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P-EXTREMAL FUNCTIONS AND BERNSTEIN-MARKOV PROPERTIES ASSOCIATED TO COMPACT SETS IN $\mathbb{R}^d$

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Abstract. Given a compact subset $P \subset (\mathbb{R}^+)^d$ and a compact set K in $\mathbb{C}^d$. We concern with the Bernstein-Markov properties of the triple (P, K, μ) where μ is a finite positive Borel measure with compact support K. Our approach uses (global) P-extremal functions which is inspired by the classical case (when $P = \Sigma$ the unit simplex) in [7].

1. Introduction

Let K be a compact subset of $\mathbb{C}^d$ and μ be a positive Borel measure on $K \subset \mathbb{C}^d$. Obviously the $L^2(\mu)$-norm on K of a polynomial p is majorized by its sup-norm. It is a natural problem to see whether this estimate can be reversed. For this purpose, we say that the pair (K, μ) has the *Bernstein-Markov property* if for each $\varepsilon > 0$ there exists a positive constant $C = C_\varepsilon > 0$ such that

$$\|p\|_K := \sup_{z \in K} |p(z)| \leq C e^{\varepsilon \deg p} \|p\|_{L^2(\mu)}, \ \forall p \in \mathbb{C}[z_1, \ldots, z_d]. \tag{1.1}$$

The Bernstein-Markov property is a classical concept and was studied thoroughly in [6, 7, 11]. The reader is referred to [8] for an authoritative survey on Bernstein-Markov property and its numerous applications to approximation theory. One use of this property is the possibility to approximate the (global) extremal function of K by functions of the form $\frac{1}{\deg p} \log |p|$ where p are polynomials that form an orthonormal system for $L^2(K, \mu)$ (see [9]). In [7], T. Bloom and N. Levenberg found, among other things, an interesting mass density condition that guarantees the Bernstein-Markov property of (K, μ). The result below is a consequence of Theorem 1.2 and Theorem 2.2 in [7].

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Theorem 1.1. *Let K be a compact regular subset of $\mathbb{C}^d$ and μ be a finite positive Borel measure on K. Set*

$$E_r = \{z \in K : \mu(K \cap B(z,r)) \geq r^T\},\ \forall r > 0.$$

Let $V^*_{E_r}$ and V^*_K be the extremal function of E_r and K, respectively. Suppose that there exists a positive constant T such that one of the following (*equivalent*) *conditions occurs*:

(i) $V^*_{E_r} \to 0$ *q.e. on K, i.e., outside a pluripolar set*;

(ii) $V^*_{E_r} \to V^*_K$ (*globally*) *uniformly on $\mathbb{C}^d$ as $r \to 0$.*

Then (K, μ) has the Bernstein-Markov property (1.1).

Let $\mathbb{R}^+ := [0, +\infty)$ and $P \subset (\mathbb{R}^+)^N$ be a convex body, that is a compact, convex set with non-empty interior. In recent years, a pluripotential theory associated to P has been developed (see [1–3, 10]). It is attached to the notions of P-extremal functions (see the definition in the next section) and of P-polynomials. Our principal goal is to extend Theorem 1.1 to the context of P-polynomials and P-extremal functions where P is a compact subset (not necessarily a convex body) of $(\mathbb{R}^+)^d$ that contains the origin. Let us now recall the notion of P-polynomials associated to such compact set P. Following [1], for each $n \geq 1$ we consider the finite-dimensional polynomial space

$$\text{Poly}(nP) := \{p \in \mathbb{C}[z_1, \dots, z_d] : p(z) = \sum_{J \in nP \cap (Z^+)^d} a_J z^J\}.$$

Here we use the multi-dimensional notation $z^J = z_1^{j_1} \cdots z_d^{j_d}$ for $J = (j_1, \dots, j_d)$.

In the case $P = \Sigma := \{(x_1, \dots, x_d) \in (\mathbb{R}^+)^d : x_1 + \cdots + x_d \leq 1\}$, the standard unit simplex in $\mathbb{R}^d$ we have $\text{Poly}(n\Sigma) = \mathcal{P}_n$ the usual space of holomorphic polynomials of degree at most n in $\mathbb{C}^d$. On the other hand, since there exists $A \in \mathbb{Z}^+$ such that $P \subset A\Sigma$ we get

$$\text{Poly}(nP) \subset \text{Poly}(nA\Sigma) = \mathcal{P}_{nA},\ \forall n \geq 1.$$

Sometimes we also assume further that P is *admissible* in the sense that

$$\Sigma \subset kP \quad \text{for some } k \in \mathbb{Z}^+.$$

The above restriction was emphasized in [1] and [2] to exploit the approximability of the P-extremal functions by (normalized) logarithms of P-polynomials.

Our article is organized as follows. In the next section we gather up auxiliary facts about P-extremal functions. The most important notion is that of the logarithmic indicator for a compact set $P \subset (\mathbb{R}^+)^d$. Using this concept we define the P-extremal functions and formulate their basic properties (Bernstein-Walsh's estimate and Siciak-Zakharyuta's theorem). Our (new) results appear in the later sections. In Section 3 we prove in Theorem 3.2 equivalent conditions for convergence of sequence of P-extremal functions V_{P,K_j} towards $V_{P,K}$ where K_j are subsets of a compact set K in $\mathbb{C}^d$. In the case $P = \Sigma$ this result is essentially the equivalence (i)$\Leftrightarrow$(ii) given in Theorem 1.1. Section 4 starts with definition the Bernstein-Markov properties for the triple (P, K, μ). After presenting some easy consequences of the Bernstein-Markov properties in Proposition 4.3 and Proposition 4.5, we move to Theorem 4.6. In this result we follow the approach indicated in [7] and deduce the Bernstein-Markov property for P-polynomials from the convergence of certain sequences of P-extremal functions which are defined by some mass density estimate. The next result, Proposition 4.8 serves as a prototype in which the above (somewhat) abstract results can be applied.

2. Preliminaries

Throughout this paper, unless otherwise specified, we always denote by K a compact subset of $\mathbb{C}^d$, μ a positive finite measure whose support equals to K and for P a compact subset of $(\mathbb{R}^+)^d$ satisfying $0 \in P$. We claim no originality in this section since most of the material that follows is taken from [14], [5] and especially [12] (in the case $P = \Sigma$) and from [1], [10] (in the case P is an admissible convex body).

We first recall some elements about global P-extremal functions associated to P. The first function to be defined is *the logarithmic indicator function* of P

$$\begin{aligned} H_P(z) &:= \sup_{J=(j_1,\dots,j_d)\in P} \log(|z_1|^{j_1} \cdots |z_d|^{j_d}) \\ &= \sup_{J=(j_1,\dots,j_d)\in P} (j_1 \log|z_1| + \cdots + j_d \log|z_d|),\ z \in \mathbb{C}^d. \end{aligned}$$

Here we use the convention that $0 \cdot \infty = 0$. We also consider *the support function* of P

$$h_P(x) := \sup_{J=(j_1,\dots,j_d)\in P} (j_1 x_1 + \cdots + j_d x_d),\ x \in \mathbb{R}^d.$$

By Cauchy-Schwarz inequality we get

$$|h_P(x) - h_P(y)| \le \|x - y\| \sup_{t\in P} \|t\|,\ \forall x, y \in \mathbb{R}^d.$$

This implies continuity of H_P on $\mathbb{C}^d$, and so $H_P \in \mathrm{PSH}(\mathbb{C}^d)$ being the upper envelope of a family of plurisubharmonic functions on $\mathbb{C}^d$. Moreover, by the above convention $H_P \ge 0$ on $\mathbb{C}^d$. In the standard case $P = \Sigma$, an easy reasoning yields

$$H_\Sigma(z) = \max_{1\le j\le d} \log^+ |z_j|,\ \forall z \in \mathbb{C}^d.$$

If P is *admissible* (but not necessarily convex), i.e., $\Sigma \subset kP$ for some $k \in \mathbb{Z}^+$, then we have the lower bound

$$\begin{aligned} H_P(z) &\ge \frac{1}{k} H_\Sigma(z) \\ &= \frac{1}{k} \max_{1\le j\le d} \log^+ |z_j|. \end{aligned} \tag{2.1}$$

We will now use $H_P(z)$ to provide a generalization of the standard Lelong classes

$$\begin{aligned}\mathcal{L}_P &:= \mathcal{L}_P(\mathbb{C}^d)\\ &= \{u \in \mathrm{PSH}(\mathbb{C}^d) : u(z) \le c_u + H_P(z), z \in \mathbb{C}^d\},\\ \mathcal{L}_{P,+} &:= \mathcal{L}_{P,+}(\mathbb{C}^d)\\ &= \{u \in \mathrm{PSH}(\mathbb{C}^d) : -c_u + H_P(z) \le u(z) \le c_u + H_P(z), z \in \mathbb{C}^d\},\end{aligned}$$

where c_u is a constant depending only on u. The above definition makes sense since $H_P \in \mathcal{L}_P$ for every P. In the special case $P = \Sigma$ we recover the standard Lelong classes in $\mathbb{C}^d$. For a bounded subset $E \subset \mathbb{C}^d$, the P-global extremal function of E is defined by

$$V_{P,E}(z) := \sup\{u(z) : u \in \mathcal{L}_P(\mathbb{C}^d), u \le 0 \text{ on } E\}. \tag{2.2}$$

We also let $V_E^*(z) := \limsup_{\xi\to z} V_E(\xi)$ be the upper semicontinous regularization of $V_{P,E}$. For $P = \Sigma$ we have $V_{\Sigma,E} = V_E$, the standard Siciak global extremal function.

It is well-known that $V_E^* \equiv +\infty$ if and only if E is pluripolar, i.e., there exists a plurisubharmonic function u on $\mathbb{C}^d$ such that $E \subset \{z \in \mathbb{C}^d : u(z) = -\infty\}$. According to a result of Siciak in [14] (see also [12]), in this case we can even choose $u \in \mathcal{L}(\mathbb{C}^d)$. One use of these extremal functions is to define certain concepts of regularity.

Definition 2.1. A compact set $K \subset \mathbb{C}^d$ is said to be L-(resp. PL-)regular if V_K (resp. $V_{P,K}$) is continuous on $\mathbb{C}^d$.

By simple considerations as in [2], p. 17 (see also our Proposition 3.1(iv)) we see that under some mild restriction on P, the set K is L-regular if and only if it is PL-regular. Indeed, it is a consequence of the following comparison lemma.

Lemma 2.2. *If P is admissible, then there exist constants $a, b > 0$ such that for all compact set K we have*

$$aV_K \le V_{P,K} \le bV_K.$$

We omit the straightforward proof which is based on (2.2). The definition of $V_{P,K}$ also yields the following useful Bernstein-Walsh estimate.

Proposition 2.3. *Let E be a non-pluripolar subset of $\mathbb{C}^d$. Then for any $p \in$ Poly(nP) we obtain*

$$|p(z)| \le \|p\|_E e^{nV_{P,E}(z)},\ z \in \mathbb{C}^d.$$

We have the following simple facts which will be useful in the sequel.

Proposition 2.4. (i) *Let $P(a, r)$ be the open polydisk with center $a = (a_1, \ldots, a_d)$, radius r. Then*

$$V_{P,\overline{P}(a,r)} = H_P\Big(\frac{z-a}{r}\Big) = \sup_{J\in P} \log^+ \Big|\frac{z-a}{r}\Big|^J,\ z \in \mathbb{C}^d.$$

(ii) *If* $u \in \mathcal{L}_P$, *then*

$$u(z) \le \max_{\overline{P}(a,r)} u + H_P(\frac{z-a}{r}),\ \forall z \in \mathbb{C}^d.$$

(iii) $E \subset \mathbb{C}^d$ *is* PL*-pluripolar if and only if* $V^*_{P,E} \equiv \infty$. *Moreover, if* E *is non-pluripolar, then* $V^*_{P,E} \in \mathcal{L}_P$.

(iv) E *is pluripolar if and only if* E *is* PL*-pluripolar.*

(v) *Let* $\{u_\alpha\}_{\alpha\in I} \subset \mathcal{L}_P$ *and* $u := \sup_{\alpha\in I} u_\alpha$. *Set* $A := \{z \in \mathbb{C}^d : u(z) < +\infty\}$. *Then* $u^* \in \mathcal{L}_P$ *if and only if* A *is non-pluripolar.*

Proof. (i) After a translation and dilation of coordinates we may assume that $a = 0$ and $r = 1$. It is then enough to show

$$V_{P,\overline{P}(0,1)}(z) = H_P(z) = \sup_{J\in P} \log^+ |z|^J,\ z \in \mathbb{C}^d.$$

Since $H_P \in \mathrm{PSH}(\mathbb{C}^d)$, $H_P = 0$ on $\overline{P}(0,1)$, it is clear that $H_P \le V_{P,\overline{P}(0,1)}$ on $\mathbb{C}^d$. For the reverse inequality, we take $z \in \mathbb{C}^d$. If $|z| := \max(|z_1|, \dots, |z_d|) \le 1$, then the inequality is obvious. Consider the case $|z| > 1$. Then for every $u \in \mathcal{L}_P$, $u \le 0$ on $\overline{P}(0,1)$ the function

$$\varphi(\lambda) = u(\lambda z) - H_P(\lambda z)$$

is *bounded* and subharmonic on $\{\lambda \in \mathbb{C} : |\lambda| > \frac{1}{|z|}\}$. Moreover $\varphi(\lambda) \le 0$ on the circle $|\lambda| = \frac{1}{|z|}$. Using an extended version of the maximum principle we get $\varphi(\lambda) \le 0$ for all $|\lambda| \ge \frac{1}{|z|}$. By setting $\lambda := 1$ we obtain the required inequality.

(ii) Set $v(z) = u(z) - \max\limits_{\overline{P}(a,r)} u$, $z \in \mathbb{C}^d$. Then $v \in \mathcal{L}_P$, $v \le 0$ on $\overline{P}(a,r)$. Then by (i),

$$v(z) \le V_{P,\overline{P}(0,1)}(z) = H_P(z),$$

thus we get (ii).

(iii) Assume first that E is PL-pluripolar. Then we can find $u \in \mathcal{L}_P$ such that $u|_E = -\infty$. It follows that $V_{P,E} \ge u + C$ for all $C > 0$. Hence $V^*_{P,E} \equiv \infty$. Conversely, if $V^*_{P,E} \equiv \infty$, then by Choquet's topological lemma we can find a sequence $u_j \in \mathcal{L}_P$ with $u_j|_E \le 0$ and $u_j \uparrow \infty$ a.e. on $\mathbb{C}^d$. By multiplying u_j with large constants we may obtain that

$$M_j := \max_{P(0,1)} u_j \ge 2^j,\ \forall j.$$

By the same reasoning as in [5] we infer that

$$u := \sum_{j\ge1} 2^j (u_j - M_j)$$

belongs to $\mathcal{L}_P$ and satisfies $u|_E \equiv -\infty$. Thus E is PL-pluripolar. Now suppose that E is not PL-pluripolar. Then by the above reasoning we have $V^*_{P,E}(a) < +\infty$ for some $a \in \mathbb{C}^d$. Then there exists a polydisk $P(a,r)$ such that $C :=$

$\sup_{\overline{P}(a,r)} V_{P,E} < +\infty$. On the other hand, (ii) implies that for every $u \in \mathcal{L}_P$ with $u|_E \leq 0$ we have

$$u(z) \leq C + H_P(\frac{z-a}{r}) \leq C' + H_P(z),\ \forall z \in \mathbb{C}^d,$$

for some constant $C' > 0$ depends only on C, a, r. It follows that $V^*_{P,E} \leq C' + H_P$. We are done.

(iv) We proceed by contradiction as in the classical case $P = \Sigma$. Assume that E is not PL-pluripolar. Then by (iii) $V^*_{P,E} \in \mathcal{L}_P$ and therefore $M := \sup_E V^*_{P,E} < +\infty$. Since E is bounded, there is a polydisk $P(0,r)$ such that $E \subset P(0,r)$. Then from (ii) we obtain

$$V^*_{P,E} \geq V^*_{P,\overline{P}(0,r)} = H_P(z) - \log r = \frac{1}{k} H_\Sigma - \log r.$$

Thus we can find $R > r$ such that $\inf_{\partial P(0,R)} V^*_{P,E} \geq 2M+1$. Now we choose $u \in \mathrm{PSH}(\mathbb{C}^d)$ such that $u = -\infty$ on E and $u < 0$ on $P(0,R)$. For each positive integer $j \geq 1$ we set

$$v_j := \begin{cases} \max\{\frac{1}{j}u + 1, \frac{1}{2M+1} V^*_{P,E}\} & \text{on } P(0,R), \\ \frac{1}{2M+1} V^*_{P,E}, & \text{otherwise.} \end{cases}$$

Then $(2M+1)v_j \in \mathcal{L}_P$ and on E we have $(2M+1)v_j \leq M$. Hence $(2M+1)v_j - M \leq V_{P,E}$ on $\mathbb{C}^d$. In particular

$$(2M+1)(\frac{1}{j}u + 1) \leq M + V_{P,E} \ \text{ in } P(0,R)$$

for all $j \geq 1$. By letting $j \to \infty$ we obtain $V^*_{P,E} \geq M+1$ on E which is absurd.

(v) If A is not pluripolar, then we can find $r, M > 0$ such that $A' := \{z \in P(0,r) : u(z) < M\}$ is not pluripolar as well. So by (iv), A' is also not PL-pluripolar. Observe that

$$u_\alpha \leq V_{P,A'} + C\ \forall \alpha \in I.$$

Hence $u^* \leq V^*_{P,A'} + C$. Since the function on the right belongs to $\mathcal{L}_P$ we conclude that $u^* \in \mathcal{L}_P$. □

In the special but important case where P is an *admissible convex* body, using a standard technique for solving $\bar\partial$-equation with L^2-estimates, Bayraktar et al. proved (see Theorem 1.1 in [3]) that $V_{P,K}$ can be defined by means of polynomials. In case $P = \Sigma$, this result of course reduces to the famous Siciak-Zakharyuta approximation theorem. We state below a slight generalization of this fundamental result. We only need it in Proposition 4.3 of the next section.

Theorem 2.5. *Let P be an admissible compact subset of $(\mathbb{R}^+)^d$ which is in general position, i.e., P is not included in a real hyperplane in $\mathbb{R}^d$. Then for any non-pluripolar compact subset K in $\mathbb{C}^d$ we have*

$$V_{P,K} = \lim_{n\to\infty} \frac{1}{n} \log \Phi_{n,K}$$

pointwise on $\mathbb{C}^d$. *Here*

$$\Phi_{n,K}(z) := \sup\{|p_n(z)| : p_n \in Poly(n\tilde{P}),\ \|p_n\|_K \le 1\}$$

and $\tilde{P}$ *is the closure of the convex hull of* P.

We will sometimes write Φ_n instead of $\Phi_{n,K}$ if there is no danger of confusion. The above result follows immediately from the mentioned above Theorem 1.1 in [3], Proposition 3.1(v) (in the next section) and the following simple fact.

Lemma 2.6. *Let* P *and* $\tilde{P}$ *be given as in Theorem 2.5. Then* $\tilde{P}$ *is an admissible convex body.*

Proof. First we let $P' \subset P$ be a maximal set of linearly independent vectors. By the assumption we see that P' contains exactly d elements. By a linear change of coordinates we may achieve that $P' = \{e_1, \ldots, e_d\}$, the standard basic of $\mathbb{R}^d$. Thus the convex hull of P' coincides with the standard simplex Σ. Hence $\tilde{P}$ has non-empty interior. This implies the desired conclusion. □

Finally we mention the following useful *domination principle* in $\mathcal{L}_{P,+}$: *If* $u, v \in \mathcal{L}_{P,+}$ *and satisfies* $u \le v$ *a.e. with respect to* $(dd^c v)^d$, *then* $u \le v$ *on* $\mathbb{C}^d$.

In the case $P = \Sigma$, the above result is proved by Bedford and Taylor in [4]. The general case can be demonstrated exactly in the same fashion (see Proposition 2.2 in [13]).

3. Convergence of P-extremal functions

We start by collecting below some continuity properties of P-extremal functions that will be used later on.

Proposition 3.1. *Let* E *be a bounded set in* $\mathbb{C}^d$ *and* K *be a compact set. Then we have the following assertions*:

(i) *If* $K_j \downarrow K$ *and if* K_j *are compact, then* $V_{P,K_j} \uparrow V_{P,K}$.

(ii) *If* $E_j \uparrow E$, *then* $V^*_{P,E_j} \downarrow V^*_{P,E}$.

(iii) $V^*_{P,E\setminus F} = V^*_{P,E}$ *if* F *is pluripolar.*

(iv) *If* $V^*_{P,K} \equiv 0$ *on* K, *then* $V_{P,K}$ *is continuous on* $\mathbb{C}^d$. *Thus, if* P *is admissible, then* K *is* L*-regular if and only if it is* PL*-regular.*

(v) *If* $\tilde{P}$ *is the closure of the convex hull of* P, *then we have*

$$V_{P,E} = V_{\tilde{P},E} \text{ on } \mathbb{C}^d.$$

(vi) *Let* P_j *be a sequence of bounded sets in* $\mathbb{R}^d$ *that decreases to* P. *Suppose that* E *has non-empty interior. Then we have* $V^*_{P_j,E} \downarrow V^*_{P,E}$.

We do not know if (vi) still holds for general bounded sets E.

Proof. Using (iv) of Proposition 2.4 we see that all the assertions (i)-(iv) can be proved in the same fashion as in the classical case $P = \Sigma$ (see the discussion after Propositions 2.1 and 2.3 in [10]). Note that Lemma 2.2 is needed for the second assertion of (iv).

(v) Using the definitions of support functions we obtain $h_P = h_{\tilde{P}}$ on $\mathbb{R}^d$. It follows that $H_P = H_{\tilde{P}}$ on $\mathbb{C}^d$. So we are done.

(vi) The definition of P-extremal function (2.2) implies that $V^*_{P_j,E} \downarrow u \in PSH(\mathbb{C}^d)$ and $u \geq V^*_{P,E}$. After a dilation we may assume E contains a small ball $\mathbb{B}(0,r)$. By Proposition 2.4(i) we get

$$V^*_{P_j,E}(z) \leq H_{P_j}(z/r),\ \forall j.$$

Since $h_{P_j} \downarrow h_P$ on $\mathbb{R}^d$ we conclude that $H_{P_j} \downarrow H_P$ on $\mathbb{C}^d$. It follows that

$$u(z) \leq H_P(z/r),\ \forall z \in \mathbb{C}^d.$$

Observe that there exists a pluripolar subset X of $\mathbb{C}^d$ such that $u = 0$ on $E \setminus X$. Thus, by Proposition 2.4(iv) we can find $v \in \mathcal{L}_P$ with $v|_E < 0$ and $v|_X = -\infty$. Then by (2.2), for every $\varepsilon > 0$ we get

$$\frac{1}{1+\varepsilon}(u + \varepsilon v) \leq V_{P,E}.$$

Therefore, by letting $\varepsilon \to 0$ we obtain $u \leq V_{P,E}$ on $\mathbb{C}^d \setminus v^{-1}(-\infty)$. Hence $u \leq V^*_{P,E}$ entirely on $\mathbb{C}^d$. So $u = V^*_{P,E}$ as desired. □

Now we are able to present the main result of this section about convergence of P-extremal functions.

Theorem 3.2. *Let $\{K_j\}$ be a sequence of subsets of K. Consider the following assertions*:

(i) $V^*_{P,K_j} \to 0$ *q.e. on* K.

(ii) $V^*_{P,K_j} \to V^*_{P,K}$ *pointwise on* $\mathbb{C}^d$;

(iii) $V^*_{P,K_j} \to V^*_{P,K}$ *uniformly on* $\mathbb{C}^d$;

(iv) $V^*_{K_j} \to 0$ *q.e. on* K.

(v) $V^*_{K_j} \to V^*_K$ *pointwise on* $\mathbb{C}^d$;

(vi) $V^*_{K_j} \to V^*_K$ *uniformly on* $\mathbb{C}^d$.

Then (i)⇔(ii)⇔(iii) *if K is PL-regular,* (iv)⇔(v)⇔(vi) *if K is L-regular, and* (i)⇔(iv) *if P is admissible.*

Proof. First we consider the case K is PL-regular.

(i) ⇒ (ii) We can assume that K_j is non-pluripolar for all $j \geq 1$. Then $V^*_{P,K_j} \in \mathcal{L}_{P,+},\ \forall j \geq 1$. For $s \geq 1$, define

$$v_s(z) := \sup_{j \geq s} V^*_{P,K_j}(z),\ z \in \mathbb{C}^d.$$

Then the set $\{v_1 < +\infty\}$ contains a non-pluripolar subset of K. Thus Proposition 2.4(v) implies that $v^*_s \in \mathcal{L}_P$ for every $s \geq 1$. Therefore

$$V^*_{P,K} \leq v := \lim \downarrow v^*_s.$$

In particular $v \in \mathcal{L}_P$, $v = 0$ q.e. on K. Here the latter equality follows from the fact that $v_s = v_s^*$ q.e. on $\mathbb{C}^d$. By Proposition 2.4(iii) we obtain $v \leq V_{P,K}^*$ on $\mathbb{C}^d$. Moreover, since $K_j \subset K$ we have

$$v \leq V_{P,K}^* \leq V_{P,K_j}^*, \ \forall j \geq 1.$$

Putting all this together we concludes that

$$\lim_{j\to\infty} V_{P,K_j}^*(z) = V_{P,K}^*(z), \ \forall z \in \mathbb{C}^d.$$

(ii) $\Rightarrow$ (iii) Since K is PL-regular it follows that $V_{P,K_j}^* \to V_{P,K}^* = 0$ on K. On the other hand, by Proposition 2.4(v), the sequence V_{P,K_j}^* is locally uniformly bounded on $\mathbb{C}^d$. Then using Hartogs' lemma we infer that $V_{P,K_j}^* \to 0$ uniformly on K. Hence, from (2.2) we deduce easily that $V_{P,K_j}^* \to V_{P,K}^*$ uniformly on $\mathbb{C}^d$.

(iii) $\Rightarrow$ (i) is trivial.

If K is L-regular, then by setting $P = \Sigma$ in the above proof we have (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi).

Finally, in case P is admissible we may apply Proposition 3.1(iv) and the comparison lemma (Lemma 2.2) to see that (i) $\Leftrightarrow$ (iv). □

Remark 3.3. 1. We do not need PL-regularity of K for the implication (i) $\Rightarrow$ (ii).

2. The assumption $V_{K_j}^* \to 0$ q.e. on K does not imply L-regularity of K. For a simple example we let K be the union of a closed disk Δ and an isolated point a while K_j is taken to be a sequence of closed disks increasing to Δ.

3. Under the assumptions that P is an admissible convex body and $V_{K_j}^* \to 0$ *pointwise* on K then by adapting the proof of the implication (i) $\Rightarrow$ (ii) to the case $P = \Sigma$ we can show that K is indeed L-regular. So in this case all the equivalent conditions in Theorem 3.2 holds true.

4. Bernstein-Markov properties

The following basic notions are central to our work. Note that the first one is a direct generalization of the classical Bernstein-Markov property and was studied in [2].

Definition 4.1. The triple (P, K, μ) is said to have the Bernstein-Markov property if for each $\varepsilon > 0$, there exists a constant $C = C_\varepsilon > 0$ such that

$$\|p\|_K \leq Ce^{n\varepsilon}\|p\|_{L^2(\mu)}, \quad \forall p \in \text{Poly}(nP), \quad n \geq 1; \tag{4.1}$$

Remark 4.2. (a) We present a class of pairs (K, μ) having the Bernstein-Markov property. Let γ be smooth, closed curve on $\{1 \leq |z| \leq 2\}$ and K be the compact set $\gamma \cup \{|z| = 1\}$. Let μ be a finite positive Borel measure on K whose support coincides with K such that $\mu|_{|z|=1}$ is the normalized Lebesgue measure. Consider an arbitrary polynomial $p(z) := a_0 + a_1 z + \cdots + a_n z^n$. By Cauchy-Schwarz's inequality and maximum principle, we obtain

$$\|p\|_K^2 \leq \frac{4^{n+1}-1}{3}(|a_0|^2 + \cdots + |a_n|^2) \leq \frac{4^{n+1}}{3}\int_{\partial\Delta} |p|^2 d\mu.$$

Thus (K,μ) enjoy the Bernstein-Markov property.

(b) If $P = \Sigma$, then (4.1) becomes (1.1). Note that in general the exponent n in (4.1) may be less than the (standard) degree of p.

We first give an easy result illustrating one use of the above notions.

Proposition 4.3. *Assume that P is an admissible compact subset of $(\mathbb{R}^+)^d$ which is in general position. Define*

$$W_{P,K,\mu}(z) := \sup\Big\{\frac{1}{n}\log|p(z)| : p \in Poly(nP), \|p\|_{L^2(\mu)} \leq 1\Big\},\ z \in \mathbb{C}^d.$$

Then, if (P,K,μ) has the strong Bernstein-Markov property, then $W^*_{P,K,\mu} = V^*_{P,K}$.

Proof. First, we note that for any $p \in \text{Poly}(nP)$ with $\|p\|_K \leq 1$ we have

$$\|p'\|_{L^2(\mu)} \leq 1 \text{ where } p' := \mu(K)^{-1/2}p.$$

It follows that

$$W_{P,K,\mu} \geq \frac{1}{n}\log|p'| = -\frac{1}{2n}\log\mu(K) + \frac{1}{n}\log|p|.$$

Now Theorem 2.5 implies that

$$W_{P,K,\mu} \geq V_{P,K}. \tag{4.2}$$

For the other direction, fix $z \in \mathbb{C}^d$, then we can find a sequence $p_j \in \text{Poly}(n_j P)$ such that $\|p_j\|_{L^2(\mu)} \leq 1$ and

$$\lim_{j\to\infty}\frac{1}{n_j}\log|p_j(z)| = W_{P,K,\mu}(z).$$

Then for $\varepsilon > 0$ we can find $C_\varepsilon > 0$ depending only on ε such that

$$\|p_j\|_K \leq C_\varepsilon e^{n_j(\varepsilon)}\|p_j\|_{L^2(\mu)} \leq C_\varepsilon e^{n_j(\varepsilon)}.$$

Set $p'_j := C_\varepsilon^{-1}e^{-n_j(\varepsilon)}p_j$. Then we have

$$V_{P,K}(z) \geq \frac{1}{n_j}\log|p_j(z)| = -\frac{\log C_\varepsilon}{n_j} - \varepsilon + \frac{1}{n_j}\log|p_j(z)|.$$

By letting $j \to \infty$ and then $\varepsilon \to 0$ we obtain

$$V_{P,K}(z) \geq W_{P,K,\mu}(z). \tag{4.3}$$

The result is desired from (4.2) and (4.3). □

Remark 4.4. The extremal function $W_{P,K,\mu}$ can also be defined by n-th Bergman function on $\text{Poly}(nP)$ (see Section 5.1 in [2]).

The Bernstein-Markov property also yields the following curious result.

Proposition 4.5. *Let P be as in the above proposition. Suppose further that (P, K, μ) has the Bernstein-Markov property. Then for any compact subset L of $\mathbb{C}^d$ we have*

$$\sup_K V_{P,L} = \lim_{n\to\infty} \frac{1}{2n} \log\Big(\int_K \Phi_{n,L}^2 d\mu\Big).$$

Here $\Phi_{n,L}$ is the function introduced in Theorem 2.5.

Proof. First by Bernstein-Walsh inequality we have $\Phi_{n,L} \le e^{nV_{P,L}}$ on K. Hence

$$\int_K \Phi_{n,L}^2 d\mu \le \mu(K) e^{2n \sup\limits_K V_{P,L}}.$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{2n} \log\Big(\int_K \Phi_{n,L}^2 d\mu\Big) \le \sup_K V_{P,L}. \tag{4.4}$$

For the reverse direction, we first use Bernstein-Markov property of (P, K, μ) to derive the following estimate: For any $\delta > 0$ there exists a constant $C > 0$ independent of n such that

$$\sup_{z\in K} \Phi_{n,L}(z) \le C e^{n\delta} \Big(\int_K \Phi_{n,L}^2 d\mu\Big)^{1/2}.$$

Hence

$$\sup_{z\in K} \frac{1}{n} \log \Phi_{n,L}(z) \le \delta + \frac{\log C}{n} + \frac{1}{2n} \log\Big(\int_K \Phi_{n,L}^2 d\mu\Big).$$

Since $\delta > 0$ can be arbitrarily small we infer that

$$\liminf_{n\to\infty} \Big(\sup_{z\in K} \frac{1}{n} \log \Phi_{n,L}(z)\Big) \le \limsup_{n\to\infty} \frac{1}{2n} \log\Big(\int_K \Phi_{n,L}^2 d\mu\Big). \tag{4.5}$$

Next, by Theorem 2.5 the sequence $\frac{1}{n} \log \Phi_{n,L}$ converges to $V_{P,L}$ pointwise (from below) on $\mathbb{C}^d$. So we get

$$\lim_{n\to\infty} \Big(\sup_{z\in K} \frac{1}{n} \log \Phi_{n,L}(z)\Big) = \sup_K V_{P,L}. \tag{4.6}$$

Hence by combining (4.4), (4.5) and (4.6) we reach the desired conclusion. $\square$

We now give a sufficient condition, in the same spirit as Theorem 1.1, for the triple (P, K, μ) to have the Bernstein-Markov property.

Theorem 4.6. *Let K be a compact PL-regular set in $\mathbb{C}^d$, μ be a finite positive Borel measure on K. Suppose for each $\varepsilon > 0$ there exist two constants $T > 0$, $\alpha \in (0, \frac{1}{T+1})$ and a sequence $r_n \downarrow 0$ satisfying the following conditions*:

(i) $\inf\limits_{n\ge 1} r_n e^{\frac{n\varepsilon(1-\alpha)}{T}} > 0$;

(ii) $\lim\limits_{n\to\infty} r_n e^{\alpha n\varepsilon} = 0$;

(iii) $V^*_{P,E_n} \to 0$ *q.e. on K, where*

$$E_n := \{z \in K : \mu(K \cap B(z, r_n)) \ge r_n^T\}.$$

Then the triple (P, K, μ) has the Bernstein-Markov property.

Before giving the proof of theorem, we would like to state some notes.

Remark 4.7. (a) Since $\frac{1-\alpha}{T} > \alpha$, we can always find a sequence $r_n \downarrow 0$ satisfying (i) and (ii).

(b) The above result formally generalizes Theorem 1.1. Nevertheless, it is unclear to us if Theorem 4.6 is really stronger than Theorem 1.1 even in the case $P = \Sigma$. Indeed, according to Remark 3.5 in [8], there exists no known example of measure μ such that (K, μ) has the strong Bernstein-Markov property but μ does not satisfy the condition in Theorem 1.1 for *any* $T > 0$.

(c) If P is admissible in the sense of (2.1), then P is also L-regular. So using Theorem 3.2 we see that the condition $V^*_{P,E_n} \to 0$ q.e. on K is equivalent to $V^*_{E_n} \to 0$ q.e. on K. Hence it may be interpreted as the convergence of *relative* capacities of E_n towards that of K (with respect to some ball containing K), in view of Theorem 1.1 in [7].

Proof. We rely on Bloom-Levenberg's methods given in [7] and [6]. Fix $\varepsilon > 0$ and let T, α, r_n be numbers satisfying (i), (ii) and (iii). We now divide the rest of the proof into some steps.

Step 1. Set $\varepsilon' := \alpha\varepsilon$. We claim that there exists $\delta > 0$ such that for all n large enough we have

$$\|p\|_{K_\delta} \leq \|p\|_{E_n} e^{n\varepsilon'}, \ \forall p \in \text{Poly}(nP), \tag{4.7}$$

where $K_\delta := \{z \in \mathbb{C}^d : d(z, K) \leq \delta\}$. To see this, we first apply Proposition 3.1 to see that $V_{P,K_\delta} \downarrow V_{P,K}$ on $\mathbb{C}^d$. Since $V_{P,K}$ is continuous on $\mathbb{C}^d$, by Dini's theorem we can choose $\delta = \delta(\varepsilon')$ such that

$$|V_{P,K}(z) - V_{P,K_\delta}(z)| < \frac{\varepsilon'}{2}, \ \ \forall z \in K_\delta.$$

In particular, since $V_{P,K_\delta} = 0$ on K_δ we get

$$V_{P,K}(z) \leq \frac{\varepsilon'}{2}, \ \forall z \in K_\delta.$$

The Bernstein-Walsh inequality (Proposition 2.3) now implies that for any $n \geq 1$ and $p \in \text{Poly}(nP)$ we have

$$\|p\|_{K_\delta} \leq \|p\|_K e^{n\varepsilon'/2}.$$

On the other hand, by the hypothesis $V^*_{P,E_n} \to 0$ q.e. on K, so by Theorem 2.5, we see that the family V^*_{P,E_n} is locally uniformly bounded from above on $\mathbb{C}^d$. So by shrinking δ and using Theorem 3.2 we see that for all n sufficiently large

$$V^*_{P,E_n}(z) \leq \frac{\varepsilon'}{2}, \ \forall z \in K.$$

Using again the Bernstein-Walsh inequality for E_n we obtain

$$\|p\|_K \leq \|p\|_{E_n} e^{n\varepsilon'/2}, \ \forall p \in \text{Poly}(nP). \tag{4.8}$$

Combining these last estimates we obtain (4.7).

Step 2. We will show for all n large enough and all $p \in \text{Poly}(nP)$, $w \in E_n (\subset K)$

$$|p(z)| \geq |p(w)| - \frac{1}{2}\|p\|_{E_n}, \ \forall |z-w| < r_n. \tag{4.9}$$

For $z \neq w$ we put $e = \frac{z-w}{\|z-w\|} = (e_1, \dots, e_d)$. Put $q(t) := p(w_1 + e_1 t, \dots, w_d + e_d t)$. Then q is a polynomial of one complex variable t with $p(z) = q(\|z-w\|)$ and $p(w) = q(0)$. Then

$$p(z) - p(w) = q(\|z-w\|) - q(0) = \int_0^{\|z-w\|} q'(t) dt.$$

So for $r' > r := \|z-w\| > 0$ we have

$$|p(z) - p(w)| \leq r\|q'\|_{|t|<r} \leq r\frac{\|q\|_{|t|<r'}}{r'-r} \leq \frac{r}{r'-r}\|p\|_{K_{r'}}.$$

Here we use Cauchy's inequality in the second estimate. Choose the parameters

$$r := r_n, \ r' := r_n(1 + 2e^{n\varepsilon'}).$$

By Step 1 and the assumption (ii), we obtain for n large enough the following estimate

$$|p(z)| \geq |p(w)| - \frac{\|p\|_{K_{r'}}}{2e^{n\varepsilon'}} \geq |p(w)| - \frac{1}{2}\|p\|_{E_n}.$$

We finish the proof of this step.

Step 3. Completion of the proof. Fix $p \in \text{Poly}(nP)$. Then for each $w \in E_n$, from (4.9) we obtain the following chain of estimates

$$\begin{aligned}\|p\|_{L^2(\mu)} &= \Big(\int_K |p|^2 d\mu\Big)^{\frac{1}{2}} \geq \Big(\int_{B(w,r_n)\cap K} |p|^2 d\mu\Big)^{\frac{1}{2}} \\ &\geq \mu(B(w,r_n)\cap K)^{1/2} \inf_{B(w,r_n)} |p(z)| \\ &\geq r_n^{T/2}\big(|p(w)| - \frac{1}{2}\|p\|_{E_n}\big).\end{aligned}$$

Taking supremum over $w \in E_n$ and using (4.8) we get

$$\|p\|_{L^2(\mu)} \geq \frac{1}{2} r_n^{T/2}\|p\|_{E_n} \geq \frac{1}{2} r_n^{T/2} e^{-n\varepsilon'/2}\|p\|_K.$$

So in view of the property (i), there exists a constant $C_\varepsilon > 0$ such that for $n \geq n_0$ large enough we have

$$C_\varepsilon e^{n\varepsilon/2}\|p\|_{L^2(\mu)} \geq \|p\|_K.$$

Finally, since $\text{Poly}(n_0 P)$ is a finite dimension space, the norm $\|.\|_{L^2(\mu)}$ and the sup-norm are equivalent. The proof is thereby completed. □

We have the following result which gives examples of measures satisfying the condition of Theorem 4.6.

Proposition 4.8. *Let K be a compact set in $\mathbb{C}^d$, μ be a finite positive Borel measure on K and $T > 0$ be a constant. Set*

$$G := \{z \in K : \liminf_{r \to 0} \frac{\mu(B(z,r) \cap K)}{r^T} > 1\}.$$

Then if K is PL-regular and $V^*_{P,G} = V_{P,K}$, then (P, K, μ) has the Bernstein-Markov property.

Remark 4.9. The second requirement is satisfied if $(dd^c V_{P,K})^d = 0$ on $K \setminus G$ in view of the domination principle in $\mathcal{L}_{P,+}$.

Proof. For $r > 0$ we set

$$f_r(z) := \frac{\mu(B(z,r) \cap K)}{r^T},\ E_r := \{z \in K : f_r(z) \geq 1\}.$$

Then we have

$$G = \bigcup_{r>0} \Big(\bigcap_{0<s<r} E_s \Big) = \bigcup_{r>0} F_r,$$

where $F_r := \bigcap_{0<s<r} E_s$. Note that $F_r \subset E_r$ and by the above reasoning $\{F_r\}_{r>0} \uparrow G$. By Proposition 3.1(iii) we get

$$V^*_{P,F_r} \downarrow V^*_{P,G} = V^*_{P,K} \text{ on } \mathbb{C}^d.$$

Since $V^*_{P,E_r} \leq V^*_{P,F_r}$ we infer $V^*_{P,E_r} \to 0$ pointwise on K as $r \to 0$. By Theorem 4.6 we obtain the desired conclusion. □

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