilon) dx \\ &\ge \int_{|t_\epsilon u_\epsilon(x_0) \le \delta_0|} k(x) b_3 |t_\epsilon u_\epsilon|^q dx + \int_{|t_\epsilon u_\epsilon(x_0) \ge \delta_0|} k(x) b_4 |t_\epsilon u_\epsilon|^p dx \\ &\ge \int_{|t_\epsilon u_\epsilon(x_0) \ge \delta_0|} k(x) b_4 |t_\epsilon u_\epsilon|^p dx \qquad (2.15) \\ &\ge b_4 \cdot C^p \cdot \delta_1 \cdot t_\epsilon^p \int_{|x - x_0| < \rho_1} \frac{\varphi^p |x - x_0|^{-\beta} \epsilon^{\frac{p}{4}}}{(\epsilon + |x - x_0|^2)^{\frac{p}{2}}} dx \end{aligned}$$

$$\geq C_4 \cdot \epsilon^{\frac{p}{4}} \cdot t_\epsilon^p \int_0^{\rho_1} \frac{r^2}{r^\beta(\epsilon + r^2)^{\frac{p}{2}}} dr$$
$$\geq C_5 \cdot \epsilon^{\frac{3}{2} - \frac{p}{4} - \frac{\beta}{2}} \cdot t_\epsilon^p,$$

where C_4, C_5 are independent of ϵ, λ. Since $3 - \frac{p}{2} < \beta < 3$, then $\frac{6-p-2\beta}{2} < 0$. Let $\epsilon = \lambda^2$, $0 < \lambda < \lambda_0 = (\frac{C_5 T_1^2}{C_0 + C_3})^{\frac{6-p-2\beta}{2}}$, by using (2.14) and (2.15), we deduce

$$\begin{aligned} I_\lambda(t_\epsilon u_\epsilon) &= h(t_\epsilon u_\epsilon) - \lambda \int_{\mathbb{R}^3} k(x) G(t_\epsilon u_\epsilon) dx \\ &\leq \Lambda + C_3 \epsilon^{\frac{1}{2}} - \lambda C_5 \cdot \epsilon^{\frac{3}{2} - \frac{p}{4} - \frac{\beta}{2}} \cdot T_1^2 \\ &< \Lambda - C_0 \lambda. \end{aligned}$$

Then $I_\lambda(t_\epsilon u_\epsilon) < \Lambda - C_0\lambda$. Consequently, there exists $u_1 \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that (2.9) holds. This completes the proof. $\square$

Theorem 2.6. *Under the hypotheses of Theorem 1.1, the problem* (1.1) *has at least a positive solution.*

Proof. For $4 < p, q < 6$, there exists $\delta_2 > 0$ such that $\Lambda - C_0\lambda > 0$ for every $\lambda \in (0, \delta_2)$. Set $\lambda_* = \min\{\delta_2, \lambda_0\}$, then Lemmas 2.1-2.5 hold when $\lambda \in (0, \lambda_*)$. By Lemma 2.2, we can conclude that I_λ satisfies the mountain-pass geometry, so there exists a $(PS)_c$ sequence $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^3)$, such that

$$I_\lambda(u_n) \to c > \alpha, \qquad I_\lambda'(u_n) \to 0, \tag{2.16}$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

and

$$\Gamma = \{\gamma \in \mathrm{C}([0,1], \mathcal{D}^{1,2}(\mathbb{R}^3)) : \gamma(0) = 0, \gamma(1) = \mathrm{u}_0\}.$$

By Lemma 2.5, we obtain that

$$0 < \alpha < c \leq \max_{t \in [0,1]} I_\lambda(t\tilde{u}) \leq \sup_{t \geq 0} I_\lambda(t\tilde{u}) < \Lambda - C_0\lambda. \tag{2.17}$$

Furthermore, by Lemma 2.4, $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Then $\{u_n\}$ is bounded in $L^6(\mathbb{R}^3)$ and there exists $u_* \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$u_n \rightharpoonup u_*, \quad \text{weakly in } \mathcal{D}^{1,2}(\mathbb{R}^3).$$

In the following, we only need to prove $u_n \to u_*$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, then Theorem 2.6 is obtained. Let us first prove that, for all $\phi \in \mathcal{D}^{1,2}(\mathbb{R}^3)$, there holds

$$\int_{\mathbb{R}^3} k(x) g(u_n) \cdot \phi(x) dx \to \int_{\mathbb{R}^3} k(x) g(u_*) \cdot \phi(x) dx. \tag{2.18}$$

For any $\epsilon > 0$, since $k \in L^{\frac{6}{6-q}}(\mathbb{R}^3) \bigcap L^{\frac{6}{6-p}}(\mathbb{R}^3)$, there exists $\rho_2 \equiv \rho_2(\epsilon) > 0$ such that

$$|k|_{\frac{6}{6-q}, \mathbb{R}^3 \setminus B_{\rho_2}(0)} < \epsilon \ \text{ and } \ |k|_{\frac{6}{6-p}, \mathbb{R}^3 \setminus B_{\rho_2}(0)} < \epsilon.$$

On one hand, by using the boundedness of the sequence $\{u_n\}$, we deduce that

$$\int_{\mathbb{R}^3\setminus B_{\rho_2}(0)} k(x)(g(u_n)-g(u_*))\cdot\phi dx \le C(\phi)\epsilon. \tag{2.19}$$

On the other hand, since g is continuous and $u_n \to u_*$ strongly in $L^p_{loc}(\mathbb{R}^3)$, we have

$$g(u_n)\to g(u_*) \ \text{ strongly in } \ L^p_{loc}(\mathbb{R}^3).$$

By the Hölder inequality, for any $\epsilon > 0$, we have

$$\int_{B_{\rho_2}(0)} k(x)(g(u_n)-g(u_*))\cdot\phi(x)dx \le \hat{C}(\phi)\epsilon \tag{2.20}$$

for n large enough. Then from (2.19), (2.20) and the arbitrary choice of ϵ, we deduce that (2.18) holds. Similarly we also get

$$\int_{\mathbb{R}^3} u_n^5\phi(x)dx \to \int_{\mathbb{R}^3} u_*^5\phi(x)dx. \tag{2.21}$$

Set $\lim_{n\to\infty}\|u_n\| = l$, by (2.18) and (2.21), for any $\phi\in\mathcal{D}^{1,2}(\mathbb{R}^3)$ as $n\to\infty$ there holds

$$\begin{aligned}\langle I_\lambda'(u_n),\phi\rangle &= (a+b\|u_n\|^2)\int_{\mathbb{R}^3}(\nabla u_n,\nabla\phi)dx-\int_{\mathbb{R}^3}u_n^5\phi dx-\lambda\int_{\mathbb{R}^3}k(x)g(u_n)\cdot\phi dx\\ &\to (a+bl^2)\int_{\mathbb{R}^3}(\nabla u_*,\nabla\phi)dx-\int_{\mathbb{R}^3}u_*^5\phi dx-\lambda\int_{\mathbb{R}^3}k(x)g(u_*)\cdot\phi dx\\ &= \langle I_\lambda'(u_*),\phi\rangle.\end{aligned}$$

Thus from (2.16) we get that $\langle I_\lambda'(u_*),\phi\rangle = 0$ for all $\phi\in\mathcal{D}^{1,2}(\mathbb{R}^3)$.

In particular, let $u_* = \phi$, we have

$$(a+bl^2)\|u_*\|^2 - \int_{\mathbb{R}^3} u_*^6 dx - \lambda\int_{\mathbb{R}^3} k(x)g(u_*)u_* dx = 0. \tag{2.22}$$

Since $\langle I_\lambda'(u_n),u_n\rangle\to 0$, there holds

$$(a+b\|u_n\|^2)\|u_n\|^2 - \int_{\mathbb{R}^3} u_n^6 dx - \lambda\int_{\mathbb{R}^3} k(x)g(u_n)u_n dx = o(1).$$

Combining this with Lemma 2.4, we get that

$$(a+bl^2)l^2 - \int_{\mathbb{R}^3} u_*^6 dx - \lambda\int_{\mathbb{R}^3} k(x)g(u_*)u_* dx = 0. \tag{2.23}$$

Consequently, by (2.22) and (2.23), we have $\|u_*\| = l$. That is $\lim_{n\to\infty}\|u_n\| = \|u_*\|$. Moreover, $\|u_n - u_*\|\to 0$ implies that $u_n\to u_*$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Thus,

$$I_\lambda(u_*) = \lim_{n\to\infty} I_\lambda(u_n) = c > \alpha > 0. \tag{2.24}$$

It follows from (2.24) that $u_* \not\equiv 0$, then $u_*\ge 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. The strong maximum principle implies that u_* is a positive solution of (1.1). This completes the proof. □

3. Proof of Theorem 1.1

In this section, we shall prove that problem (1.1) has at least one positive ground state solution via variational method.

Proof of Theorem 1.1. By Theorem 2.6, we have obtained the existence of positive solution for (1.1). In this section, we will consider the existence of positive ground state solution.

Let

$$m = \inf\{I_\lambda(u) : u \in \mathcal{D}^{1,2}(\mathbb{R}^3),\ u \neq 0,\ I_\lambda^{'}(u) = 0\}.$$

From the definition of m, there exists $\{v_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^3)\backslash\{0\}$ such that

$$I_\lambda(v_n) \to m, \quad I_\lambda^{'}(v_n) \to 0, \quad n \to \infty. \tag{3.1}$$

Obviously from (3.1), we can easily deduce that $\{v_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Then there exist a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) and $v_* \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup v_* \quad \text{weakly in} \quad \mathcal{D}^{1,2}(\mathbb{R}^3).$$

We claim $v_* \neq 0$. Otherwise $v_n \rightharpoonup 0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$, then by Lemma 2.1 we get that

$$\int_{\mathbb{R}^3} k(x)g(v_n)v_n dx \to 0.$$

From (3.1), it follows that

$$a\|v_n\|^2 + b\|v_n\|^4 - \int_{\mathbb{R}^3} |v_n|^6 dx - \lambda \int_{\mathbb{R}^3} k(x)g(v_n)v_n dx = o(1).$$

Therefore,

$$a\|v_n\|^2 + b\|v_n\|^4 - \int_{\mathbb{R}^3} |v_n|^6 dx = o(1). \tag{3.2}$$

Set $\lim_{n\to\infty} \|v_n\| = l$. By (3.2) and (1.3), we can conclude that

$$l^2 \geq \frac{bS^3 + \sqrt{b^2S^6 + 4aS^3}}{2}.$$

Next we claim that

$$\int_{\mathbb{R}^3} k(x)G(v_n)dx \to 0. \tag{3.3}$$

Indeed, we just need to prove

$$\int_{\mathbb{R}^3} k(x)b_1|v_n|^q dx + \int_{\mathbb{R}^3} k(x)b_2|v_n|^p dx \to 0.$$

Since $k \in L^{\frac{6}{6-q}}(\mathbb{R}^3)\bigcap L^{\frac{6}{6-p}}(\mathbb{R}^3)$, for any $\epsilon > 0$, there exists $\rho_3 \equiv \rho_3(\epsilon) > 0$ such that

$$|k|_{\frac{6}{6-q},\mathbb{R}^3\backslash B_{\rho_3}(0)} < \epsilon \ \text{ and } \ |k|_{\frac{6}{6-p},\mathbb{R}^3\backslash B_{\rho_2}(0)} < \epsilon.$$

On one hand, by the boundedness of the sequence $\{v_n\}$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and Hölder inequality, one has

$$b_1 \int_{\mathbb{R}^3\setminus B_{\rho_3}(0)} k(x)|v_n|^q dx \le C_6\epsilon. \tag{3.4}$$

On the other hand, we can conclude

$$b_1 \int_{B_{\rho_3}(0)} k(x)|v_n|^q dx \le \hat{C}_6\epsilon \tag{3.5}$$

for large n. Then by (3.4), (3.5) and the arbitrary choice of ϵ, one gets

$$b_1 \int_{\mathbb{R}^3} k(x)|v_n|^q dx \to 0, \quad n \to \infty.$$

Similarly, we can derive that

$$b_2 \int_{\mathbb{R}^3} k(x)|v_n|^p dx \to 0, \quad n \to \infty.$$

Therefore (3.3) holds.

By (3.1), (3.2) and (3.3), we have

$$\begin{aligned} m &= \lim_{n\to\infty} \left\{ \frac{a}{2}\|v_n\|^2 + \frac{b}{4}\|v_n\|^4 - \frac{1}{6}\int_{\mathbb{R}^3} |v_n|^6 dx - \lambda \int_{\mathbb{R}^3} k(x)G(v_n)dx \right\} \\ &= \lim_{n\to\infty} \left\{ \frac{a}{3}\|v_n\|^2 + \frac{b}{12}\|v_n\|^4 \right\} \\ &\ge \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(b^2S^4+4aS)^{\frac{3}{2}}}{24} \\ &= \Lambda. \end{aligned}$$

Together with Lemma 2.5, $\Lambda \le m < \Lambda - C_0\lambda$, this is a contradiction. Then $v_n \rightharpoonup v_* \neq 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $I_\lambda^{'}(v_*) = 0$. From Theorem 2.6, we have concluded that $v_n \to v_*$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and v_* is a positive solution of (1.1), which implies that $I_\lambda(v_*) \ge m$. Now to show $m \ge I_\lambda(v_*)$.

Indeed, since $I_\lambda^{'}(v_*) = 0$ we have

$$\begin{aligned} I_\lambda(v_*) &= I_\lambda(v_*) - \frac{1}{6}\langle I_\lambda^{'}(v_*), v_*\rangle \\ &= \frac{a}{2}\|v_*\|^2 + \frac{b}{4}\|v_*\|^4 - \frac{1}{6}\int_{\mathbb{R}^3} |v_*|^6 dx - \lambda\int_{\mathbb{R}^3} k(x)G(v_*)dx \\ &\quad - \frac{1}{6}\left\{ a\|v_*\|^2 + b\|v_*\|^4 - \int_{\mathbb{R}^3} |v_*|^6 dx - \lambda\int_{\mathbb{R}^3} k(x)g(v_*)v_* dx \right\} \\ &= \frac{a}{3}\|v_*\|^2 + \frac{b}{12}\|v_*\|^4 - \lambda\int_{\mathbb{R}^3} k(x)\left(G(v_*) - \frac{1}{6}g(v_*)v_*\right)dx. \end{aligned}$$

By $I_\lambda(v_n) \to m$, we get

$$\begin{aligned} m + o(1) &= I_\lambda(v_n) \\ &= \frac{a}{3}\|v_n\|^2 + \frac{b}{12}\|v_n\|^4 - \lambda \int_{\mathbb{R}^3} k(x)\left(G(v_n) - \frac{1}{6}g(v_n)v_n\right) dx. \end{aligned}$$

It deduces from Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} k(x)G(v_n)dx = \int_{\mathbb{R}^3} k(x)G(v_*)dx.$$

Combining this with Fatou's Lemma, we obtain that

$$m \geq \frac{a}{3}\|v_*\|^2 + \frac{b}{12}\|v_*\|^4 - \lambda \int_{\mathbb{R}^3} k(x)\left(G(v_*) - \frac{1}{6}g(v_*)v_*\right) dx = I_\lambda(v_*).$$

Thus $v_* \neq 0$ satisfies $I'_\lambda(v_*) = 0$ and $I_\lambda(v_*) = m$, which implies that v_* is a positive ground state solution of (1.1). Therefore, we obtain the existence of positive ground state solution of (1.1). □

Remark 3.1. In the next paper, we hope to investigate the sign-changing solution for Kirchhoff type equation as paper [14, 17, 18] by using the method of invariant sets of descending flow or by means of a constraint variational methods.

Acknowledgements. The authors would like to express sincere thanks to the anonymous referee for his/her carefully reading of the manuscript and valuable comments and suggestions.

References

[1] G. Cerami and G. Vaira, *Positive solutions for some non-autonomous Schrödinger-Poisson systems*, J. Differential Equations **248** (2010), no. 3, 521–543. https://doi.org/10.1016/j.jde.2009.06.017

[2] G. M. Figueiredo, *Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument*, J. Math. Anal. Appl. **401** (2013), no. 2, 706–713. https://doi.org/10.1016/j.jmaa.2012.12.053

[3] X. He and W. Zou, *Existence and concentration behavior of positive solutions for a Kirchhoff equation in* $\mathbb{R}^3$, J. Differential Equations **252** (2012), no. 2, 1813–1834. https://doi.org/10.1016/j.jde.2011.08.035

[4] T. Hu and L. Lu, *Multiplicity of positive solutions for Kirchhoff type problems in* $\mathbb{R}^3$, Topol. Methods Nonlinear Anal. **50** (2017), no. 1, 231–252. https://doi.org/10.12775/tmna.2017.028

[5] Y. Huang, Z. Liu, and Y. Wu, *On finding solutions of a Kirchhoff type problem*, Proc. Amer. Math. Soc. **144** (2016), no. 7, 3019–3033. https://doi.org/10.1090/proc/12946

[6] L. Huang, E. M. Rocha, and J. Chen, *On the Schrödinger-Poisson system with a general indefinite nonlinearity*, Nonlinear Anal. Real World Appl. **28** (2016), 1–19. https://doi.org/10.1016/j.nonrwa.2015.09.001

[7] C.-Y. Lei, J.-F. Liao, and C.-L. Tang, *Multiple positive solutions for Kirchhoff type of problems with singularity and critical exponents*, J. Math. Anal. Appl. **421** (2015), no. 1, 521–538. https://doi.org/10.1016/j.jmaa.2014.07.031

[8] C.-Y. Lei, G.-S. Liu, and L.-T. Guo, *Multiple positive solutions for a Kirchhoff type problem with a critical nonlinearity*, Nonlinear Anal. Real World Appl. **31** (2016), 343–355. https://doi.org/10.1016/j.nonrwa.2016.01.018
[9] C.-Y. Lei, H. Suo, C. Chu, and L. Guo, *On ground state solutions for a Kirchhoff type equation with critical growth*, Comput. Math. Appl. **72** (2016), no. 3, 729–740. https://doi.org/10.1016/j.camwa.2016.05.027
[10] G. Li and H. Ye, *Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in* $\mathbb{R}^3$, J. Differential Equations **257** (2014), no. 2, 566–600. https://doi.org/10.1016/j.jde.2014.04.011
[11] S. Liang and J. Zhang, *Existence of solutions for Kirchhoff type problems with critical nonlinearity in* $\mathbb{R}^3$, Nonlinear Anal. Real World Appl. **17** (2014), 126–136. https://doi.org/10.1016/j.nonrwa.2013.10.011
[12] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana **1** (1985), no. 1, 145–201. https://doi.org/10.4171/RMI/6
[13] P.-L. Lions, *The concentration-compactness principle in the calculus of variations, the limit case, Part II*, Rev. Mat. Iberoam. **2** (1985), no. 2, 45–121. https://doi.org/10.4171/RMI/12
[14] Z. Liu and C. Luo, *Existence of positive ground state solutions for Kirchhoff type equation with general critical growth*, Topol. Methods Nonlinear Anal. **49** (2017), no. 1, 165–182. https://doi.org/10.12775/tmna.2016.068
[15] Z. Liu, Z.-Q. Wang, and J. Zhang, *Infinitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system*, Ann. Mat. Pura Appl. (4) **195** (2016), no. 3, 775–794. https://doi.org/10.1007/s10231-015-0489-8
[16] A. Ourraoui, *On a p-Kirchhoff problem involving a critical nonlinearity*, C. R. Math. Acad. Sci. Paris **352** (2014), no. 4, 295–298. https://doi.org/10.1016/j.crma.2014.01.015
[17] A. Qian, J. Liu, and A. Mao, *Ground state and nodal solutions for a Schrödinger-Poisson equation with critical growth*, J. Math. Phys. **59** (2018), no. 12, 121509, 20 pp. https://doi.org/10.1063/1.5050856
[18] M. Shao and A. Mao, *Signed and sign-changing solutions of Kirchhoff type problems*, J. Fixed Point Theory Appl. **20** (2018), no. 1, Paper No. 2, 20 pp. https://doi.org/10.1007/s11784-018-0486-9
[19] J. Wang, L. Tian, J. Xu, and F. Zhang, *Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth*, J. Differential Equations **253** (2012), no. 7, 2314–2351. https://doi.org/10.1016/j.jde.2012.05.023
[20] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser Boston, Inc., Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-4146-1

Caixia Chen
Mathematics Department of Jining University
Shandong 273155, P. R. China
Email address: 2426905518@qq.com

Aixia Qian
School of Mathematical Sciences
Qufu Normal University
Shandong 273165, P. R. China
Email address: qaixia@qfnu.edu.cn