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ON COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF COORDINATEWISE NEGATIVELY ASSOCIATED RANDOM VECTORS IN HILBERT SPACES

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Abstract. This paper establishes the Baum–Katz type theorem and the Marcinkiewicz–Zymund type strong law of large numbers for sequences of coordinatewise negatively associated and identically distributed random vectors $\{X, X_n, n \geq 1\}$ taking values in a Hilbert space H with general normalizing constants $b_n = n^\alpha \widetilde{L}(n^\alpha)$, where $\widetilde{L}(\cdot)$ is the de Bruijn conjugate of a slowly varying function $L(\cdot)$. The main result extends and unifies many results in the literature. The sharpness of the result is illustrated by two examples.

1. Introduction

The concept of negative association of random variables was introduced by Joag-Dev and Proschan [16]. A collection $\{X_1, X_2, \ldots, X_n\}$ of (real-valued) random variables is said to be negatively associated (NA) if for any disjoint subsets I, J of $\{1, 2, \ldots, n\}$ and any real coordinatewise nondecreasing functions f on $\mathbb{R}^{|I|}$ and g on $\mathbb{R}^{|J|}$,

$$\mathrm{Cov}(f(X_k, k \in I), g(X_k, k \in J)) \leq 0$$

whenever the covariance exists, where $|I|$ denotes the cardinality of I. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be negatively associated if every finite subfamily is negatively associated.

This concept was first extended to Hilbert space-valued random vectors by Burton et al. [5]. After that, a literature of investigation concerning the limit theorems for negatively associated random vectors in Hilbert spaces has emerged, including the law of the iterated logarithm (Dabrowski and Dehling [8]), strong law of large numbers (Ko et al. [18], Thanh [31]), complete convergence (Hien et al. [11], Huan et al. [15], Wu et al. [38]), and weak laws of large numbers (Anh and Hien [2], Dung et al. [19], Hien and Thanh [10], Kim [17]). The notion of coordinatewise negatively associated random vectors in Hilbert

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spaces was introduced by Huan et al. [15]. Let H be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, the corresponding norm $\| \cdot \|$, and orthonormal basis $\{e_j, j \in B\}$. A sequence $\{X_n, n \geq 1\}$ of random vectors taking values in H is said to be coordinatewise negatively associated (CNA) if for each $j \in B$, the sequence of real-valued random variables $\{\langle X_n, e_j \rangle, n \geq 1\}$ is NA.

This paper aims to establish complete convergence and strong law of large numbers for weighted sums of coordinatewise negatively associated random vectors taking values in Hilbert spaces. Let $1 \leq p < 2$, $\alpha p \geq 1$, and let $\{X_n, n \geq 1\}$ be a sequence of coordinatewise negatively associated and identically distributed random vectors in H. By using some results related to slowly varying functions and techniques developed by Anh et al. [3], we provide the sufficient conditions for

$$(1) \qquad \sum_{n \geq A^{1/\alpha}} n^{\alpha p - 2} P\left(\max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} a_{ni} X_i \right\| > \varepsilon n^{\alpha} \widetilde{L}(n^{\alpha}) \right) < \infty \quad \text{for all} \quad \varepsilon > 0,$$

where $\widetilde{L}(\cdot)$ is the de Bruijn conjugate of a slowly varying function $L(\cdot)$ defined on $[A, \infty)$ for some $A > 0$. This result is new even when the random variables are independent and identically distributed. If $\alpha p = 1$ and $L(x) = \log^{-1/\gamma}(x)$, $x \geq 2$, then we obtain optimal moment conditions for complete convergence results of Sung [30], and Chen and Sung [7]. If $L(x) = \tilde{L}(x) = 1$ for all $x \geq 0$, then our main result reduces to Theorem 2.1 of Huan et al. [15]. The Baum–Katz type theorem for real-valued random variables with regularly varying normalizing constants was recently considered in Thành [32] and Anh et al. [3]. When H is a finite dimensional Hilbert space, the sufficient conditions for (1) are also necessary. Therefore, we generalize Theorem 3.1 of Anh et al. [3] to the finite dimensional Hilbert space-valued random vectors by letting $\alpha = 1/p$.

The Baum–Katz type theorem and the law of large numbers for weighted sums of dependent random variables and their applications in statistics were studied widely in the past decades. For more details, we refer the readers to [6,7,13,30,34–37], among others. The limit theorems, including the law of large numbers and the central limit theorem, for random variables taking values in Hilbert spaces and Banach spaces were also studied by many authors. We refer to Ledoux and Talagrand [20], Marcus and Woyczyński [21], and Pisier [25] for this topic. For complete convergence and laws of large numbers, many results were developed, see by Adler et al. [1], Hong et al. [12], Hu et al. [14], Rosalsky and Thanh ([26, 27]), Rosalsky et al. [28], and the references therein.

Throughout this paper, H denotes a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, the corresponding norm $\| \cdot \|$, and orthonormal basis $\{e_j, j \in B\}$. The symbol C denotes a generic positive constant whose value may be different for each appearance. By saying $\{X_n, n \geq 1\}$ is a sequence of CNA random vectors in H, we mean that the random vectors are CNA with respect to the orthonormal basis $\{e_j, j \in B\}$. For $x \geq 0$, $\log(x)$ denotes the natural logarithm of $\max\{x, e\}$.

2. Preliminaries

In this section, we present some lemmas which will be used to prove our main result. Firstly, we present the Kolmogorov type maximal inequality for CNA random vectors, see Huan et al. [15] for a proof.

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of CNA mean 0 random vectors in H satisfying $E\|X_n\|^2 < \infty$ for all $n \geq 1$. Then for any $n \geq 1$, we have*

$$E\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} X_i\right\|^2\right) \leq 2\sum_{i=1}^{n} E\|X_i\|^2.$$

Next, we present some technical results concerning slowly varying functions. A real-valued function $R(\cdot)$ is said to be regularly varying with the index of regular variation ρ ($\rho \in \mathbb{R}$) if it is a positive and measurable function on $[A, \infty)$ for some $A \geq 0$, and for each $\lambda > 0$,

$$\lim_{x\to\infty} \frac{R(\lambda x)}{R(x)} = \lambda^\rho. \tag{2}$$

A regularly varying function with the index of regular variation $\rho = 0$ is called slowly varying. It is well known that a function $R(\cdot)$ is regularly varying with the index of regular variation ρ if and only if it can be written in the form

$$R(x) = x^\rho L(x), \tag{3}$$

where $L(\cdot)$ is a slowly varying function. On the regularly varying functions and their important role in probability, we refer to Seneta [29].

Let $L(\cdot)$ be a slowly varying function. Then by Theorem 1.5.13 of Bingham et al. [4], there exists a slowly varying function $\widetilde{L}(\cdot)$, unique up to asymptotic equivalence, satisfying

$$\lim_{x\to\infty} L(x)\widetilde{L}\big(xL(x)\big) = 1 \text{ and } \lim_{x\to\infty} \widetilde{L}(x)L\big(x\widetilde{L}(x)\big) = 1. \tag{4}$$

The function $\widetilde{L}$ is called the de Bruijn conjugate of L, and $\big(L, \widetilde{L}\big)$ is called a (slowly varying) conjugate pair (see, Bingham et al. [4], page 29). By [4, Proposition 1.5.14], if $\left(L, \widetilde{L}\right)$ is a conjugate pair, then for $a, b, \alpha > 0$, each of $\left(L(ax), \widetilde{L}(bx)\right)$, $\left(aL(x), a^{-1}\widetilde{L}(x)\right)$, $\left((L(x^\alpha))^{1/\alpha}, (\widetilde{L}(x^\alpha))^{1/\alpha}\right)$ is a conjugate pair. In this paper, for a slowly varying function $L(\cdot)$ defined on $[A, \infty)$ for some $A > 0$, we denote the de Bruijn conjugate of $L(\cdot)$ by $\widetilde{L}(\cdot)$. Without loss of generality, we assume that $\widetilde{L}(\cdot)$ is also defined on $[A, \infty)$ and that $L(x)$ and $\widetilde{L}(x)$ are both bounded on finite closed intervals.

The following lemma shows that we can approximate a slowly varying function $L(\cdot)$ by a differentiable slowly varying function $L_1(\cdot)$. See page 111 of Galambos and Seneta [9] for a proof.

Lemma 2.2. *For any slowly varying function $L(\cdot)$ defined on $[A,\infty)$ for some $A \ge 0$, there exists a differentiable slowly varying function $L_1(\cdot)$ defined on $[B,\infty)$ for some $B \ge A$ such that*

$$\lim_{x\to\infty} \frac{L(x)}{L_1(x)} = 1 \quad \textit{and} \quad \lim_{x\to\infty} \frac{xL_1'(x)}{L_1(x)} = 0.$$

Conversely, if $L(\cdot)$ is a positive differentiable function satisfying

$$\lim_{x\to\infty} \frac{xL'(x)}{L(x)} = 0, \tag{5}$$

then $L(\cdot)$ is a slowly varying function.

The first part of Lemma 2.3 below can be found on page 18 of Seneta [29], while the other parts were proved recently by Anh et al. [3].

Lemma 2.3. *Let $p > 0$ and let $L(\cdot)$ be a slowly varying function defined on $[A,\infty)$ for some $A > 0$. Then*

(i) $\lim_{x\to\infty} x^p L(x) = \infty$ *and* $\lim_{x\to\infty} x^{-p} L(x) = 0$.

(ii) *If $L(\cdot)$ is a differentiable slowly varying function satisfying*

$$\lim_{x\to\infty} \frac{xL'(x)}{L(x)} = 0,$$

then there exists $B \ge A$ such that $x^p L(x)$ is increasing on $[B,\infty)$, $x^{-p}L(x)$ is decreasing on $[B,\infty)$.

(iii) *For all $\lambda > 0$,*

$$\lim_{x\to\infty} \frac{L(x)}{L(x+\lambda)} = 1.$$

The following lemma is a direct consequence of Karamata's theorem (see Proposition 1.5.10 of Bingham et al. [4]). The first part of the lemma can also see in [24, Proposition 2.2.1].

Lemma 2.4. (i) *Let $p > -1$ and let $L(\cdot)$ be a slowly varying function defined on $[A,\infty)$ for some $A > 0$. Then*

$$\sum_{k=1}^{n} k^p L(k+A) \sim \frac{L(n+A)n^{p+1}}{p+1} \quad \textit{as } n\to\infty.$$

(ii) *Let $p > 1$, $q \in \mathbb{R}$ and $L(\cdot)$ be a differentiable slowly varying function defined on $[A,\infty)$ for some $A > 0$. Then*

$$\sum_{k=n}^{\infty} \frac{L^q(k)}{k^p} \sim \frac{L^q(n)}{(p-1)n^{p-1}}.$$

3. Main results

With the preliminaries accounted for, the main results may now be presented. In the following theorem, we establish complete convergence for weighted sums of CNA and identically distributed random vectors in Hilbert spaces. This result extends Theorem 2.1 of Huan et al. [15] to the weighted sums with regularly varying norming constants. We note that Huan et al. considered the case where the sequence $\{X_n, n \geq 1\}$ is coordinatewise weakly upper bounded by a random vector X which is more general than the case where random vectors X_n, $n \geq 1$, are identically distributed. For simplicity, we consider the identically distributed case, though our result can be generalized to coordinatewise weakly upper bounded case as considered in Huan et al. [15] by the same method.

Theorem 3.1. *Let $1 \leq p < 2$, $\alpha p \geq 1$, $\{X, X_n, n \geq 1\}$ be a sequence of CNA and identically distributed random vectors in H and $L(x)$ be a slowly varying function defined on $[A, \infty)$ for some $A > 0$. When $p = 1$, we assume further that $L(x)$ is increasing on $[A, \infty)$. Let $b_n = n^\alpha \widetilde{L}(n^\alpha)$, $n \geq A^{1/\alpha}$ and $\{a_{ni}, n \geq 1, 1 \leq i \leq n\}$ be an array of constants satisfying*

$$\sum_{i=1}^{n} a_{ni}^2 \leq Cn \quad \textit{for all } n \geq 1. \tag{6}$$

If

$$E(X) = 0, \ \sum_{j \in B} E\left(|X^{(j)}|^p L^p(|X^{(j)}| + A)\right) < \infty, \tag{7}$$

then

$$\sum_{n \geq A^{1/\alpha}} n^{\alpha p - 2} P\left(\max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} a_{ni} X_i \right\| > \varepsilon b_n\right) < \infty \quad \textit{for all} \ \ \varepsilon > 0. \tag{8}$$

Proof. For simplicity, we assume that $A^{1/\alpha}$ is an integer number since we can take $[A^{1/\alpha}] + 1$ otherwise. We may also assume that $a_{ni} \geq 0, n \geq 1, 1 \leq i \leq n$ since in the general case we can use the decomposition $a_{ni} \equiv a_{ni}^+ - a_{ni}^-$. By Lemma 2.2 and Lemma 2.3, without loss of generality, we can assume that

$$f(x) = x^p L^p(x) \ \text{ is increasing on } [A, \infty), \tag{9}$$

$$g(x) = x^{p-1} L^p(x) \ \text{ is increasing on } [A, \infty), \tag{10}$$

$$h(x) = x^{p-2} L^p(x) \ \text{ is decreasing on } [A, \infty), \tag{11}$$

$$h(x) = x^\alpha L^\alpha(x) \ \text{ is increasing on } [A^{1/\alpha}, \infty). \tag{12}$$

We may also assume that $\widetilde{L}(A) \geq 1$. For $n \geq A^{1/\alpha}$, $j \in B$, set

$$X_{ni}^{(j)} = -b_n I(X_i^{(j)} < -b_n) + X_i^{(j)} I(|X_i^{(j)}| \leq b_n) + b_n I(X_i^{(j)} > b_n),$$

$$X_{ni} = \sum_{j \in B} X_{ni}^{(j)} e_j,\ 1 \le i \le n,$$

and

$$S_{nk} = \sum_{i=1}^{k} \Big(a_{ni} X_{ni} - E(a_{ni} X_{ni}) \Big),\ 1 \le k \le n.$$

Let $\varepsilon > 0$ be arbitrary. For $n \ge A^{1/\alpha}$,

$$\begin{aligned}
&P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} a_{ni}X_i\right\| > \varepsilon b_n\right)\\
&\le P\left(\bigcup_{i=1}^{n}\bigcup_{j\in B}|X_i^{(j)}| > b_n\right) + P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} a_{ni}X_{ni}\right\| > \varepsilon b_n\right)\\
&\le P\left(\bigcup_{i=1}^{n}\bigcup_{j\in B}|X_i^{(j)}| > b_n\right) + P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} E(a_{ni}X_{ni})\right\| > \varepsilon b_n/2\right)\\
&\quad + P\left(\max_{1\le k\le n}\|S_{nk}\| > \varepsilon b_n/2\right). \qquad (13)
\end{aligned}$$

We have

$$\begin{aligned}
&\sum_{n\ge A^{1/\alpha}} n^{\alpha p-2} P\Big(\bigcup_{i=1}^{n}\bigcup_{j\in B}|X_i^{(j)}| > b_n\Big)\\
&\le \sum_{n\ge A^{1/\alpha}} n^{\alpha p-2} \sum_{j\in B}\sum_{i=1}^{n} P\Big(|X_i^{(j)}| > b_n\Big)\\
&= \sum_{n\ge A^{1/\alpha}} \sum_{j\in B} n^{\alpha p-1} P(|X^{(j)}| > b_n)\\
&= \sum_{j\in B}\sum_{n\ge A^{1/\alpha}} n^{\alpha p-1} \sum_{i=n}^{\infty} P\left(b_i < |X^{(j)}| \le b_{i+1}\right)\\
&= \sum_{j\in B}\sum_{i\ge A^{1/\alpha}} \sum_{n=A^{1/\alpha}}^{i} n^{\alpha p-1} P\left(b_i < |X^{(j)}| \le b_{i+1}\right)\\
&\le C\sum_{j\in B}\sum_{i\ge A^{1/\alpha}} i^{\alpha p} P\left(b_i < |X^{(j)}| \le b_{i+1}\right)\\
&= C\sum_{j\in B}\sum_{i\ge A^{1/\alpha}} E i^{\alpha p} I\left(b_i < |X^{(j)}| \le b_{i+1}\right). \qquad (14)
\end{aligned}$$

For $j \in B$, $i \ge A^{1/\alpha}$ and for $\omega \in (b_i < |X^{(j)}| \le b_{i+1})$,

$$i^{\alpha p} = \frac{i^{\alpha p} b_i^p L^p(b_i)}{b_i^p L^p(b_i)} = \frac{b_i^p L^p(b_i)}{\widetilde{L}^p(i^\alpha) L^p(i^\alpha \widetilde{L}(i^\alpha))}$$

$$\leq Cb_i^p L^p(b_i) \leq C|X^{(j)}(\omega)|^p L^p(|X^{(j)}(\omega)| + A). \tag{15}$$

Combining (14) and (15), we have

$$\begin{aligned}
&\sum_{n\geq A^{1/\alpha}} n^{\alpha p-2} P\Big(\bigcup_{i=1}^{n}\bigcup_{j\in B} |X_i^{(j)}| > b_n\Big) \\
&\leq C\sum_{j\in B}\sum_{i\geq A^{1/\alpha}} E\Big(|X^{(j)}|^p L^p(|X^{(j)}|+A) I(b_i < |X^{(j)}| \leq b_{i+1})\Big) \\
&\leq C\sum_{j\in B} E\Big(|X^{(j)}|^p L^p(|X^{(j)}|+A)\Big) < \infty. \qquad (16)
\end{aligned}$$

For $n \geq 1$, by the Cauchy–Schwarz inequality and (6),

$$\Big(\sum_{i=1}^{n} |a_{ni}|\Big)^2 \leq n\Big(\sum_{i=1}^{n} a_{ni}^2\Big) \leq Cn^2. \tag{17}$$

For $n \geq A^{1/\alpha}$, the first half of (7) and (17) imply that

$$\begin{aligned}
&\frac{1}{b_n}\max_{1\leq k\leq n}\Big\|\sum_{i=1}^{k} E(a_{ni}X_{ni})\Big\| \\
&\leq \frac{1}{b_n}\sum_{i=1}^{n} \|E(a_{ni}X_{ni})\| \\
&\leq \frac{1}{b_n}\sum_{i=1}^{n}\sum_{j\in B} |a_{ni}| \left(\Big|EX_i^{(j)} I(|X_i^{(j)}| \leq b_n)\Big| + b_n P(|X_i^{(j)}| > b_n)\right) \\
&\leq \frac{1}{b_n} Cn \sum_{j\in B} \left(\Big|E(X^{(j)} I(|X^{(j)}| > b_n))\Big| + b_n P(|X^{(j)}| > b_n)\right) \\
&\leq \frac{1}{b_n} Cn \sum_{j\in B} E|X^{(j)}| I(|X^{(j)}| > b_n). \qquad (18)
\end{aligned}$$

For $j \in B$, n large enough and for $\omega \in (|X^{(j)}| > b_n)$,

$$\begin{aligned}
\frac{n}{b_n} &\leq \frac{n^{\alpha p}}{n^\alpha \widetilde{L}(n^\alpha)} \text{ (by } \alpha p \geq 1) \\
&= \frac{\left(n^\alpha \widetilde{L}(n^\alpha)\right)^{p-1} L^p\left(n^\alpha \widetilde{L}(n^\alpha)\right)}{\widetilde{L}^p(n^\alpha) L^p\left(n^\alpha \widetilde{L}(n^\alpha)\right)} \\
&\leq Cb_n^{p-1} L^p(b_n) \\
&\leq C|X^{(j)}(\omega)|^{p-1} L^p(|X^{(j)}(\omega)| + A). \qquad (19)
\end{aligned}$$

Combining (18), (19) and the second half of (7), we have

$$\begin{aligned}&\frac{1}{b_n}\max_{1\le k\le n}\left\|\sum_{i=1}^{k}E(a_{ni}X_{ni})\right\|\\&\le C\sum_{j\in B}E\left(|X^{(j)}|^pL^p(|X^{(j)}|+A)I\big(|X^{(j)}|>b_n\big)\right)\\&\to 0\ \text{ as } n\to\infty.\end{aligned}\tag{20}$$

From (13), (16) and (20), to obtain (8), it remains to show that

$$\sum_{n\ge A^{1/\alpha}}n^{\alpha p-2}P\Big(\max_{1\le k\le n}\|S_{nk}\|>b_n\varepsilon/2\Big)<\infty.\tag{21}$$

Set $b_{A^{1/\alpha}-1}=0$. We have

$$\begin{aligned}&\sum_{n\ge A^{1/\alpha}}n^{\alpha p-2}P\Big(\max_{1\le k\le n}\|S_{nk}\|>b_n\varepsilon/2\Big)\\
&\le\sum_{n\ge A^{1/\alpha}}\frac{4}{\varepsilon^2b_n^2}n^{\alpha p-2}E\Big(\max_{1\le k\le n}\|S_{nk}\|\Big)^2\qquad\text{(by Chebyshev's inequality)}\\
&\le\sum_{n\ge A^{1/\alpha}}\frac{8}{\varepsilon^2b_n^2}n^{\alpha p-2}\sum_{i=1}^{n}E\Big(a_{ni}X_{ni}-E(a_{ni}X_{ni})\Big)^2\qquad\text{(by Lemma 2.1)}\\
&\le\sum_{n\ge A^{1/\alpha}}\sum_{j\in B}\frac{8n^{\alpha p-2}\left(\sum_{i=1}^{n}a_{ni}^2\right)\left(E(X^{(j)})^2I(|X^{(j)}|\le b_n)+b_n^2P(|X^{(j)}|>b_n)\right)}{\varepsilon^2b_n^2}\\
&\le C\sum_{n\ge A^{1/\alpha}}\sum_{j\in B}n^{\alpha p-1}\left(\frac{E((X^{(j)})^2I(|X^{(j)}|\le b_n))}{b_n^2}+P(|X^{(j)}|>b_n)\right)\quad\text{(by (6))}\\
&\le C+C\sum_{n\ge A^{1/\alpha}}\sum_{j\in B}n^{\alpha p-2\alpha-1}\widetilde{L}^{-2}(n^\alpha)\sum_{A^{1/\alpha}\le i\le n}E\left((X^{(j)})^2I(b_{i-1}<|X^{(j)}|\le b_i)\right)\\
&\qquad\text{(by (14) and (16))}\\
&\le C+C\sum_{i\ge A^{1/\alpha}}\sum_{j\in B}\left(\sum_{n\ge i}n^{\alpha p-2\alpha-1}\widetilde{L}^{-2}(n^\alpha)\right)E\left((X^{(j)})^2I(b_{i-1}<|X^{(j)}|\le b_i)\right)\\
&\le C+C\sum_{j\in B}\sum_{i\ge A^{1/\alpha}}i^{\alpha p-2\alpha}\widetilde{L}^{-2}(i^\alpha)E\left((X^{(j)})^2I(b_{i-1}<|X^{(j)}|\le b_i)\right)\qquad(22)\\
&\qquad\text{(by Lemma 2.4).}\end{aligned}$$

For $j\in B$, $i\ge A^{1/\alpha}$ and for $\omega\in(b_{i-1}<|X^{(j)}|\le b_i)$,

$$\begin{aligned}i^{\alpha p-2\alpha}\widetilde{L}^{-2}(i^\alpha)&=\frac{i^{\alpha p-2\alpha}\widetilde{L}^{p-2}(i^\alpha)L^p\big(i^\alpha\widetilde{L}(i^\alpha)\big)}{\widetilde{L}^p(i^\alpha)L^p\big(i^\alpha\widetilde{L}(i^\alpha)\big)}\\&\le Ci^{\alpha p-2\alpha}\widetilde{L}^{p-2}(i^\alpha)L^p\big(i^\alpha\widetilde{L}(i^\alpha)\big)=Cb_i^{p-2}L^p(b_i)\\&\le C|X^{(j)}(\omega)|^{p-2}L^p\big(|X^{(j)}(\omega)|+A\big).\end{aligned}\tag{23}$$

Combining (22), (23) and the second half of (7), we have

$$\sum_{n\geq A^{1/\alpha}} n^{\alpha p-2}P\Big(\max_{1\leq k\leq n}\|S_{nk}\|>b_n\varepsilon/2\Big)\leq C+C\sum_{j\in B}E\left(|X^{(j)}|^pL^p(|X^{(j)}|+A)\right) < \infty$$

thereby proving (21). □

The following corollary is the Marcinkiewicz–Zymund type strong law of large number for sequences of CNA and identically distributed random vectors in Hilbert spaces.

Corollary 3.2. *Let $1\leq p<2$, $\{X, X_n, n\geq 1\}$ be a sequence of CNA and identically distributed random vectors in H and $L(x)$ be a slowly varying function defined on $[A,\infty)$ for some $A>0$. When $p=1$, we assume further that $L(x)$ is increasing on $[A,\infty)$. Let $b_n=n^{1/p}\widetilde{L}(n^{1/p})$, $n\geq A^p$. If (7) holds, then*

$$\lim_{n\to\infty}\frac{1}{b_n}\max_{1\leq k\leq n}\left\|\sum_{i=1}^k X_i\right\|=0 \quad a.s. \tag{24}$$

Proof. Without loss of generality, we can assume that

$$f(x)=x^{1/p}\widetilde{L}(x^{1/p}) \text{ is increasing on } [A,\infty). \tag{25}$$

For any $\varepsilon>0$, by applying Theorem 3.1, we have

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1}P\left(\max_{1\leq k\leq n}\left\|\sum_{j=1}^k X_j\right\|>\varepsilon b_n\right)\\
&= \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} n^{-1}P\left(\max_{1\leq k\leq n}\left\|\sum_{j=1}^k X_j\right\|>\varepsilon b_n\right)\\
&\geq \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} 2^{-i-1}P\left(\max_{1\leq k\leq 2^i}\left\|\sum_{j=1}^k X_j\right\|>\varepsilon b_{2^{i+1}}\right)\\
&= \frac{1}{2}\sum_{i=0}^{\infty}P\left(\max_{1\leq k\leq 2^i}\left\|\sum_{j=1}^k X_j\right\|>\varepsilon b_{2^{i+1}}\right). \qquad (26)
\end{aligned}$$

By the Borel–Cantelli lemma, (26) ensures that

$$\frac{1}{b_{2^{i+1}}}\max_{1\leq k\leq 2^i}\left\|\sum_{j=1}^k X_j\right\|\to 0 \text{ a.s. as } i\to\infty. \tag{27}$$

For $2^{i-1} \le n < 2^i$, we have

$$0 \le \frac{1}{b_n} \max_{1\le k\le n} \left\| \sum_{j=1}^{k} X_j \right\| \le \frac{C}{b_{2^{i+1}}} \max_{1\le k\le 2^i} \left\| \sum_{j=1}^{k} X_j \right\|. \tag{28}$$

The conclusion of the corollary follows from (27) and (28). □

If H is a finite dimensional Hilbert space, then the following theorem shows that (7) is also necessary for (24). This result extends Theorem 2.1 of Anh et al. [3] to the CNA case.

Theorem 3.3. *Let $1 \le p < 2$, $\alpha p \ge 1$ and H be a d-dimensional Hilbert space with orthognormal basis $\{e_1, e_2, \ldots, e_d\}$. Let $\{X, X_n, n \ge 1\}$ be a sequence of CNA and identically distributed random vectors in H and $L(x)$ be a slowly varying function defined on $[A,\infty)$ for some $A > 0$. When $p = 1$, we assume further that $L(x) \ge 1$ and is increasing on $[A,\infty)$. Let $b_n = n^\alpha \widetilde{L}(n^\alpha)$, $n \ge A^{1/\alpha}$. Then the following four statements are equivalent.*

(i) *The random vector X satisfies*

$$E(X) = 0,\ E\left(\|X\|^p L^p(\|X\| + A)\right) < \infty. \tag{29}$$

(ii) *For every array of constants $\{a_{ni}, n \ge 1, 1 \le i \le n\}$ satisfying*

$$\sum_{i=1}^{n} a_{ni}^2 \le Cn \quad \text{for all } n \ge 1, \tag{30}$$

we have

$$\sum_{n\ge A^{1/\alpha}} n^{\alpha p-2} P\left(\max_{1\le k\le n} \left\| \sum_{i=1}^{k} a_{ni} X_i \right\| > \varepsilon b_n\right) < \infty \text{ for all } \varepsilon > 0. \tag{31}$$

(iii)

$$\sum_{n\ge A^{1/\alpha}} n^{\alpha p-2} P\left(\max_{1\le k\le n} \left\| \sum_{i=1}^{k} X_i \right\| > \varepsilon b_n\right) < \infty \text{ for all } \varepsilon > 0. \tag{32}$$

(iv) *The strong law of large numbers*

$$\lim_{n\to\infty} \frac{\max_{1\le k\le n} \left\| \sum_{i=1}^{k} X_i \right\|}{n^{1/p}\widetilde{L}(n^{1/p})} = 0 \quad a.s. \tag{33}$$

holds.

Proof. Firstly, we prove the implication "(i)⇒(ii)". Assume that (29) holds and $\{a_{ni}, n \ge 1, 1 \le i \le n\}$ are constants satisfying (30). By Lemma 2.2 and Lemma 2.3, without loss of generality, we can assume that

$$f(x) = x^p L^p(x) \ \text{ is increasing on } [A,\infty). \tag{34}$$

For $j = 1, 2, \ldots, d$, by the second half of (29), we have

$$E\left(|X^{(j)}|^p L^p(|X^{(j)}| + A)\right) \leq CE\left(\|X\|^p L^p(\|X\| + A)\right) < \infty. \tag{35}$$

Combining (35) and the second half of (29), we have

$$\sum_{j=1}^{d} E\left(|X^{(j)}|^p L^p(|X^{(j)}| + A)\right) < \infty.$$

By Theorem 3.1, we obtain (31).

The implication "(ii)⇒(iii)" is immediate by letting $a_{ni} \equiv 1$. By the same argument in the proof of Corollary 3.2, we have "(iii)⇒(iv)".

Finally, we prove the implication "(iv)⇒(i)". We assume that (33) holds. For $n \geq 1$, set $c_n = n^{1/p}\widetilde{L}(n^{1/p})$. It follows from (33) that

$$\lim_{n\to\infty} \frac{\max_{1\leq k\leq n} \left|X_k^{(j)}\right|}{c_n} = 0 \text{ a.s. for all } j = 1, 2, \ldots, d. \tag{36}$$

Since $\{X_n, n \geq 1\}$ is a sequence of CNA random vectors, for all $j = 1, 2, \ldots, d$, we have $\{(X_n^{(j)} > c_n/d), n \geq 1\}$ are pairwise negatively correlated events, and so are $\{(X_n^{(j)} < -c_n/d), n \geq 1\}$. By the generalized Borel–Cantelli lemma (see Petrov [23]), it follows from (36) that

$$\sum_{n\geq A^{1/\alpha}} P(|X^{(j)}| > c_n/d) = \sum_{n\geq A^{1/\alpha}} P(|X_n^{(j)}| > c_n/d) < \infty \text{ for all } j \in B. \tag{37}$$

Hence,

$$\begin{aligned}
\sum_{n\geq A^{1/\alpha}} P(\|X\| > c_n) &\leq \sum_{n\geq A^{1/\alpha}} P\left(\sum_{j=1}^{d} |X^{(j)}| > c_n\right) \\
&\leq \sum_{n\geq A^{1/\alpha}} P\left(\bigcup_{j=1}^{d} (|X^{(j)}| > c_n/d)\right) \\
&\leq \sum_{n\geq A^{1/\alpha}} \sum_{j=1}^{d} P(|X^{(j)}| > c_n/d) \\
&= \sum_{j=1}^{d} \sum_{n\geq A^{1/\alpha}} P(|X^{(j)}| > c_n/d) < \infty.
\end{aligned}$$

Following the arguments in the proof of Theorem 3.1 to get "(iv) ⇒ (i)" of Anh et al. [3], we obtain (29). □

Remark 3.4. (i) If H is a finite dimensional Hilbert space, then condition (7) is equivalent to condition (29).

(ii) By letting $\alpha p = 1$, $\{X, X_n,\, n \geq 1\}$ be a sequence of negatively associated and identically distributed random variables, we see that Theorem 3.3 is an extension of Theorem 3.1 of Anh et al. [3].

(iii) By considering a special case that $\alpha = 1/p$ and $L(n) \equiv \log^{\beta/p}(n)$ for some $\beta \geq 0$, we obtain a generalization of a recent result of Miao et al. [22] which established the Marcinkiewicz–Zygmund-type strong law of large numbers for sequences of negatively associated identically distributed real-valued random variables with the norming constants are of the forms $n^{1/\alpha} \log^{\beta/p}(n)$.

(iv) Let $1 < p \leq 2$ and $\gamma > 0$. Sung [30] proved

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni} X_i\right| > \varepsilon n^{1/p} \log^{1/\gamma}(n)\right) < \infty \quad \text{for all} \quad \varepsilon > 0 \tag{38}$$

under moment conditions $E(X) = 0$ and

$$\begin{cases} E|X|^{\gamma} < \infty \text{ for } \ \gamma > p, \\ E|X|^{p} \log(|X|+1) < \infty \text{ for } \ \gamma = p, \\ E|X|^{p} < \infty \text{ for } \ \gamma < p. \end{cases}$$

Under the setting of Theorem 3.3, we see that the necessary and sufficient conditions for (38) are $E(X) = 0$ and $E\left(X/(\log(X))^{p/\gamma}\right) < \infty$. However, the array $\{a_{ni}, n \geq 1, i \geq 1\}$ considered in Sung [30] satisfies the condition

$$\sum_{i=1}^{n} |a_{ni}|^{p} \leq Cn, \ n \geq 1, \tag{39}$$

which is slightly weaker than our condition (30). Sung [30] noted that the moment condition in the case $\gamma > p$ is optimal, and he also raised an interesting open problem: what is optimal moment conditions for (38) with the weights satisfying (39) in the cases $p = \gamma$ and $p > \gamma$? Later, Chen and Sung [7] improved the mentioned result of Sung [30] by proving that in the case $p > \gamma$, the conditions $E(X) = 0$ and $E\left(|X|^{p}/\log^{-1+p/\gamma}(|X|)\right) < \infty$ imply (38). The above open problem was mentioned again by Chen and Sung [7]. To our best knowledge, this open problem remains unsolved.

4. Examples

In this section, we will present two examples to illustrate the sharpness of the main result. In Theorem 3.3 we see that if H is a finite dimensional Hilbert space, then (7) is necessary and sufficient for (24). The first example, which was inspired by Example 2.5 of Huan et al. [15], shows that if H is an infinite dimensional Hilbert space, the reverse of Theorem 3.1 is not true in general.

Example 4.1. Let $1 \le p < 2$, $\alpha p \ge 1$. Consider the real Hilbert space ℓ_2 of all square summable real sequences with inner product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i \quad \text{for } x = (x_1, x_2, \dots),\ y = (y_1, y_2, \dots) \in \ell_2.$$

Let $\{X_n,\, n \ge 1\}$ be a sequence of ℓ_2-valued i.i.d. random vectors with

$$P\left(X_1^{(j)} = -j^{-1/p}\right) = P\left(X_1^{(j)} = j^{-1/p}\right) = \frac{1}{2}.$$

In Example 2.5 of Huan et al. [15] proved that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} X_i\right\| > \varepsilon n^{\alpha}\right) < \infty \quad \text{for all} \quad \varepsilon > 0 \tag{40}$$

but

$$\sum_{j=1}^{\infty} E|X_i^{(j)}|^p = \infty.$$

Let $\beta > 0$. By (40), we have

$$\sum_{n=2}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} X_i\right\| > \varepsilon n^{\alpha} \log^{\beta} n\right) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

We have,

$$\sum_{j=1}^{\infty} E\frac{|X_i^{(j)}|^p}{\log^{\alpha\beta}(|X|+2)} = \sum_{j=1}^{\infty} \frac{1}{j\log^{\alpha\beta}(j^{-1/p}+2)} = \infty.$$

The second example, which is inspired by Example 5.6 of Thanh and Yin [33], shows that in Theorem 3.3, we cannot replace (30) by the weaker condition (39). We note that, by applying Hölder's inequality, if

$$\sum_{i=1}^{n} |a_{ni}|^q \le Cn,\ n \ge 1 \quad \text{for some } q > 0,$$

then

$$\sum_{i=1}^{n} |a_{ni}|^r \le Cn,\ n \ge 1 \quad \text{for all } 0 < r < q.$$

Example 4.2. Let H be the real line and $\{X_n, n \ge 1\}$ a sequence of independent identically distributed random variables with $P(X_1 = -1) = P(X_1 = 1) = 1/2$. Let $\{a_{ni}, n \ge 1, 1 \le i \le n\}$ be an array of constants such that for all $n \ge 1$,

$$a_{ni} = 0 \text{ for } 1 \le i < n \text{ and } a_{nn} = n^{1/p}.$$

Then (39) holds but (30) fails.

Let $\alpha = 1/p$ and $L(x) = \tilde{L}(x) = 1$ for all $x \geq 0$. Since $X_n, n \geq 1$ are bounded independent identically distributed random variables and $E(X_1) = 0$, (29) holds. However, for all $0 < \varepsilon < 1$, we have for all $A > 0$ that

$$\sum_{n\geq A} n^{\alpha p-2} P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} a_{ni}X_i\right\| > \varepsilon n^{\alpha}\right) = \sum_{n\geq A} n^{-1} P\left(\left\|n^{1/p}X_n\right\| > \varepsilon n^{1/p}\right)$$
$$= \sum_{n\geq A} n^{-1} = \infty.$$

Therefore (31) fails.

5. Further remark

The Baum–Katz-type theorem for sequences of pairwise independent identically distributed real-valued random variables under optimal moment conditions was recently established in Thành [32]. We note that two dependence structures negative association and pairwise independence do not imply each other, but the pairwise independence is much stronger than the pairwise negative dependence. The notion of coordinatewise pairwise negative dependence was recently introduced by Hien et al. [11] and independently introduced by Le et al. [19]. It is an open problem as to whether Theorem 3.3 holds for sequences of pairwise and coordinatewise negatively dependent Hilbert space-valued random vectors. The main difficulty is that random variables which are pairwise independent or pairwise negatively dependent do not satisfy the Kolmogorov maximal inequality. It would be an interesting problem for future research.

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