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MEAN VALUES OF DERIVATIVES OF QUADRATIC PRIME DIRICHLET L-FUNCTIONS IN FUNCTION FIELDS

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Abstract. In this paper, we establish an asymptotic formula for mean value of $L^{(k)}(\frac{1}{2}, \chi_P)$ averaging over $\mathbb{P}_{2g+1}$ and over $\mathbb{P}_{2g+2}$ as $g \to \infty$ in odd characteristic. We also give an asymptotic formula for mean value of $L^{(k)}(\frac{1}{2}, \chi_u)$ averaging over $\mathcal{I}_{g+1}$ and over $\mathcal{F}_{g+1}$ as $g \to \infty$ in even characteristic.

1. Introduction

The study on mean values of central values of derivative of quadratic Dirichlet L-functions in function fields was initiated by Andrade and Rajagopal [3]. Let $\mathbb{F}_q[t]$ be the polynomial ring over a finite field $\mathbb{F}_q$, where q is odd, and denote by $\mathcal{H}_n$ the set of monic square-free polynomials in $\mathbb{F}_q[t]$ of degree n. In [3], Andrade and Rajagopal gave an asymptotic formula for mean value of $L''(\frac{1}{2}, \chi_D)$ averaging over $\mathcal{H}_{2g+1}$ as $g \to \infty$. Here, $L(s, \chi_D)$ is the quadratic Dirichlet L-function associated to a quadratic character χ_D and $L''(s, \chi_D)$ is the second derivative of $L(s, \chi_D)$. It is shown by Bae and Jung [5] that $L'(\frac{1}{2}, \chi_D) = (-\ln q)g \cdot L(\frac{1}{2}, \chi_D)$ for any $D \in \mathcal{H}_{2g+1}$. Hence the moment of $L'(\frac{1}{2}, \chi_D)$ over $\mathcal{H}_{2g+1}$ is a constant multiple of that of $L(\frac{1}{2}, \chi_D)$ over $\mathcal{H}_{2g+1}$. Applying the results of Florea on k-th moment of $L(\frac{1}{2}, \chi_D)$ over $\mathcal{H}_{2g+1}$ ([7–9]), they obtained the k-th moment of $L'(\frac{1}{2}, \chi_D)$ over $\mathcal{H}_{2g+1}$ for $1 \leq k \leq 4$. They also improved the error term and showed that there is an extra term in the asymptotic formula of Andrade and Rajagopal [3] for the first moment of $L''(\frac{1}{2}, \chi_D)$. Andrade and Jung [2] established an asymptotic formula for mean value of $L^{(k)}(\frac{1}{2}, \chi_D)$ averaging over $\mathcal{H}_{2g+1}$ and over $\mathcal{H}_{2g+2}$ as $g \to \infty$, where $L^{(k)}(s, \chi_D)$ is the k-th derivative of $L(s, \chi_D)$. Recently, Bae and Jung [6] established an even characteristic analogue of the result of Andrade and Jung. The aim of this paper is to give asymptotic formulas for mean values

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of central values of k-th derivative of prime Dirichlet L-functions in both odd characteristic and even characteristic.

Let us fix some notations. Let $\mathrm{k} = \mathbb{F}_q(t)$ be the rational function field over a finite field $\mathbb{F}_q$ and $\mathbb{A} = \mathbb{F}_q[t]$. Denote by $\mathbb{A}^+$ the set of monic polynomials in $\mathbb{A}$ and by $\mathbb{P}$ the set of monic irreducible polynomials in $\mathbb{A}$. Write $\mathbb{A}_n^+ = \{f \in \mathbb{A}^+ : \deg(f) = n\}$ and $\mathbb{P}_n = \mathbb{P} \cap \mathbb{A}_n^+$. Let $\pi_q(n) = |\mathbb{P}_n|$ be the number of monic irreducible polynomials in $\mathbb{A}$ of degree n. The prime polynomial theorem says that (see [10, Theorem 2.2])

$$\pi_q(n) = \frac{q^n}{n} + O\left(\frac{q^{\frac{n}{2}}}{n}\right). \tag{1.1}$$

Let $\zeta_{\mathbb{A}}(s)$ be the zeta function of $\mathbb{A}$ defined by

$$\zeta_{\mathbb{A}}(s) = \sum_{f \in \mathbb{A}^+} \frac{1}{|f|^s}, \quad \mathrm{Re}(s) > 1,$$

where $|f| = q^{\deg(f)}$. It is well known that $\zeta_{\mathbb{A}}(s) = \frac{1}{1-q^{1-s}}$.

2. Odd characteristic case

In this section we assume that q is odd. For any monic square-free polynomial D in $\mathbb{A}$, the quadratic Dirichlet L-function $L(s, \chi_D)$ attached to a quadratic character χ_D is defined as follows:

$$L(s, \chi_D) = \sum_{f \in \mathbb{A}^+} \frac{\chi_D(f)}{|f|^s}, \quad \mathrm{Re}(s) > 1.$$

Let $J_k(n) = \sum_{m=1}^n m^k$ for any positive integer n. We establish an asymptotic formula for mean value of $L^{(k)}(\frac{1}{2}, \chi_P)$ averaging over $\mathbb{P}_{2g+1}$ and over $\mathbb{P}_{2g+2}$ as $g \to \infty$.

Theorem 2.1. *Let k be a fixed positive integer. As $g \to \infty$, we have that*

(1)

$$\sum_{P \in \mathbb{P}_{2g+1}} \frac{L^{(k)}(\frac{1}{2}, \chi_P)}{(\ln q)^k}$$
$$= 2^k \frac{q^{2g+1}}{2g+1} \left((-1)^k J_k([\tfrac{g}{2}]) + \sum_{m=0}^{k} \binom{k}{m} (-g)^{k-m} J_m([\tfrac{g-1}{2}]) \right) + O\left(g^k q^{\frac{3g}{2}}\right),$$

(2)

$$\sum_{P \in \mathbb{P}_{2g+2}} \frac{L^{(k)}(\frac{1}{2}, \chi_P)}{(-\ln q)^k}$$
$$= 2^{k-1} \frac{q^{2g+2}}{g+1} \left(J_k([\tfrac{g}{2}]) + \sum_{a+b+c=k} \frac{(-1)^{b+c} k!}{a!b!c!} \frac{g^a \delta^{(b)}(\frac{1}{2})}{(2 \ln q)^b} J_c([\tfrac{g-1}{2}]) \right)$$

$$- \zeta_{\mathbb{A}}(2) \frac{q^{\frac{3g}{2}+1}}{2g+2} \left((g+1)^k q^{[\frac{g}{2}]-\frac{1}{2}} + q^{[\frac{g-1}{2}]} \sum_{m=0}^{k} \binom{k}{m} \frac{g^{k-m} \delta^{(m)}(\frac{1}{2})}{(-\ln q)^m} \right) + O\left(g^k q^{\frac{3g}{2}} \right),$$

where $\delta(s) = \frac{1-q^{-s}}{1-q^{s-1}}$.

2.1. Preparations for the proof

In this subsection we present some auxiliary lemmas which are needed in proof of Theorem 2.1.

The following lemma is quoted from Rudnick [11, (2.5)], and it is proved by using the explicit formula for $L(s, \chi_P)$ and the Riemann hypothesis for curves.

Lemma 2.2. *For any non-constant* $f \in \mathbb{A}^+$*, which is not a perfect square, we have*

$$\sum_{P \in \mathbb{P}_n} \chi_P(f) \ll \frac{q^{\frac{n}{2}}}{n} \deg(f).$$

Let k be a fixed positive integer. For $h \in \{g-1, g\}$, $m \in \{0, 1, \ldots, k\}$ and $n \in \{2g+1, 2g+2\}$, we define two sums $\mathcal{S}_{h,m}(\mathbb{P}_n)$ and $\mathcal{T}_h(\mathbb{P}_n)$ as follows:

$$\mathcal{S}_{h,m}(\mathbb{P}_n) = \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{f \in \mathbb{A}_\ell^+} \sum_{P \in \mathbb{P}_n} \chi_P(f)$$

and

$$\mathcal{T}_h(\mathbb{P}_n) = q^{-\frac{h+1}{2}} \sum_{\ell=0}^{h} \sum_{f \in \mathbb{A}_\ell^+} \sum_{P \in \mathbb{P}_n} \chi_P(f).$$

We first give an asymptotic formula of $\mathcal{S}_{h,m}(\mathbb{P}_n)$.

Lemma 2.3. *For* $h \in \{g-1, g\}$*,* $m \in \{0, 1, \ldots, k\}$ *and* $n \in \{2g+1, 2g+2\}$*, we have*

$$\mathcal{S}_{h,m}(\mathbb{P}_n) = 2^m J_m([\tfrac{h}{2}]) \frac{q^n}{n} + O\left(g^m q^{\frac{3g}{2}} \right).$$

Proof. We split the sum $\mathcal{S}_{h,m}(\mathbb{P}_n)$ over f with f being a square or a non-square to obtain

$$\mathcal{S}_{h,m}(\mathbb{P}_n) = \mathcal{S}_{h,m}(\mathbb{P}_n)_{\square} + \mathcal{S}_{h,m}(\mathbb{P}_n)_{\neq\square}, \tag{2.1}$$

where

$$\mathcal{S}_{h,m}(\mathbb{P}_n)_{\square} = \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{\square = f \in \mathbb{A}_\ell^+} \sum_{P \in \mathbb{P}_n} \chi_P(f)$$

and

$$\mathcal{S}_{h,m}(\mathbb{P}_n)_{\neq\square} = \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{\square\neq f\in\mathbb{A}_\ell^+} \sum_{P\in\mathbb{P}_n} \chi_P(f).$$

For the contribution of non-squares, we use Lemma 2.2 to deduce that

$$\left|\mathcal{S}_{h,m}(\mathbb{P}_n)_{\neq\square}\right| \ll \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{f\in\mathbb{A}_\ell^+} \left|\sum_{P\in\mathbb{P}_n} \chi_P(f)\right|$$

$$\ll \frac{q^{\frac{n}{2}}}{n} \sum_{\ell=0}^{h} \ell^{m+1} q^{\frac{\ell}{2}} \ll g^m q^{\frac{3g}{2}}. \tag{2.2}$$

For $f \in \mathbb{A}_\ell^+$ with $0 \leq \ell \leq h$, if f is a square, that is, $f = L^2$ for some $L \in \mathbb{A}_\alpha^+$, then $\ell = 2\alpha$ and $\chi_P(f) = \chi_P(L^2) = 1$. Thus we have

$$\mathcal{S}_{h,m}(\mathbb{P}_n)_{\square} = \sum_{\alpha=0}^{[\frac{h}{2}]} (2\alpha)^m q^{-\alpha} \sum_{L\in\mathbb{A}_\alpha^+} \sum_{P\in\mathbb{P}_n} \chi_P(L^2) = 2^m J_m([\tfrac{h}{2}])\pi_q(n).$$

Then, by the prime polynomial theorem (1.1), we deduce that

$$\mathcal{S}_{h,m}(\mathbb{P}_n)_{\square} = 2^m J_m([\tfrac{h}{2}]) \frac{q^n}{n} + O(g^m q^g), \tag{2.3}$$

since

$$2^m J_m([\tfrac{h}{2}]) \frac{q^{\frac{n}{2}}}{n} \ll g^m q^g.$$

Inserting (2.2) and (2.3) into (2.1), we complete the proof. □

We now give an asymptotic formula of $\mathcal{T}_h(n)$.

Lemma 2.4. *For $h \in \{g-1, g\}$ and $n \in \{2g+1, 2g+2\}$, we have*

$$\mathcal{T}_h(\mathbb{P}_n) = \zeta_{\mathbb{A}}(2) q^{[\frac{h}{2}]-\frac{h+3}{2}} \frac{q^n}{n} + O\left(q^{\frac{3g}{2}}\right).$$

Proof. Splitting the sum $\mathcal{T}_h(\mathbb{P}_n)$ over f with f being a square or a non-square, we can write

$$\mathcal{T}_h(\mathbb{P}_n) = \mathcal{T}_h(\mathbb{P}_n)_{\square} + \mathcal{T}_h(\mathbb{P}_n)_{\neq\square}, \tag{2.4}$$

where

$$\mathcal{T}_h(\mathbb{P}_n)_{\square} = q^{-\frac{h+1}{2}} \sum_{\ell=0}^{h} \sum_{\square=f\in\mathbb{A}_\ell^+} \sum_{P\in\mathbb{P}_n} \chi_P(f)$$

and

$$\mathcal{T}_h(\mathbb{P}_n)_{\neq\square} = q^{-\frac{h+1}{2}} \sum_{\ell=0}^{h} \sum_{\square\neq f\in\mathbb{A}_\ell^+} \sum_{P\in\mathbb{P}_n} \chi_P(f).$$

Using Lemma 2.2, we obtain that

$$|\mathcal{T}_h(\mathbb{P}_n)_{\neq\Box}| \ll q^{-\frac{h+1}{2}} \sum_{\ell=0}^{h} \sum_{f\in\mathbb{A}_\ell^+} \left| \sum_{P\in\mathbb{P}_n} \chi_P(f) \right| \ll q^{-\frac{h+1}{2}} \frac{q^{\frac{n}{2}}}{n} \sum_{\ell=0}^{h} \ell q^\ell \ll q^{\frac{3g}{2}}. \tag{2.5}$$

As in the proof of Lemma 2.3, if $f \in \mathbb{A}_\ell^+$ is a square, then $f = L^2$ for some $L \in \mathbb{A}_\alpha^+$ with $\ell = 2\alpha$ and $\chi_P(f) = \chi_P(L^2) = 1$. Thus we have

$$\mathcal{T}_h(\mathbb{P}_n)_\Box = \zeta_\mathbb{A}(2) q^{-\frac{h+3}{2}} \left(q^{[\frac{h}{2}]} - 1\right) \pi_q(n).$$

Then, by the prime polynomial theorem (1.1), we deduce that

$$\mathcal{T}_h(\mathbb{P}_n)_\Box = \zeta_\mathbb{A}(2) q^{-\frac{h+3}{2}} (q^{[\frac{h}{2}]} - 1) \frac{q^n}{n} + O\left(\frac{q^g}{g}\right), \tag{2.6}$$

since

$$\zeta_\mathbb{A}(2) q^{-\frac{h+3}{2}} (q^{[\frac{h}{2}]} - 1) \frac{q^{\frac{n}{2}}}{n} \ll \frac{q^g}{g}.$$

We can complete the proof by inserting (2.5) and (2.6) into (2.4) and arranging the terms. $\Box$

2.2. Proof of Theorem 2.1(1)

In this subsection, we give a proof of first part of Theorem 2.1. Let $P \in \mathbb{P}_{2g+1}$. From Lemma 5.1 in [2], we have

$$\begin{aligned} \frac{L^{(k)}(\frac{1}{2}, \chi_P)}{(\ln q)^k} &= \sum_{\ell=0}^{g} (-\ell)^k q^{-\frac{\ell}{2}} \sum_{f\in\mathbb{A}_\ell^+} \chi_P(f) \\ &+ \sum_{m=0}^{k} \binom{k}{m} (-2g)^{k-m} \sum_{\ell=0}^{g-1} \ell^m q^{-\frac{\ell}{2}} \sum_{f\in\mathbb{A}_\ell^+} \chi_P(f). \end{aligned} \tag{2.7}$$

Summing (2.7) over $\mathbb{P}_{2g+1}$, we obtain that

$$\begin{aligned} \sum_{P\in\mathbb{P}_{2g+1}} \frac{L^{(k)}(\frac{1}{2}, \chi_P)}{(\ln q)^k} &= (-1)^k \mathcal{S}_{g,k}(\mathbb{P}_{2g+1}) \\ &+ \sum_{m=0}^{k} \binom{k}{m} (-2g)^{k-m} \mathcal{S}_{g-1,m}(\mathbb{P}_{2g+1}). \end{aligned} \tag{2.8}$$

It follows from Lemma 2.3 that

$$\mathcal{S}_{g,k}(\mathbb{P}_{2g+1}) = 2^k J_k([\tfrac{g}{2}]) \frac{q^{2g+1}}{2g+1} + O\left(g^k q^{\frac{3g}{2}}\right) \tag{2.9}$$

and

$$\mathcal{S}_{g-1,m}(\mathbb{P}_{2g+1}) = 2^m J_m([\tfrac{g-1}{2}]) \frac{q^{2g+1}}{2g+1} + O\left(g^m q^{\frac{3g}{2}}\right). \tag{2.10}$$

Inserting (2.9) and (2.10) into (2.8) and arranging the terms, we complete the proof.

2.3. Proof of Theorem 2.1(2)

In this subsection, we give a proof of second part of Theorem 2.1. Let $P \in \mathbb{P}_{2g+2}$. From Lemma 6.1 in [2], we have

$$\begin{aligned}\frac{L^{(k)}(\frac{1}{2},\chi_P)}{(-\ln q)^k} &= \sum_{\ell=0}^{g} \ell^k q^{-\frac{\ell}{2}} \sum_{f\in\mathbb{A}_\ell^+} \chi_P(f) - (g+1)^k q^{-\frac{g+1}{2}} \sum_{\ell=0}^{g}\sum_{f\in\mathbb{A}_\ell^+}\chi_P(f) \\ &\quad + \sum_{a+b+c=k} \frac{k!}{a!b!c!} \frac{(2g)^a\delta^{(b)}(\frac{1}{2})}{(-\ln q)^b} \sum_{\ell=0}^{g-1}(-\ell)^c q^{-\frac{\ell}{2}} \sum_{f\in\mathbb{A}_\ell^+}\chi_P(f) \\ &\quad - q^{-\frac{g}{2}} \sum_{m=0}^{k}\binom{k}{m}\frac{g^{k-m}\delta^{(m)}(\frac{1}{2})}{(-\ln q)^m}\sum_{\ell=0}^{g-1}\sum_{f\in\mathbb{A}_\ell^+}\chi_P(f). \qquad (2.11)\end{aligned}$$

Summing (2.11) over $\mathbb{P}_{2g+2}$, we have

$$\begin{aligned}\sum_{P\in\mathbb{P}_{2g+2}} \frac{L^{(k)}(\frac{1}{2},\chi_P)}{(-\ln q)^k} &= \mathcal{S}_{g,k}(\mathbb{P}_{2g+2}) + \sum_{a+b+c=k}\frac{(-1)^c k!}{a!b!c!}\frac{(2g)^a\delta^{(b)}(\frac{1}{2})}{(-\ln q)^b}\mathcal{S}_{g-1,c}(\mathbb{P}_{2g+2}) \\ &\quad - (g+1)^k\mathcal{T}_g(\mathbb{P}_{2g+2}) - \sum_{m=0}^{k}\binom{k}{m}\frac{g^{k-m}\delta^{(m)}(\frac{1}{2})}{(-\ln q)^m}\mathcal{T}_{g-1}(\mathbb{P}_{2g+2}). \qquad (2.12)\end{aligned}$$

It follows from Lemma 2.3 that

$$S_{g,k}(\mathbb{P}_{2g+2}) = 2^k J_k([\tfrac{g}{2}])\frac{q^{2g+2}}{2g+2} + O\left(g^k q^{\frac{3g}{2}}\right) \qquad (2.13)$$

and

$$\mathcal{S}_{g-1,c}(\mathbb{P}_{2g+2}) = 2^c J_c([\tfrac{g-1}{2}])\frac{q^{2g+2}}{2g+2} + O\left(g^c q^{\frac{3g}{2}}\right). \qquad (2.14)$$

It also follows from Lemma 2.4 that

$$\mathcal{T}_g(\mathbb{P}_{2g+2}) = \zeta_{\mathbb{A}}(2) q^{[\frac{g}{2}]-\frac{g+3}{2}}\frac{q^{2g+2}}{2g+2} + O\left(q^{\frac{3g}{2}}\right) \qquad (2.15)$$

and

$$\mathcal{T}_{g-1}(\mathbb{P}_{2g+2}) = \zeta_{\mathbb{A}}(2) q^{[\frac{g-1}{2}]-\frac{g}{2}-1}\frac{q^{2g+2}}{2g+2} + O\left(q^{\frac{3g}{2}}\right). \qquad (2.16)$$

We can complete the proof by inserting (2.13), (2.14), (2.15) and (2.16) into (2.12) and arranging terms.

3. Even characteristic case

In this section q will be assumed to be even. Let us first recall some basic facts on the quadratic function fields of prime conductor briefly. For more details, we refer [1, §2.2] or [4, §2.2, §2.3]. Let $\wp : \mathrm{k} \to \mathrm{k}$ be the additive homomorphism defined by $\wp(x) = x^2 + x$. For any $u \in \mathrm{k} \setminus \wp(\mathrm{k})$, let $K_u = \mathrm{k}(x_u)$, where x_u is a zero of $X^2 + X + u = 0$. Fix an element $\xi \in \mathbb{F}_q \setminus \wp(\mathbb{F}_q)$. Then any separable quadratic extension K of k of prime conductor is of the form $K = K_u$, where $u \in \mathrm{k}$ can be uniquely normalized as follows:

$$u = \frac{A}{P} + \sum_{\ell=1}^{s} \alpha_\ell T^{2\ell-1} + \alpha,$$

where $P \in \mathbb{P}$, $0 \neq A \in \mathbb{A}$ with $\deg(A) < \deg(P)$, $\alpha \in \{0, \xi\}$, $\alpha_\ell \in \mathbb{F}_q$ and $\alpha_s \neq 0$ for $s > 0$. The infinite prime $(1/T)$ of k splits, is inert or ramified in K_u according as $s = 0$ and $\alpha = 0$, $s = 0$ and $\alpha = \xi$, or $s > 0$. Then the field K_u is called real, inert imaginary, or ramified imaginary, respectively. The discriminant D_u of K_u is P^2 if $s = 0$ and $P^2 \cdot (1/T)^{2s}$ otherwise, and the genus g_u of K_u is $\deg(P) - 1$ if $s = 0$ and $\deg(P) + s - 1$ otherwise.

Let $\mathcal{F}$ be the set of non-zero rational functions $u \in \mathrm{k}$ such that $u = \frac{A}{P}$ for some $P \in \mathbb{P}$ and $0 \neq A \in \mathbb{A}$ with $\deg(A) < \deg(P)$. Then, under the correspondence $u \mapsto K_u$, $\mathcal{F}$ corresponds to the set of all real separable quadratic extensions K_u of k of prime conductor. For $P \in \mathbb{P}$, let $\mathcal{F}_P$ be the set of rational functions $u \in \mathcal{F}$ whose denominator is P. Then $\mathcal{F}$ is the disjoint union of $\mathcal{F}_P$ with $P \in \mathbb{P}$. For $u \in \mathcal{F}_P$, the genus g_u of K_u is $\deg(P) - 1$. For $n \geq 1$, let $\mathcal{F}_n$ be the union of $\mathcal{F}_P$ with $P \in \mathbb{P}_n$. Then $\mathcal{F}_n$ corresponds to the set of all real separable quadratic extensions K_u of k of prime conductor with genus $n - 1$. For a positive integer s, let $\mathcal{G}_s$ be the set of polynomials $F(T) \in \mathbb{A}$ of the form

$$F(T) = \alpha + \sum_{i=1}^{s} \alpha_i T^{2i-1},$$

where $\alpha \in \{0, \xi\}, \alpha_i \in \mathbb{F}_q$ and $\alpha_s \neq 0$. For any two subsets U, V of k, write $U + V = \{u + v : u \in U, v \in V\}$. Let $\mathcal{I} = (\mathcal{F} \cup \{0\}) + \mathcal{G}$, where $\mathcal{G} = \bigcup_{s \geq 1} \mathcal{G}_s$. Then, under the correspondence $u \mapsto K_u$, $\mathcal{I}$ corresponds to the set of all ramified imaginary separable quadratic extensions K_u of k of prime conductor. For $w \in \mathcal{F}_P + \mathcal{G}_s$, the genus g_w of K_w is $\deg(P) + s - 1$. Let $\mathcal{F}_0 = \{0\}$. For any $r \geq 0$ and $s \geq 1$, let $\mathcal{I}_{(r,s)} = \mathcal{F}_r + \mathcal{G}_s$. If $w \in \mathcal{I}_{(r,s)}$, the genus g_w of K_w is $r + s - 1$. For $n \geq 1$, let $\mathcal{I}_n$ be the union of all $\mathcal{I}_{(r,s)}$, where (r, s) runs over all pairs of non-negative integers such that $s > 0$ and $r + s = n$. Then $\mathcal{I}_n$ corresponds to the set of all ramified imaginary separable quadratic extensions K_u of k of prime conductor with genus $n - 1$.

Let $u \in \mathrm{k}$ be normalized as above and χ_u be the character defined by $\chi_u(f) = \{\frac{u}{f}\}$. The L-function $L(s, \chi_u)$ associated to χ_u is defined as follows:

$$L(s,\chi_u)=\sum_{f\in\mathbb{A}^+}\frac{\chi_u(f)}{|f|^s},\quad \mathrm{Re}(s)>1.$$

We establish an asymptotic formula for mean value of $L^{(k)}(\frac{1}{2},\chi_u)$ averaging over $\mathcal{I}_{g+1}$ and over $\mathcal{F}_{g+1}$ as $g\to\infty$.

Theorem 3.1. *Let k be a fixed positive integer. As $g\to\infty$, we have that*

(1)

$$\begin{aligned}&\sum_{u\in\mathcal{I}_{g+1}}\frac{L^{(k)}(\frac{1}{2},\chi_u)}{(\ln q)^k}\\&=2^{k+1}\frac{q^{2g+1}}{g}\left((-1)^kJ_k([\tfrac{g}{2}])+\sum_{m=0}^{k}\binom{k}{m}(-g)^{k-m}J_m([\tfrac{g-1}{2}])\right)\\&\quad+O(g^{k-2}q^{2g}),\end{aligned}$$

(2)

$$\begin{aligned}&\sum_{u\in\mathcal{F}_{g+1}}\frac{L^{(k)}(\frac{1}{2},\chi_u)}{(-\ln q)^k}\\&=2^k\frac{q^{2g+2}}{g+1}\left(J_k([\tfrac{g}{2}])+\sum_{a+b+c=k}\frac{(-1)^ck!}{a!b!c!}\frac{g^a\delta^{(b)}(\frac{1}{2})}{(-2\ln q)^b}J_c([\tfrac{g-1}{2}])\right)\\&\quad-\zeta_{\mathbb{A}}(2)\frac{q^{\frac{3g}{2}+\frac{3}{2}}}{g+1}\left((g+1)^kq^{[\frac{g}{2}]}+q^{[\frac{g-1}{2}]+\frac{1}{2}}\sum_{m=0}^{k}\binom{k}{m}\frac{g^{k-m}\delta^{(m)}(\frac{1}{2})}{(-\ln q)^m}\right)\\&\quad+O\left(g^{k-1}q^{\frac{3g}{2}}\right),\end{aligned}$$

where $\delta(s)=\frac{1-q^{-s}}{1-q^{s-1}}$.

3.1. Preparations for the proof

In this subsection we present some auxiliary lemmas which are needed in proof of Theorem 3.1.

The following lemma is quoted from Lemma 4.4 and Lemma 4.6 in [1].

Lemma 3.2. *For $f\in\mathbb{A}^+$ with $\deg(f)\le 2g+1$, which is not a perfect square in $\mathbb{A}$, we have*

$$\left|\sum_{u\in\mathcal{I}_{g+1}}\chi_u(f)\right|\ll(\ln g)q^g \tag{3.1}$$

and

$$\left|\sum_{u\in\mathcal{F}_{g+1}}\chi_u(f)\right|\ll\frac{q^g}{g}. \tag{3.2}$$

Let k be a fixed positive integer. For $h \in \{g-1, g\}$ and $m \in \{0, 1, \ldots, k\}$, we define three sums $\mathcal{S}_{h,m}(\mathcal{I}_{g+1})$ and $\mathcal{S}_{h,m}(\mathcal{F}_{g+1})$, $\mathcal{T}_h(\mathcal{F}_{g+1})$ as

$$\mathcal{S}_{h,m}(\mathcal{I}_{g+1}) = \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{f \in \mathbb{A}_\ell^+} \sum_{u \in \mathcal{I}_{g+1}} \chi_u(f)$$

and

$$\mathcal{S}_{h,m}(\mathcal{F}_{g+1}) = \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{f \in \mathbb{A}_\ell^+} \sum_{u \in \mathcal{F}_{g+1}} \chi_u(f),$$

$$\mathcal{T}_h(\mathcal{F}_{g+1}) = q^{-\frac{h+1}{2}} \sum_{\ell=0}^{h} \sum_{f \in \mathbb{A}_\ell^+} \sum_{u \in \mathcal{F}_{g+1}} \chi_u(f).$$

We give an asymptotic formula of $\mathcal{S}_{h,m}(\mathcal{I}_{g+1})$.

Lemma 3.3. *For $h \in \{g-1, g\}$ and $m \in \{0, 1, \ldots, k\}$, we have*

$$\mathcal{S}_{h,m}(\mathcal{I}_{g+1}) = 2^{m+1} J_m([\tfrac{h}{2}]) \frac{q^{2g+1}}{g} + O(g^{m-2} q^{2g}).$$

Proof. We split the sum $\mathcal{S}_{h,m}(\mathcal{I}_{g+1})$ over f with f being a square or a non-square to obtain

$$\mathcal{S}_{h,m}(\mathcal{I}_{g+1}) = \mathcal{S}_{h,m}(\mathcal{I}_{g+1})_{\square} + \mathcal{S}_{h,m}(\mathcal{I}_{g+1})_{\neq\square},$$

where

$$\mathcal{S}_{h,m}(\mathcal{I}_{g+1})_{\square} = \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{\square = f \in \mathbb{A}_\ell^+} \sum_{u \in \mathcal{I}_{g+1}} \chi_u(f)$$

and

$$\mathcal{S}_{h,m}(\mathcal{I}_{g+1})_{\neq\square} = \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{\square \neq f \in \mathbb{A}_\ell^+} \sum_{u \in \mathcal{I}_{g+1}} \chi_u(f).$$

We use (3.1) to deduce that

$$\left|\mathcal{S}_{h,m}(\mathcal{I}_{g+1})_{\neq\square}\right| \ll \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{\square \neq f \in \mathbb{A}_\ell^+} \left| \sum_{u \in \mathcal{I}_{g+1}} \chi_u(f) \right|$$

$$\ll (\ln g) q^g \sum_{\ell=0}^{h} \ell^m q^{\frac{\ell}{2}} \ll g^m q^{\frac{3g}{2}}. \tag{3.3}$$

For $f \in \mathbb{A}_\ell^+$ with $0 \leq \ell \leq h$, if f is a square, that is, $f = L^2$ for some $L \in \mathbb{A}_\alpha^+$, then $\ell = 2\alpha$ and $\chi_u(f) = \chi_u(L^2) = 1$ for any $u \in \mathcal{I}_{g+1}$. Thus we have

$$\mathcal{S}_{h,m}(\mathcal{I}_{g+1})_{\square} = 2^m J_m([\tfrac{h}{2}]) |\mathcal{I}_{g+1}|.$$

From [1, Lemma 4.3], we have

$$|\mathcal{I}_{g+1}| = \frac{2q^{2g+1}}{g} + O\left(\frac{q^{2g}}{g^2}\right).$$

Thus we get

$$\mathcal{S}_{h,m}(\mathcal{I}_{g+1})_{\Box} = 2^{m+1} J_m([\tfrac{h}{2}]) \frac{q^{2g+1}}{g} + O(g^{m-2}q^{2g}). \tag{3.4}$$

Combining (3.3) and (3.4), we complete the proof. □

We give an asymptotic formula of $\mathcal{S}_{h,m}(\mathcal{F}_{g+1})$.

Lemma 3.4. *For $h \in \{g-1, g\}$ and $m \in \{0, 1, \ldots, k\}$, we have*

$$\mathcal{S}_{h,m}(\mathcal{F}_{g+1}) = 2^m J_m([\tfrac{h}{2}]) \frac{q^{2g+2}}{g+1} + O\left(g^{m-1} q^{\frac{3g}{2}}\right).$$

Proof. We split the sum $\mathcal{S}_{h,m}(\mathcal{F}_{g+1})$ over f with f being a square or a non-square to obtain

$$\mathcal{S}_{h,m}(\mathcal{F}_{g+1}) = \mathcal{S}_{h,m}(\mathcal{F}_{g+1})_{\Box} + \mathcal{S}_{h,m}(\mathcal{F}_{g+1})_{\neq\Box},$$

where

$$\mathcal{S}_{h,m}(\mathcal{F}_{g+1})_{\Box} = \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{\Box = f \in \mathbb{A}_\ell^+} \sum_{u \in \mathcal{F}_{g+1}} \chi_u(f)$$

and

$$\mathcal{S}_{h,m}(\mathcal{F}_{g+1})_{\neq\Box} = \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{\Box \neq f \in \mathbb{A}_\ell^+} \sum_{u \in \mathcal{F}_{g+1}} \chi_u(f).$$

Using (3.2), we have

$$\begin{aligned} \left|\mathcal{S}_{h,m}(\mathcal{F}_{g+1})_{\neq\Box}\right| &\ll \sum_{\ell=0}^{h} \ell^m q^{-\frac{\ell}{2}} \sum_{\Box \neq f \in \mathbb{A}_\ell^+} \left|\sum_{u \in \mathcal{F}_{g+1}} \chi_u(f)\right| \\ &\ll \frac{q^g}{g} \sum_{\ell=0}^{h} \ell^m q^{\frac{\ell}{2}} \ll g^{m-1} q^{\frac{3g}{2}}. \end{aligned} \tag{3.5}$$

As in the proof of Lemma 3.3, if $f \in \mathbb{A}_\ell^+$ is a square, then $f = L^2$ for some $L \in \mathbb{A}_\alpha^+$ with $\ell = 2\alpha$ and $\chi_P(f) = \chi_P(L^2) = 1$. Thus we have

$$\mathcal{S}_{h,m}(\mathcal{F}_{g+1})_{\Box} = 2^m J_m([\tfrac{h}{2}]) |\mathcal{F}_{g+1}|.$$

From [1, Lemma 4.3], we have

$$|\mathcal{F}_{g+1}| = \frac{q^{2g+2}}{g+1} + O\left(\frac{q^{\frac{3g}{2}}}{g}\right).$$

Thus we get

$$\mathcal{S}_{h,m}(\mathcal{F}_{g+1})_{\square} = 2^m J_m([\tfrac{h}{2}])\frac{q^{2g+2}}{g+1} + O\left(g^{m-1}q^{\frac{3g}{2}}\right). \tag{3.6}$$

Combining (3.5) and (3.6), we complete the proof. □

We give an asymptotic formula of $\mathcal{T}_h(\mathcal{F}_{g+1})$.

Lemma 3.5. *For $h \in \{g-1, g\}$ and $m \in \{0, 1, \ldots, k\}$, we have*

$$\mathcal{T}_h(\mathcal{F}_{g+1}) = \frac{\zeta_{\mathbb{A}}(2)}{g+1} q^{2g-\frac{h}{2}+[\frac{h}{2}]+\frac{3}{2}} + O\left(\frac{q^{\frac{3g}{2}}}{g}\right).$$

Proof. Splitting the sum $\mathcal{T}_h(\mathcal{F}_{g+1})$ over f with f being a square or a non-square, we can write

$$\mathcal{T}_h(\mathcal{F}_{g+1}) = \mathcal{T}_h(\mathcal{F}_{g+1})_{\square} + \mathcal{T}_h(\mathcal{F}_{g+1})_{\neq\square},$$

where

$$\mathcal{T}_h(\mathcal{F}_{g+1})_{\square} = q^{-\frac{h+1}{2}} \sum_{\ell=0}^{h} \sum_{\square=f\in\mathbb{A}_\ell^+} \sum_{u\in\mathcal{F}_{g+1}} \chi_u(f)$$

and

$$\mathcal{T}_h(\mathcal{F}_{g+1})_{\neq\square} = q^{-\frac{h+1}{2}} \sum_{\ell=0}^{h} \sum_{\square\neq f\in\mathbb{A}_\ell^+} \sum_{u\in\mathcal{F}_{g+1}} \chi_u(f).$$

For the contribution of non-squares, we use (3.2) to deduce that

$$\begin{aligned}\left|\mathcal{T}_h(\mathcal{F}_{g+1})_{\neq\square}\right| &\ll q^{-\frac{h+1}{2}} \sum_{\ell=0}^{h} \sum_{\square\neq f\in\mathbb{A}_\ell^+} \left|\sum_{u\in\mathcal{F}_{g+1}} \chi_u(f)\right| \\ &\ll \frac{q^{-\frac{h+1}{2}+g}}{g} \sum_{\ell=0}^{h} q^\ell \ll \frac{q^{\frac{3g}{2}}}{g}. \end{aligned} \tag{3.7}$$

As in the proof of Lemma 3.3, if $f \in \mathbb{A}_\ell^+$ is a square, then $f = L^2$ for some $L \in \mathbb{A}_\alpha^+$ with $\ell = 2\alpha$ and $\chi_P(f) = \chi_P(L^2) = 1$. Thus we have

$$\mathcal{T}_h(\mathcal{F}_{g+1})_{\square} = q^{-\frac{h+1}{2}} \frac{q^{[\frac{h}{2}]+1}-1}{q-1} |\mathcal{F}_{g+1}|.$$

From [1, Lemma 4.3], we have

$$|\mathcal{F}_{g+1}| = \frac{q^{2g+2}}{g+1} + O\left(\frac{q^{\frac{3g}{2}}}{g}\right).$$

Thus we get

$$\mathcal{T}_h(\mathcal{F}_{g+1})_{\square} = \frac{\zeta_{\mathbb{A}}(2)}{g+1} q^{2g-\frac{h}{2}+[\frac{h}{2}]+\frac{3}{2}} + O\left(\frac{q^{\frac{3g}{2}}}{g}\right). \tag{3.8}$$

Combining (3.7) and (3.8), we complete the proof. □

3.2. Proof of Theorem 3.1(1)

In this subsection, we give a proof of first part of Theorem 3.1. Let $u \in \mathcal{I}_{g+1}$. From Lemma 5.1 in [6], we have

$$\frac{L^{(k)}(\frac{1}{2},\chi_u)}{(\ln q)^k} = \sum_{\ell=0}^{g}(-\ell)^k q^{-\frac{\ell}{2}} \sum_{f\in\mathbb{A}_\ell^+}\chi_u(f) + \sum_{m=0}^{k}\binom{k}{m}(-2g)^{k-m}\sum_{\ell=0}^{g-1}\ell^m q^{-\frac{\ell}{2}}\sum_{f\in\mathbb{A}_\ell^+}\chi_u(f). \tag{3.9}$$

Summing (3.9) over $\mathcal{I}_{g+1}$, we obtain that

$$\sum_{u\in\mathcal{I}_{g+1}}\frac{L^{(k)}(\frac{1}{2},\chi_u)}{(\ln q)^k} = (-1)^k\mathcal{S}_{g,k}(\mathcal{I}_{g+1}) + \sum_{m=0}^{k}\binom{k}{m}(-2g)^{k-m}\mathcal{S}_{g-1,m}(\mathcal{I}_{g+1}). \tag{3.10}$$

It follows from Lemma 3.3 that

$$\mathcal{S}_{g,k}(\mathcal{I}_{g+1}) = 2^{k+1}J_k([\tfrac{g}{2}])\frac{q^{2g+1}}{g} + O(g^{k-2}q^{2g}) \tag{3.11}$$

and

$$\mathcal{S}_{g-1,m}(\mathcal{I}_{g+1}) = 2^{m+1}J_m([\tfrac{g-1}{2}])\frac{q^{2g+1}}{g} + O(g^{m-2}q^{2g}). \tag{3.12}$$

Inserting (3.11) and (3.12) into (3.10) and arranging the terms, we complete the proof.

3.3. Proof of Theorem 3.1(2)

In this subsection, we give a proof of second part of Theorem 3.1. Let $u \in \mathcal{F}_{g+1}$. From Lemma 6.1 in [6], we have

$$\begin{aligned}\frac{L^{(k)}(\frac{1}{2},\chi_u)}{(-\ln q)^k} &= \sum_{\ell=0}^{g}\ell^k q^{-\frac{\ell}{2}}\sum_{f\in\mathbb{A}_\ell^+}\chi_u(f) - (g+1)^k q^{-\frac{g+1}{2}}\sum_{\ell=0}^{g}\sum_{f\in\mathbb{A}_\ell^+}\chi_u(f)\\ &+ \sum_{a+b+c=k}\frac{k!}{a!b!c!}\frac{(2g)^a\delta^{(b)}(\frac{1}{2})}{(-\ln q)^b}\sum_{\ell=0}^{g-1}(-\ell)^c q^{-\frac{\ell}{2}}\sum_{f\in\mathbb{A}_\ell^+}\chi_u(f)\\ &- q^{-\frac{g}{2}}\sum_{m=0}^{k}\binom{k}{m}\frac{g^{k-m}\delta^{(m)}(\frac{1}{2})}{(-\ln q)^m}\sum_{\ell=0}^{g-1}\sum_{f\in\mathbb{A}_\ell^+}\chi_u(f).\end{aligned} \tag{3.13}$$

Summing (3.13) over $\mathcal{F}_{g+1}$, we obtain that

$$\sum_{u\in\mathcal{F}_{g+1}} \frac{L^{(k)}(\frac{1}{2},\chi_u)}{(-\ln q)^k} = \mathcal{S}_{g,k}(\mathcal{F}_{g+1}) + \sum_{a+b+c=k} \frac{(-1)^c k!}{a!b!c!} \frac{(2g)^a \delta^{(b)}(\frac{1}{2})}{(-\ln q)^b} \mathcal{S}_{g-1,c}(\mathcal{F}_{g+1})$$

$$- (g+1)^k \mathcal{T}_g(\mathcal{F}_{g+1}) - \sum_{m=0}^{k} \binom{k}{m} \frac{g^{k-m}\delta^{(m)}(\frac{1}{2})}{(-\ln q)^m} \mathcal{T}_{g-1}(\mathcal{F}_{g+1}). \tag{3.14}$$

It follows from Lemma 3.4 that

$$\mathcal{S}_{g,k}(\mathcal{F}_{g+1}) = 2^k J_k([\tfrac{g}{2}]) \frac{q^{2g+2}}{g+1} + O\left(g^{k-1} q^{\frac{3g}{2}}\right) \tag{3.15}$$

and

$$\mathcal{S}_{g-1,c}(\mathcal{F}_{g+1}) = 2^c J_c([\tfrac{g-1}{2}]) \frac{q^{2g+2}}{g+1} + O\left(g^{c-1} q^{\frac{3g}{2}}\right). \tag{3.16}$$

It also follows from Lemma 3.4 that

$$\mathcal{T}_g(\mathcal{F}_{g+1}) = \frac{\zeta_{\mathbb{A}}(2)}{g+1} q^{\frac{3g}{2}+[\frac{g}{2}]+\frac{3}{2}} + O\left(\frac{q^{\frac{3g}{2}}}{g}\right) \tag{3.17}$$

and

$$\mathcal{T}_{g-1}(\mathcal{F}_{g+1}) = \frac{\zeta_{\mathbb{A}}(2)}{g+1} q^{\frac{3g}{2}+[\frac{g-1}{2}]+2} + O\left(\frac{q^{\frac{3g}{2}}}{g}\right). \tag{3.18}$$

We finally complete the proof by inserting (3.15), (3.16), (3.17) and (3.18) into (3.14) and arranging the terms.

References

[1] J. Andrade, S. Bae, and H. Jung, *Average values of L-series for real characters in function fields*, Res. Math. Sci. **3** (2016), Paper No. 38, 47 pp. https://doi.org/10.1186/s40687-016-0087-4

[2] J. Andrade and H. Jung, *Mean values of derivatives of L-functions in function fields: IV*, Submitted.

[3] J. Andrade and S. Rajagopal, *Mean values of derivatives of L-functions in function fields: I*, J. Math. Anal. Appl. **443** (2016), no. 1, 526–541. https://doi.org/10.1016/j.jmaa.2016.05.019

[4] S. Bae and H. Jung, *Average values of L-functions in even characteristic*, J. Number Theory **186** (2018), 269–303. https://doi.org/10.1016/j.jnt.2017.10.006

[5] S. Bae and H. Jung, *Note on the mean values of derivatives of quadratic Dirichlet L-functions in function fields*, Finite Fields Appl. **57** (2019), 249–267. https://doi.org/10.1016/j.ffa.2019.02.010

[6] S. Bae and H. Jung,*Mean values of derivatives of L-functions in even characteristic*, Submitted.

[7] A. Florea, *Improving the error term in the mean value of $L(\frac{1}{2},\chi)$ in the hyperelliptic ensemble*, Int. Math. Res. Not. IMRN **2017** (2017), no. 20, 6119–6148. https://doi.org/10.1093/imrn/rnv387

[8] A. Florea, *The second and third moment of $L(1/2,\chi)$ in the hyperelliptic ensemble*, Forum Math. **29** (2017), no. 4, 873–892. https://doi.org/10.1515/forum-2015-0152

[9] A. Florea, *The fourth moment of quadratic Dirichlet L-functions over function fields*, Geom. Funct. Anal. **27** (2017), no. 3, 541–595. https://doi.org/10.1007/s00039-017-0409-8

[10] M. Rosen, *Number theory in function fields*, Graduate Texts in Mathematics, 210, Springer-Verlag, New York, 2002. https://doi.org/10.1007/978-1-4757-6046-0

[11] Z. Rudnick, *Traces of high powers of the Frobenius class in the hyperelliptic ensemble*, Acta Arith. **143** (2010), no. 1, 81–99. https://doi.org/10.4064/aa143-1-5

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