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ON THE EXISTENCE OF THE TWEEDIE POWER PARAMETER IMPLICIT ESTIMATOR

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Abstract. A special class of exponential dispersion models is the class of Tweedie distributions. This class is very significant in statistical modeling as it includes a number of familiar distributions such as Gaussian, Gamma and compound Poisson. A Tweedie distribution has a power parameter p, a mean m and a dispersion parameter ϕ. The value of the power parameter lies in identifying the corresponding distribution of the Tweedie family. The basic objective of this research work resides in investigating the existence of the implicit estimator of the power parameter of the Tweedie distribution. A necessary and sufficient condition on the mean parameter m, suggesting that the implicit estimator of the power parameter p exists, was established and we provided some asymptotic properties of this estimator.

1. Introduction

Exponential dispersion models (EDMs) were mainly introduced as a field of study by Jørgensen [11, 12]. An EDM is characterized by its unit variance function V which describes the relationship between the mean and the variance of the distribution when the dispersion parameter ϕ is held constant. If a random variable Y follows an EDM distribution with a mean m, a unit variance function $V(m)$ and a dispersion ϕ, then the variance of Y can be expressed as follows:

$$Var(Y) = \phi V(m).$$

In particular, if we assume that $Var(Y) = \phi m^p$, then Y follows a Tweedie distribution with parameters ϕ, m and p denoted by $Tw_p(m, \phi)$ ([12,19]). It should be reminded that the Tweedie models class (TMs) includes such important distributions as the normal ($p = 0$), the Poisson ($p = 1$), the Gamma ($p = 2$) and the inverse Gaussian ($p = 3$) distributions. TMs exist for all values of p outside the interval $(0, 1)$. Apart from the already mentioned four well-known distributions, the TMs probability density function cannot be written in a closed form

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and requires numerical methods for its evaluation [5,6]. For some p, Dunn and Smyth [5,6] elaborated a method to evaluate the Tweedie probability density function. For $p \in \mathbf{I}_1 = (2, +\infty)$, the Tweedie distribution is generated by a stable distribution and has the positive reals as support. The Tweedie distribution with power parameter $p \in \mathbf{I}_2 = (1, 2)$ corresponds to the compound Poisson distribution, expressed as a Poisson mixture of Gamma distributions with mass at zero. In this case, its support is the non-negative reals. Jørgensen [11] demonstrated that a Tweedie distribution, with $p \in \mathbf{I}_3 = (-\infty, 0)$, has a support concentrated on the whole real line. For more details, refer back to Table 1. All Tweedie distributions, with $p \in (-\infty, 0) \bigcup (1, +\infty)$, have strictly positive means, $m > 0$. These distributions are useful because they are the prototype response for generalized linear models [10].

Statistical estimation of the parameters of these distributions is a thorny issue, mainly when selecting the appropriate power parameter p [15]. In this case, multiple authors have taken p for a specified a priori. Jørgensen [11] chose $p = 1.75$ to analyze an amount of spent money. This choice is quiet arbitrary. Similarly, in [16], Nelder arbitrarily set $p = 1.5$ when analyzing the time spent splicing cables.

In this research work, we addressed the existence of the implicit estimator $\hat{p}_I$ of the power parameter p [7,8]. We displayed a necessary and sufficient condition on the mean parameter m such that the implicit estimator $\hat{p}_I$ of the power parameter p exists. The implicit estimation method is a non-informative Bayesian approach [13,18]. The basic merit of this approach resides in providing a substantial computational way of learning from observations without any prior knowledge. From this perspective, it offers a more efficient alternative to the classical inference in the Bayesian method when priors are missing [3,7,9]. As a matter of fact, we opted to apply the implicit estimation approach [3,8,9]. The remaining of this paper is organized as follows. In the second section, the Tweedie distributions are introduced and their properties are identified. In the third section, an overview of the implicit estimation method is exhibited. Section 4 tackles the theoretical foundation of the power parameter implicit estimator as well as its asymptotic normality.

2. Tweedie models

The Tweedie distribution belongs to the class of exponential dispersion models (EDM) [11, 12]. Let Y be a random variable distributed as a Tweedie distribution $Tw_p(m, \phi)$. Its density function indicated in terms of

$$f_p(y; m, \phi) = a_p(y, \phi) \exp\left\{\frac{(y\theta - k_p(\theta))}{\phi}\right\},$$

where $m = \mathbb{E}(Y) = k'_p(\theta) \in M_p$ is the mean (M_p is the domain of means), $\phi > 0$ is the dispersion parameter, θ is the canonical parameter and $k_p(\theta)$ is the cumulant function. The function $a_p(y, \phi)$ cannot be written in a closed form apart from the special cases stated in the introduction. The variance of Y is denoted by $Var(Y) = \phi V_p(m)$, where $V_p(m) = k_p''(\theta)$ is called the unit variance function. Tweedie models are characterized by their unit power variance functions having the form

$$V_p(m) = m^p, \tag{1}$$

where $p \in (-\infty, 0] \bigcup [1, +\infty)$ is the power parameter determining the distribution. The power parameter p is related to the parameter $\alpha \in (-\infty, 2]$ as indicated through the following equation

$$\alpha = \frac{p-2}{p-1}.$$

The cumulant function k_p (see, [11, 14]) is expressed by

$$\begin{aligned} k_p(\theta) &= \frac{\alpha-1}{\alpha}(\frac{\theta}{\alpha-1})^{\alpha}, \ (p \neq 1, 2), \\ k_1(\theta) &= \exp(\theta), \\ k_2(\theta) &= -\log(-\theta), \end{aligned}$$

where, $\theta = \begin{cases} (\alpha-1)\ m^{\frac{1}{\alpha-1}}, & p \neq 1; \\ \log(m), & p = 1. \end{cases}$

Therefore, the density function of Y is stated as follows:

$$f_p(y; m, \phi) = a_p(y, \phi) \exp\{\frac{y\frac{m^{1-p}}{1-p} - \frac{m^{2-p}}{2-p}}{\phi}\}. \tag{2}$$

Table 1 illustrates all Tweedie models with unit variance function (1).

Table 1. Summary of Tweedie models with mean domain $\mathbf{M}_p$ and support $\mathbf{S}_p$ of distributions

Distribution(s)	p	α	$\mathbf{M}_p$	$\mathbf{S}_p$
Extreme stable	$p < 0$	$1 < \alpha < 2$	$(0, \infty)$	$\mathbb{R}$
Normal	$p = 0$	$\alpha = 2$	$\mathbb{R}$	$\mathbb{R}$
Poisson	$p = 1$	$\alpha = -\infty$	$(0, \infty)$	$\mathbb{N}$
Compound Poisson	$1 < p < 2$	$\alpha < 0$	$(0, \infty)$	$[0, \infty)$
Gamma	$p = 2$	$\alpha = 0$	$(0, \infty)$	$(0, \infty)$
Positive stable	$p > 2$	$0 < \alpha < 1$	$(0, \infty)$	$(0, \infty)$
Extreme stable	$p = \infty$	$\alpha = 1$	$\mathbb{R}$	$\mathbb{R}$

In [12], Jørgensen asserted that if $Tw_p(m, \phi)$ is a Tweedie compound Poisson distribution ($p \in \mathbf{I}_2 = (1, 2)$), the function $a_p(y, \phi)$ is provided as follows:

$$a_p(y, \phi) = \frac{1}{y}\sum_{k=1}^{\infty} \frac{y^{-k\alpha}(p-1)^{\alpha k}}{\phi^{k(1-\alpha)}(2-p)^k k! \Gamma(-k\alpha)}; y > 0.$$

A similar series expansion exists for the Tweedie positive stable as well as the Tweedie extreme stable model. For $p \in \mathbf{I}_1 = (2, +\infty)$ and $y > 0$, it follows that

$$a_p(y, \phi) = \frac{1}{y\pi} \sum_{i=1}^{\infty} \frac{\Gamma(1+\alpha k)\phi^{k(\alpha-1)}(p-1)^{\alpha k}}{\Gamma(1+k)(p-2)^k y^{\alpha k}} (-1)^k \sin(-k\pi\alpha).$$

For $p \in \mathbf{I}_3 = (-\infty, 0)$ and $y \in \mathbb{R}$, we have

$$a_p(y, \phi) = \frac{1}{y\pi} \sum_{k=1}^{\infty} \frac{\Gamma(1+\frac{k}{\alpha})(-y)^k (\alpha\phi)^{\frac{k}{\alpha}}}{k!(\alpha-1)^{\frac{k(\alpha-1)}{\alpha}}} \sin(\frac{-k\pi}{\alpha}).$$

In [12], Jørgensen considered the cases of $p < 0$ and $p > 2$ and argued that the density function $a_p(y, \phi)$ corresponds to extreme stable distribution. Consult Dunn and Smyth [5, 6] and Jørgensen [12, page 141] for more details.

3. General view of the implicit estimation method

The implicit estimation method was set forward by Hassairi et al. [8] as an alternative to the non informative Bayesian approach. Relying upon some prior information, the Bayesian theory ([1, 17]) takes into account an unknown parameter θ as a random variable and specifies its posterior distribution given data. Similarly, the implicit distribution is regarded as a posterior distribution of a parameter θ given data. Indeed, considering a family of probability distributions $\{p(x/\theta),\ \theta \in \Theta\}$ parameterized by an unknown parameter θ in a set Θ; where x stands for the observed data, the implicit distribution $p(\theta/x)$ is computed by multiplying the likelihood function $p(x/\theta)$ through a counting measure σ if Θ is a countable set and through a Lebesgue measure σ if Θ is an open set (σ depends only on the topological structure of Θ) and subsequently dividing by the norming constant $c(x) = \int_\Theta p(x/\theta)\sigma(d\theta)$. Hence, the implicit distribution is expressed by the following formula $p(\theta/x)(d\theta) = (c(x))^{-1} p(x/\theta)\sigma(d\theta)$ and acts as a posterior distribution of θ given x in the Bayesian method. This represents a specific improper prior which is based only on the topology of Θ (without any statistical assumption).

Particularly, if the set of parameters Θ is bounded, then σ is proportional to the uniform distribution on Θ. If Θ is unbounded, σ is an improper prior.

Provided its existence (which holds for most statistical models), the implicit distribution can be invested for the estimation of the parameter θ following a Bayesian methodology. The implicit estimator $\widehat{\theta}$ of θ is a Bayes estimator with regard to the squared error loss function given by the expected posterior mean, that is

$$\widehat{\theta} = \mathbb{E}(\theta/x) = \int_\Theta \theta p(\theta/x)(d\theta).$$

An intrinsic issue in Bayesian estimation lies in how to define the prior distribution. If the prior information about the parameter θ is obtainable, it should be incorporated in the prior density. If we don't have prior information, we need apply non informative Bayesian estimation or implicit estimation.

4. Implicit Tweedie model selection

Let $\mathbf{y} = (y_1, \ldots, y_n)$ be a sample from a Tweedie distribution $Tw_p(m, \phi)$. Assuming that $p \in \mathbf{I}_j,\ j \in \{1, 2, 3\}$, the likelihood function is expressed in terms of

$$l_n(\mathbf{y}; m, \phi, p) = \prod_{i=1}^{n} f_p(y_i; m, \phi).$$

The implicit distribution of the power parameter p exists if and only if the integral

$$\int_{\mathbf{I}_j} l_n(\mathbf{y}; m, \phi, p) dp < +\infty$$

converges. In this case, the implicit density function $f(p; m, \phi, \mathbf{y})$ of the power parameter p is indicated as

$$f(p; m, \phi, \mathbf{y}) = \frac{l_n(\mathbf{y}; m, \phi, p)}{\int_{\mathbf{I}_j} l_n(\mathbf{y}; m, \phi, p) dp},\ j \in \{1, 2, 3\}.$$

The implicit estimator $\hat{p}_{I_j}$ of p is a non informative Bayesian one and

$$\hat{p}_{I_j} = \mathbb{E}(p|m, \phi, \mathbf{y}) = \int_{\mathbf{I}_j} p f_p(m, \phi, \mathbf{y}) dp,\ j \in \{1, 2, 3\}$$

minimizes the squared error loss function.

4.1. Existence of the power parameter implicit estimator

In this subsection, we set forward a necessary and sufficient condition on the mean parameter m such that the implicit estimator $\hat{p}_I$ of the power parameter p exists. The convergence of the integral of the likelihood function over a bounded interval is highlighted in the following theorem.

Theorem 4.1. *Let $f_p(y; m, \phi)$ be the density function of the Tweedie distribution $Tw_p(m, \phi)$ and let $[a, b] \subset \mathbf{I}_j$ for $j = \{1, 2, 3\}$. Hence, for all $q \geq 0$, one has*

$$\int_a^b |p|^q \prod_{i=1}^{n} f_p(y_i; m, \phi) dp < +\infty,$$

which converges almost surely.

Proof. Let $Y_i \sim Tw_p(m, \phi)$. Then,

$$\begin{aligned} Var(Y_i) &= \int_{\mathbb{R}} (y - m)^2 f_p(y; m, \phi) dy \\ &= \phi m^p. \end{aligned}$$

Hence,

$$|p|^q \prod_{i=1}^{n} \int_{\mathbb{R}} (y_i - m)^2 f_p(y_i; m, \phi) dy_i = |p|^q \prod_{i=1}^{n} \phi m^p$$
$$= |p|^q \phi^n m^{np}.$$

One can easily verify that the integral $\int_a^b |p|^q \phi^n m^{np} dp < +\infty$ is finite. Thus, the integral $\int_a^b |p|^q \prod_{i=1}^n \int_{\mathbb{R}} (y_i - m)^2 f_p(y_i; m, \phi) dy_i dp < +\infty$ is finite. Relying upon the Fubini-Tonelli theorem, we obtain

$$H_1 = \int_{\mathbb{R}} \int_a^b |p|^q \prod_{i=1}^{n} (y_i - m)^2 f_p(y_i; m, \phi) dp dy_1 \cdots dy_n$$
$$= \int_{\mathbb{R}} \{\prod_{i=1}^{n} (y_i - m)^2\} \{\int_a^b |p|^q \prod_{i=1}^{n} f_p(y_i; m, \phi) dp\} dy_1 \cdots dy_n < +\infty.$$

From this perspective, we infer that the integral

$$\int_a^b |p|^q \prod_{i=1}^{n} f_p(y_i; m, \phi) dp < +\infty$$

converges almost surely. □

Our main result is the following.

Theorem 4.2. *Let $f_p(y; m, \phi)$ be the density function of a the Tweedie distribution $Tw_p(m, \phi)$. Then, for all $q \geq 0$, one has*

(i) *The integral*

$$\int_2^{+\infty} |p|^q \prod_{i=1}^{n} f_p(y_i; m, \phi) dp < +\infty \tag{3}$$

converges almost surely if and only if $0 < m < 1$.

(ii) *The integral*

$$\int_{-\infty}^{0} |p|^q \prod_{i=1}^{n} f_p(y_i; m, \phi) dp < +\infty \tag{4}$$

converges almost surely if and only if $m > 1$.

To corroborate the main theorem, we shall introduce the following lemma ([20, page 81]).

Lemma 4.3. *Let Y be a random stable variable with density function $a_p(y, \phi)$ and stability index $0 < \alpha = \frac{p-2}{p-1} < 2$. Therefore,*

$$a_p(y, \phi) \underset{p \to \pm\infty}{\longrightarrow} a_\infty(y, \phi),$$

where $a_\infty(y, \phi)$ is the density function of the stable distribution with a stability index equal to 1.

Proof of Theorem 4.2. (i) Let $Y_1, \ldots, Y_n$ be n independent and identically distributed random variables with common Tweedie distribution $Tw_p(m, \phi)$. Therefore, for each $1 \leq i \leq n$, one has

$$\begin{aligned} Var(Y_i) &= \int_{\mathbb{R}} (y-m)^2 f_p(y; m, \phi) dy \\ &= \phi m^p. \end{aligned}$$

It follows that, for any $q \geq 0$,

$$\begin{aligned} |p|^q \prod_{i=1}^{n} \int_{\mathbb{R}} (y_i - m)^2 f_p(y_i; m, \phi) dy_i &= |p|^q \prod_{i=1}^{n} \phi m^p \\ &= |p|^q \phi^n m^{np}. \end{aligned}$$

By assuming that the mean $m < 1$, one can easily verify that the integral

$$\textstyle\int_2^{\infty} |p|^q \phi^n m^{np} dp < +\infty$$

converges. Thus, if $m < 1$, the integral

$$H_2 = \int_2^{+\infty} |p|^q \prod_{i=1}^{n} \int_{\mathbb{R}} (y_i - m)^2 f_p(y_i; m, \phi) dy_i dp < +\infty$$

converges. By applying the Fubini-Tonelli theorem one gets, for all $m < 1$

$$\begin{aligned} H_2 &= \int_{\mathbb{R}^n} \int_2^{+\infty} |p|^q \prod_{i=1}^{n} (y_i - m)^2 f_p(y_i; m, \phi) dp dy_1 \cdots dy_n \\ &= \int_{\mathbb{R}^n} \{\prod_{i=1}^{n} (y_i - m)^2\} \{\int_2^{+\infty} |p|^q \prod_{i=1}^{n} f_p(y_i; m, \phi) dp\} dy_1 \cdots dy_n < +\infty. \end{aligned}$$

Hence, we conclude that $\int_2^{+\infty} |p|^q \prod_{i=1}^{n} f_p(y_i; m, \phi) dp$ converges almost surely.

Conversely, if $m \geq 1$ the integral (3) diverges. Indeed, as it is well known, the Tweedie distribution with $p \geq 2$ is generated by a stable distribution and has a support on the positive reals. Moreover, when $m \geq 1$,

$$\lim_{p \to +\infty} \exp\{\frac{y \frac{m^{1-p}}{1-p} - m^{2-p}}{\phi(2-p)}\} = 1.$$

According to Equation (2), it follows that

$$f_p(y; m, \phi) \underset{p \to +\infty}{\sim} a_p(y, \phi). \tag{5}$$

Resting on (5) and Lemma 4.3, one can deduce easily that the integral (3) diverges.

(ii) Proceeding in the same way, as it is demonstrated the proof of (i), we have

$$\int_{-\infty}^{0} |p|^q \phi m^p dp < +\infty.$$

Therefore, for all $m > 1$, we get

$$\begin{aligned} H_3 &= \int_{\mathbb{R}^n} \int_{-\infty}^{0} |p|^q \prod_{i=1}^{n} (y_i - m)^2 f_p(y_i; m, \phi) dp dy_1 \cdots dy_n \\ &= \int_{\mathbb{R}^n} \{\prod_{i=1}^{n} (y_i - m)^2\} \{\int_{-\infty}^{0} |p|^q \prod_{i=1}^{n} f_p(y_i; m, \phi) dp\} dy_1 \cdots dy_n < +\infty. \end{aligned}$$

Hence, we conclude that if $m > 1$, the integral (4) converges almost surely. Conversely if $m \leq 1$, the integral (4) diverges. Indeed, investing similar the arguments to those used in (i), we shall conform that

$$f_p(y; m, \phi) \underset{p \to -\infty}{\longrightarrow} a_\infty(y, \phi) > 0.$$

Thus, the integral (4) diverges. □

Corollary 4.4. *Let $Y_1, \ldots, Y_n$ be n independent and identically distributed random variables with common Tweedie distribution $Tw_p(m, \phi)$. Then, the implicit distribution of the power parameter p exists if and only if one of the following assertions holds*:

(i) $p \in [2, +\infty)$ *and* $m < 1$.
(ii) $p \in (1, 2)$.
(iii) $p \in (-\infty, 0)$ *and* $m > 1$.

Proof. It is sufficient to check that the following integral is convergent.

$$\int_{\mathbf{I}_j} \prod_{i=1}^{n} f_p(y_i; m, \phi) dp.$$

For (i) and (iii), it is sufficient to apply Theorem 4.2 for $q = 0$. By applying Theorem 4.1, for $a = 1$ and $b = 2$, we get (ii). □

In these cases, the implicit density distribution of p is denoted by

$$\begin{aligned} f(p; \mathbf{y}, m, \phi) &= \frac{\prod_{i=1}^{n} f_p(y_i; m, \phi) dp}{\int_{\mathbf{I}_j} \prod_{i=1}^{n} f_p(y_i; m, \phi) dp}, \; j \in \{1, 2, 3\} \\ &\propto \prod_{i=1}^{n} f_p(y_i; m, \phi) dp. \end{aligned}$$

Then, the implicit estimator of p is expressed as

$$\hat{p}_{I_j} = \frac{J_{1j}}{J_{2j}}, \; j = 1, 2, 3, \tag{6}$$

where

$$J_{1j} = \int_{\mathbf{I}_j} p \prod_{i=1}^{n} f_p(y_i; m, \phi) dp,$$

$$J_{2j} = \int_{\mathbf{I}_j} \prod_{i=1}^{n} f_p(y_i; m, \phi) dp < +\infty.$$

As a matter of fact, the implicit estimator $\hat{p}_I$ can be calculated, through the use of the Monte-Carlo estimation method. The sub-classes of Tweedie models are characterized by their supports (see Table 1). The compound Poisson distribution is a frequent choice for the modelling of non-negative data with a probability mass at zero. In this case, we proved that the implicit estimator of the power parameter exists without any condition. In positive-stable and the extreme stable Tweedie models, we proved that the implicit estimator of the power parameter p exists provided that the mean $m < 1$ when $p \in [2, +\infty)$ and $m > 1$ when $p \in (-\infty, 0)$. The selection of sub-class $\mathbf{I}_j$, $(j = 1, 2, 3)$ refers basically to the nature of data sample $\mathbf{y} = (y_1, \ldots, y_n)$. If the $\mathbf{y}$ components are non-negative and have exact zeros, the model will be the compound-Poisson one $(j = 2)$. Otherwise if the components of data sample are non-negative and haven't any zero, the sub-class will be the Tweedie positive-stable one. When the data sample is an interesting mixture of negative and non-negative components, the appropriate sub-class corresponds to the Tweedie negative-stable one.

4.2. Asymptotic behavior of the power parameter implicit estimator

In this section, we shall determine some asymptotic properties of the implicit estimator of the power parameter p. Consider a sample $\mathbf{y} = (y_1, \ldots, y_n)$ from a Tweedie distribution $Tw_p(m, \phi)$ with a density function $f_p(\cdot; m, \phi)$. The log-likelihood function is expressed by

$$\mathcal{L}_n(\mathbf{y}; p) = \sum_{i=1}^{n} \log(f_p(y_i; m, \phi)).$$

The maximum likelihood estimator $\hat{p}_M$ of the power parameter p is obtained by maximizing the log-likelihood function $\mathcal{L}(\mathbf{y}; p)$. The score function for the power parameter p is stated as

$$\mathcal{U}_n(\mathbf{y}; p) = \frac{\partial \mathcal{L}_n(\mathbf{y}; p)}{\partial p}.$$

The Fisher information function of the power parameter p is indicated as

$$\mathcal{F}_{n,p} = -\mathbb{E}\left(\frac{\partial^2 \mathcal{L}_n(\mathbf{y}; p)}{\partial p^2}\right).$$

Since $y_1, y_2, \ldots, y_n$ are independent variables and following the same Tweedie distribution $Tw_p(m, \phi)$, we have

$$\mathcal{F}_{n,p} = -\sum_{i=1}^{n} \frac{\partial^2 \mathcal{L}(y_i; p)}{\partial p^2} = n\mathcal{F}_p, \text{ where } \mathcal{F}_p = -\mathbb{E}\left(\frac{\partial^2 \mathcal{L}(y_1; p)}{\partial p^2}\right).$$

According to Bonat and Kokonendji [2], the asymptotic distribution of $\hat{p}_M$ is normal:

$$\hat{p}_M \sim \mathbf{N}\left(p, \frac{\mathcal{F}_p^{-1}}{n}\right),$$

where $\mathcal{F}_p^{-1}$ denotes the inverse of the Fisher information and p is the true power parameter. Now, we shall display the asymptotic normality of the implicit estimator $\hat{p}_I$.

Proposition 4.5. *The implicit estimator $\hat{p}_I$ of the power parameter p is asymptotically normal implying that*

$$\hat{p}_I \sim \mathbf{N}(p, \frac{\mathcal{F}_p^{-1}}{n}).$$

To specify the asymptotic properties of $\hat{p}_I$, we need to consider the following lemmas.

Lemma 4.6 (Laplace approximation [4]). *Consider the integral*

$$I = \int_{\mathbf{D}} h(y) \exp\{-g_n(y)\} dy,$$

where h is a positive continuous function on an open set $\mathbf{D} \subset \mathbb{R}$, and g_n is a twice continuously differentiable on $\mathbf{D}$. Assume that g_n has a strict minimizer $y^\star$ on $\mathbf{D}$. Then the Laplace's approximation is well-defined and is provided by

$$I \approx (2\pi)^{1/2} |g_n''(y^\star)|^{-1/2} h(y^\star) \exp\{-g_n(y^\star)\}(1 + \mathcal{O}(\frac{1}{n})),$$

where g_n'' is the second derivative of the function g_n.

Lemma 4.7 (Slutsky Lemma). *Let $(X_n)_n$, $(Y_n)_n$ be two sequences of random variables. If $(X_n)_n$ converges in distribution to a random variable X and $(Y_n)_n$ converges in probability to a real constant c, then the sequence $(X_n+Y_n)_n$ converges in distribution to $X + c$ and the sequence $(X_nY_n)_n$ converges in distribution to cX.*

Proof of Proposition 4.5. Since

$$\exp\{\mathcal{L}_n(\mathbf{y}; p)\} = l_n(\mathbf{y}; m, \phi, p) = \prod_{i=1}^{n} f_p(y_i; m, \phi)$$

and according to (6), the implicit estimator $\hat{p}_{I_j}$ is stated as

$$\hat{p}_{I_j} = \frac{\int_{\mathbf{I}_j} p \exp\{\mathcal{L}_n(\mathbf{y}; p)\} dp}{\int_{\mathbf{I}_j} \exp\{\mathcal{L}_n(\mathbf{y}; p)\} dp}, \; j = 1, 2, 3. \tag{7}$$

The Laplace's method is well suitable for application to both the numerator ($g_n(p) = -\mathcal{L}_n(\mathbf{y}; p)$ and $h(p) = p$) and denominator ($g_n(p) = -\mathcal{L}_n(\mathbf{y}; p)$ and $h(p) = 1$) of (7). Therefore

$$\int_{\mathbf{I}_j} p \exp\{\mathcal{L}_n(\mathbf{y}; p)\} dp = \sqrt{2\pi} \mid \frac{\partial^2 \mathcal{L}_n(\mathbf{y}; \hat{p}_M)}{\partial p^2} \mid^{-\frac{1}{2}} \hat{p}_M \exp\{\mathcal{L}_n(\mathbf{y}; \hat{p}_M)\}(1 + \mathcal{O}(\frac{1}{n}))$$

and

$$\int_{\mathbf{I}_j} \exp\{\mathcal{L}_n(\mathbf{y}; p)\} dp = \sqrt{2\pi} \mid \frac{\partial^2 \mathcal{L}_n(\mathbf{y}; \hat{p}_M)}{\partial p^2} \mid^{-\frac{1}{2}} \exp\{\mathcal{L}_n(\mathbf{y}; \hat{p}_M)\}(1 + \mathcal{O}(\frac{1}{n})).$$

Consequently, we have

$$\hat{p}_I = \hat{p}_M \frac{1+\mathcal{O}(\frac{1}{n})}{1+\mathcal{O}(\frac{1}{n})} = \hat{p}_M(1+\mathcal{O}(\frac{1}{n})). \tag{8}$$

Now, we shall prove that the implicit estimator of the power parameter converges in probability to p. Indeed, based upon the fact that the maximum likelihood estimator $\hat{p}_M$ is asymptotically normal [2] which implies that

$$\hat{p}_M \sim \mathbf{N}(p, \frac{\mathcal{F}_p^{-1}}{n}).$$

Indeed,

$$\sqrt{n}(\hat{p}_M - p) \sim \mathbf{N}(0, \mathcal{F}_p^{-1}) \text{ as } n \to +\infty. \tag{9}$$

Therefore, referring to Equation (9), $\hat{p}_M - p = \frac{1}{\sqrt{n}}\sqrt{n}(\hat{p}_M - p)$, and by applying Slutsky Lemma, we infer that $\hat{p}_M$ converges in distribution to p (which is constant) then it converges also in probability to p .

According to Equation (8), we have

$$\begin{aligned}\sqrt{n}(\hat{p}_I - p) &= \sqrt{n}[(\hat{p}_M - p) + \hat{p}_M\mathcal{O}(\frac{1}{n})] \\ &= \sqrt{n}(\hat{p}_M - p) + \hat{p}_M\mathcal{O}(\frac{1}{\sqrt{n}})).\end{aligned} \tag{10}$$

According to the Slutsky Lemma, one verifies that $\hat{p}_M\mathcal{O}(1/\sqrt{n})$ converges in probability to 0. From Equation (10) and using again Slutsky Lemma, we deduce that $\sqrt{n}(\hat{p}_I - p)$ converges in distribution to $\mathbf{N}(0, \mathcal{F}_p^{-1})$. □

5. Conclusion and perspectives

In this work, we elaborated a necessary and sufficient condition, on the mean parameter p so that the implicit estimator of the power parameter p, of the Tweedie distribution $Tw_p(m, \phi)$, exists. We demonstrated that this estimator is asymptotically normal. Our research could be regarded as valuable in terms of opening further fruitful lines of investigation and offering promising future research directions. In perspective, we shall implement this implicit estimator and apply it in the Tweedie regression model on a given real dataset.

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