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METRIZABILITY AND SUBMETRIZABILITY FOR POINT-OPEN, OPEN-POINT AND BI-POINT-OPEN TOPOLOGIES ON $C(X,Y)$

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Abstract. We characterize metrizability and submetrizability for point-open, open-point and bi-point-open topologies on $C(X,Y)$, where $C(X,Y)$ denotes the set of all continuous functions from space X to Y; X is a completely regular space and Y is a locally convex space.

1. Introduction

Let $C_p(X,Y)$, $C_h(X,Y)$ and $C_{ph}(X,Y)$ denote the spaces of all continuous functions from space X to space Y, equipped with point-open, open-point and bi-point-open topology, respectively. While studying $C_\tau(X,Y)$, it is fundamental problem to establish the correspondence between topological and algebraic properties of spaces $C_\tau(X,Y)$, X and Y, where $C_\tau(X,Y)$ denotes the space $C(X,Y)$ equipped with topology τ. There are various studies in which authors used the topological and algebraic properties of $\mathbb{R}$ and examined the properties of $C_\tau(X,\mathbb{R})$ (see [2] and [1]). For locally convex space $\mathbb{R}$, it has been studied that X is countable if and only if $C_p(X,\mathbb{R})$ is metrizable and X is separable if and only if $C_p(X,\mathbb{R})$ is submetrizable (see [2]). Also, in [3] and [4], A. Jindal, R. A. McCoy and S. Kundu characterized the metrizability and submetrizability on $C_p(X,\mathbb{R})$, $C_h(X,\mathbb{R})$ and $C_{ph}(X,\mathbb{R})$ with some conditions on X. In this paper, we generalize the theorems of metrizability and submetrizability on $C_p(X,Y)$, $C_h(X,Y)$ and $C_{ph}(X,Y)$, where Y is a locally convex space.

2. Definitions and notations

A topological linear space Y (over $\mathbb{R}$) is called a locally convex space [6] if every neighborhood of identity element $\bar{0}$ contains a convex neighborhood of $\bar{0}$. Let $V_1(p) = \{x \in Y : p(x) < 1\}$, where p is a seminorm. Note that, $V_1(p)$

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is a convex set. A topological vector space Y is locally convex if and only if there exists a family $\mathcal{P}$ of seminorms that generates the topology on Y. The collection $\{\bigcap_{i=1}^{n} r_i V(p_i) : p_i \in \mathcal{P},\ r_i > 0\}$ is a neighborhood base at $\bar{0} \in Y$ for the topology on Y (see Theorems 11.3 and 12.4 in [6]).

The point-open topology p on $C(X, Y)$ has a subbase consisting of the sets of the form $[x, V]^- = \{f \in C(X, Y) : f(x) \in V\}$, where V is an open set in Y and $x \in X$. The space $C(X, Y)$ equipped with point-open topology is denoted by $C_p(X, Y)$.

The open-point topology h [3] on $C(X, Y)$ has a subbase consisting of the sets of the form $[U, y]^- = \{f \in C(X, Y) : f^{-1}(y) \cap U \neq \phi\}$, where U is an open set in X and $y \in Y$. The space $C(X, Y)$ equipped with open-point topology is denoted by $C_h(X, Y)$.

The bi-point-open topology ph [3] has subbasic open sets of both kinds: $[U, y]^- = \{f \in C(X, Y) : f^{-1}(y) \cap U \neq \phi\}$ and $[x, V]^+ = \{f \in C(X, Y) : f(x) \in V\}$, where U is an open set in X and $y \in Y$; $x \in X$ and V is open in Y. The space $C(X, Y)$ equipped with bi-point-open topology is denoted by $C_{ph}(X, Y)$.

A nonempty subset of a space X is said to be G_δ-dense [3] provided that it intersects every nonempty G_δ-subset of X.

Throughout this paper, X is a completely regular and Hausdorff space and $(Y, +, \times)$ denotes a locally convex Hausdorff space, where $+$ and $\times$ denote vector addition and scalar multiplication, respectively. $r \times y$ is a scalar multiplication of r and y, where $r \in \mathbb{R}$ and $y \in Y$. Let $-y$ denote additive inverse of point y in Y and $-U = \{-1 \times u : u \in U\}$, where $-u = -1 \times u$. $V + U = \{v + u : u \in U, v \in V\}$, where $u + v$ denotes the vector addition in Y. Let $\bar{0}$ denote the identity element in the space Y. Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural numbers and set of all real numbers, respectively. $\mathcal{P}$ denotes the collection of all seminorms that generates the topology of Y. Let X and Y denote two spaces with same underlying set, and $X \leq Y$ denote that X and Y have the same topology. Also, $\bar{0}_X$ denotes the constant function which maps all points of X to $\bar{0}$. $|Z|$ denotes the cardinality of set Z. Let X^0 denote the set of all isolated points in X.

3. Metrizability on $C_p(X, Y)$, $C_h(X, Y)$ and $C_{ph}(X, Y)$

We know that, each metrizable space is first countable, so first we will discuss about first countability. Recall that, character of a point $x \in X$, denoted by $\chi(X, x)$, is defined by

$$\chi(X, x) = \aleph_0 + \min\{|\mathcal{B}_x| : \mathcal{B}_x \text{ is a local base for } X \text{ at } x\}.$$

If $\chi(X, x)$ is countable, then X is first countable.

Theorem 3.1. *Let X and Y be any spaces. Then $|X| \leq \chi(C_p(X, Y))$.*

Proof. Let the family $\mathcal{B}$ form a base at $\bar{0}_X$ in $C_p(X, Y)$ such that $|\mathcal{B}| \leq \chi(C_p(X, Y), 0_X)$. We may assume that each member $B \in \mathcal{B}$ is of the form

$[x_1, U_1]^+ \cap \cdots \cap [x_n, U_n]^+$. For each $B \in \mathcal{B}$, let $K(B) = \{x_1, \ldots, x_n\}$. Let $T = \bigcup_{B \in \mathcal{B}} K(B)$. Clearly, $|T| \leq \chi(C_p(X,Y), 0_X)$. Let if possible, there exists $x_0 \in X \setminus T$. Then let $[x_0, V_1(p)]$ be a subbasic open set in $C_p(X,Y)$, where $p \in \mathcal{P}$. Consider an arbitrary $B \in \mathcal{B}$, where $B = [x_1, U_1]^+ \cap \cdots \cap [x_n, U_n]^+$. Since X is completely regular, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x_0) = 1$ and $f(x_i) = 0$ for all $i \in 1, \ldots, n$. Take $y \in Y$, where $p(y) = 1$. Also, since Y is a locally convex space, there exists a continuous function $h : \mathbb{R} \to Y$ defined by $h(a) = ay$. So the composition $h \circ f : X \to Y$ is also continuous, where $h \circ f(x_i) = \bar{0}$ for all $i \in 1, \ldots, n$ and $h \circ f(x_0) = y$. Note that, $h \circ f \in B \setminus [x_0, V_1(p)]$. This contradicts the fact that $\mathcal{B}$ forms a base at $\bar{0}_X$ for $C_p(X,Y)$. □

Corollary 3.2. *If $C_p(X,Y)$ is first countable, then X is countable.*

Theorem 3.3. *If $C_h(X,Y)$ is first countable, then X has a countable π-base.*

Proof. Since $C_h(X,Y)$ is first countable, $C_h(X,Y)$ has a countable pseudocharacter. Then there exists countable family α of open sets in $C_h(X,Y)$ such that $\{\bar{0}_X\} = \cap\alpha$. Without loss of generality, we can assume that each member A of α is of the form $[V_1, x_1] \cap \cdots \cap [V_n, x_n]$, where for each $i \in \{1, \ldots, n\}$, V_i is open in X and $x_i \in Y$. Let $K(A) = \{V_1, \ldots, V_n\}$ and let $\mathcal{B} = \bigcup_{A \in \alpha} K(A)$. Note that, $\mathcal{B}$ is a countable family of open sets. We show that $\mathcal{B}$ forms a π-base for the space X. Let V be an open set in X containing x. Then there exists a continuous function $f : X \to \mathbb{R}$ such that$f(x) = 1$ and $f(v) = 0$ for all v in V^c. Take $y \in Y$, where $p(y) = 1$ for some $p \in \mathcal{P}$. Since Y is a locally convex space, there exists a continuous function $h : \mathbb{R} \to Y$ defined by $h(a) = ay$, where $p(y) = 1$. So the composition $h \circ f : X \to Y$ is also continuous, where $h \circ f(v) = \bar{0}$ for all v in V^c and $h \circ f(x) = y$. There exists $A \in \alpha$ such that $h \circ f \notin A$. It follows that there exists $U \in \mathcal{B}$ such that $\bar{0} \notin g \circ f(U)$. We claim that, $U \subset V$. Let $u \in U$, then $g \circ f(u) \neq \bar{0}$, this implies $u \in V$. Hence $U \subset V$ and $\mathcal{B}$ forms a π-base for X. □

Recall that, for any space X and any point $x \in X$, the pseudocharacter of x in X, denoted by $\psi(X,x)$, is defined by $\psi(X,x) = \aleph_0 + \min\{|\gamma| :$ γ is a family of nonempty open subsets in X such that $\cap\gamma = \{x\}\}$.

The pseudocharacter $\psi(X)$ of X is given by

$$\psi(X) = \sup\{\psi(X,x) : x \in X\}.$$

If $C_p(X,Y)$ is metrizable, then Y is metrizable (see Exercise 1 on page 68 in [5]). But a locally convex space Y is metrizable if and only if it has a countable base at $\bar{0}$ (see Theorem 13.1 in Chapter 2 of [6]). So, the first countability of Y is necessary condition for metrizability of $C_p(X,Y)$.

Theorem 3.4. *Let X^0 be G_δ-dense in X and Y has a countable base at $\bar{0}$. Then the following statements are equivalent*:

(a) *$C_h(X,Y)$ is metrizable.*
(b) *$C_h(X,Y)$ is first countable.*
(c) *X has a countable π-base.*
(d) *X^0 is countable.*
(e) *X is countable and discrete.*
(f) *$C_p(X,Y)$ is metrizable.*

Proof. (a)⇔(b) Since Y has a countable base at $\bar{0}$, it is first countable. It implies that each singleton set in Y is a G_δ-set. So, $C_h(X,Y)$ is a topological group (the proof is similar to proof for $C_h(X,\mathbb{R})$ in [3]). A topological group is metrizable if and only if it is first countable (see Theorem 3.3.12 in [1]).

(b)⇒(c) It follows from Theorem 3.3.

(c)⇒(b) We show that the constant function $\bar{0}_X$ in $C_h(X,Y)$ has a countable base. Let $\mathcal{B}' = \{\cap_{i=1}^n [B_i,\bar{0}] : B_i \in \mathcal{B},\ n \in \mathbb{N}\}$ be the collection of open sets containing $\bar{0}_X$, where $\mathcal{B}$ denotes the countable π-base of X. Note that, $\mathcal{B}'$ is a countable collection. We show that $\mathcal{B}'$ forms a local base at $\bar{0}_X$ in $C_h(X,Y)$. Let $B = [V_1,\bar{0}]^- \cap \cdots \cap [V_n,\bar{0}]^-$ be any open set in $C_h(X,Y)$ containing $\bar{0}_X$, where V_i is open in X for each $i \in \{1,\ldots,n\}$. Since $\mathcal{B}$ forms the countable π-base for X, for each $i \in \{1,\ldots,n\}$, there exists $B_i \in \mathcal{B}$ such that $B_i \subset V_i$. Clearly, $\bar{0}_X \in [B_1,\bar{0}]^- \cap \cdots \cap [B_n,\bar{0}]^- \subset B$. Hence $\mathcal{B}'$ forms a local base at $\bar{0}_X$ in $C_h(X,Y)$. So each element in $C_h(X,Y)$ has a countable local base, because $C_h(X,Y)$ is a topological group.

(c)⇒(d), (d)⇒(e) and (e)⇒(c) These are easy to prove.

(e)⇒(f) If Y has a countable base at $\bar{0}$, then it is metrizable (see Theorem 13.1 in Chapter 2 of [6]). Since X is countable, $C_p(X,Y)$ is metrizable (see Exercise 1 on page 68 in [5]).

(f)⇒(d) Since $C_p(X,Y)$ is metrizable, it is first countable. By Corollary 3.2, X is countable. So X^0 is countable. □

Theorem 3.5. *Let X^0 be G_δ-dense in X and Y has a countable base at $\bar{0}$. Then following statements are equivalent*:

(a) *$C_{ph}(X,Y)$ is metrizable.*
(b) *$C_{ph}(X,Y)$ is first countable.*
(c) *X is a countable and discrete space.*
(d) *$C_h(X,Y)$ is metrizable.*
(e) *$C_p(X,Y)$ is metrizable.*

Proof. (a)⇔(b) Since Y has a countable base at $\bar{0}$, it is first countable. It implies that each singleton set in Y is a G_δ-set. So, $C_{ph}(X,Y)$ is a topological group (the proof is similar to proof for $C_{ph}(X,\mathbb{R})$ in [3]). A topological group is metrizable if and only if it is first countable (see Theorem 3.3.12 in [1]).

(c)⇔(d) It follows from Theorem 3.4.

(a)⇒(d) Being a metrizable space, $C_{ph}(X,Y)$ has a countable pseudocharacter. There exists a countable family α of open sets such that $\cap\alpha = \{\bar{0}_X\}$. Let each member A of α be of the form $A = [V_1,\bar{0}]^- \cap \cdots \cap [V_n,\bar{0}]^- \cap [x_1,U_1]^+ \cap$

$\cdots \cap [x_m, U_m]^+$. Since X^0 is G_δ-dense in X for each $i \in \{1, \ldots, n\}$, there exists $v_i \in X^0 \cap V_i$. So $[\{v_1\}, \bar{0}]^- \cap \cdots \cap [\{v_n\}, \bar{0}]^- \cap [x_1, U_1]^+ \cap \cdots \cap [x_m, U_m]^+ \subset A$. Let $K(A) = \{v_1, \ldots, v_n, x_1, \ldots, x_m\}$ and $\mathcal{W} = \bigcup_{A \in \alpha} K(A)$. Note that, $\mathcal{W}$ is countable. Now we show that $\overline{\mathcal{W}} = X$. Let if possible $\overline{\mathcal{W}} \neq X$, there exists $x_0 \in X \setminus \overline{\mathcal{W}}$. Since X is completely regular, there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x_0) = 1$ and $f(a) = 0$ for all a in $\overline{\mathcal{W}}$. Take $y \in Y$, where $p(y) = 1$ for some $p \in \mathcal{P}$. Since Y is a locally convex space, there exists a continuous function $h : \mathbb{R} \to Y$ defined by $h(r) = ry$. So the composition $h \circ f : X \to Y$ is also continuous, where $h \circ f(a) = \bar{0}$ for all a in $\overline{\mathcal{W}}$ and $h \circ f(x_0) = y$. This implies that $h \circ f \in \cap\alpha$. We arrive at a contradiction. So X is separable. It follows that X^0 is countable. Therefore, $C_h(X, Y)$ is metrizable (using Theorem 3.4).

(c)⇒(b) Since X is a countable and discrete space, $C_p(X, Y)$ and $C_h(X, Y)$ are first countable (use Theorem 3.4). So by definition of bi-point-open topology, $C_{ph}(X, Y)$ is first countable.

(d)⇔(e) It follows from Theorem 3.4. □

4. Submetrizability on $C_p(X, Y)$, $C_h(X, Y)$ and $C_{ph}(X, Y)$

Recall that, the weight of a space X is defined by $w(X) = \aleph_0 + \min\{\mathcal{B} : \mathcal{B}$ is a base for $X\}$. The i-weight of a space X is defined by $iw(X) = \aleph_0 + \min\{w(Z)$: there is a continuous one-to-one map from X onto $Z\}$. The density of X is given by $d(X) = \aleph_0 + \min\{|D| : D$ is a dense subset of $X\}$.

Theorem 4.1. *Let X and Y be spaces. Then $iw(C_p(X, Y)) \leq d(X) \cdot w(Y)$.*

Proof. Let $d(X) = \tau$. Let Z be a subset of X such that $\overline{Z} = X$ and $\tau = |Z|$. Let $\pi_Z : C_p(X, Y) \to \pi_Z(C_p(X, Y)) \subset C_p(Z, Y)$ be the restriction map defined by $\pi_Z(f) = f|_Z$, where $f|_Z$ denotes the restriction of function f on subspace Z. It is easy to see that π_Z is continuous and onto. Let $f_1, f_2 \in C_p(X, Y)$, $f_1 \neq f_2$. Then continuity of f_1,f_2 and $\overline{Z} = X$ imply that $f|_1(Z) \neq f|_2(Z)$. So $\pi_Z(f_1) \neq \pi_Z(f_2)$. It follows that map π_Z is one to one. We have $w(\pi_Z(C_p(X, Y))) \leq w(C_p(Z, Y)) \leq |Z| \cdot w(Y) = d(X) \cdot w(Y)$, and hence $iw(C_p(X, Y)) \leq w(\pi_Z(C_p(X, Y))) \leq d(X) \cdot w(Y)$. □

Corollary 4.2. *Let X be separable and Y be a second countable space. Then $C_p(X, Y)$ is submetrizable.*

Example 4.3. Let $X = \{a\}$ and $Y = C_p(\mathbb{R}, \mathbb{R})$. Note that, X being a singleton set with discrete topology is separable and $C_p(\mathbb{R}, \mathbb{R})$ is not second countable (use Theorem I.1.1 in [2]). But, by Theorem I.1.4 in [2], $C_p(\mathbb{R}, \mathbb{R})$ is submetrizable. It is easy to see that $C_p(X, Y)$ is homeomorphic to $C_p(\mathbb{R}, \mathbb{R})$. So, $C_p(X, Y)$ is also submetrizable. Hence, the second countability is not a necessary condition for submetrizability of $C_p(X, Y)$.

In Example 4.3, Y is not even first countable but it has a countable pseudocharacter (use Theorem I.1.4 in [2]). Later, we will see that it is necessary for

Y to have a countable pseudocharacter for submetrizability of $C_p(X, Y)$. Now, we will discuss the converse of Corollary 4.2. Since each submetrizable space has a countable pseudocharacter, so first, we will discuss about pseudocharacter character.

Theorem 4.4. *Let $C_p(X, Y)$ has a countable pseudocharacter. Then Y has a countable pseudocharacter at $\bar{0}$.*

Proof. Since $C_p(X, Y)$ has a countable pseudocharacter, there exists a countable family γ of open sets in $C_p(X, Y)$ such that $\cap\{\gamma\} = \{0_X\}$. Without loss of generality, we can assume that each W in γ is of the form $[x_1, U_1]^+ \cap \cdots \cap [x_m, U_m]^+$, where $\bar{0} \in U_i$ for $1 \leq i \leq m$. Let $S(W) = \{U_1, \ldots, U_m\}$ and $\mathcal{C} = \cup\{S(W) : W \in \gamma\}$. Since γ is countable, so is $\mathcal{C}$. Note that, $\mathcal{C}$ is a countable collection of nonempty open sets in Y. Let if possible, there exists $y_0 \in \cap\{\mathcal{C}\}$ such that $y_0 \neq \bar{0}$. Let f be the constant function mapping each member of X to y_0. Then, it is easy to see that, $f \in \cap\gamma$ and $f \neq \bar{0}$. We arrive at contradiction. Hence, Y has a countable pseudocharacter at $\bar{0}$. □

Theorem 4.5. *Let X and Y be spaces. Then $d(X) \leq \psi(C_{ph}(X, Y))$.*

Proof. Let γ be a family of open sets containing $\bar{0}_X$ such that $\cap\gamma = \{\bar{0}_X\}$ and $|\gamma| \leq \psi(C_{ph}(X, Y), \bar{0}_X)$. We can assume that each member G of γ can be written as $[x_1, U_1]^+ \cap \cdots \cap [x_n, U_n]^+ \cap [V_1, \bar{0}]^- \cap \cdots \cap [V_m, \bar{0}]^-$. Let $y_i \in V_i$ for $1 \leq i \leq m$. For each $G \in \gamma$, let $K(G) = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$. Let $Z = \bigcup_{G \in \gamma} K(G)$. Then $|Z| \leq |\gamma|$. We show that $\overline{Z} = X$. Suppose $x_0 \in X \setminus \overline{Z}$, then there exists a continuous function f such that $f(\overline{Z}) = \{0\}$ and $f(x_0) = 1$. Take $y \in Y$, where $p(y) = 1$ for some $p \in \mathcal{P}$. Since Y is a locally convex space, there exists a continuous function $h : \mathbb{R} \to Y$ defined by $h(r) = ry$. So the composition $h \circ f : X \to Y$ is also continuous, where $h \circ f(a) = \bar{0}$ for all a in $\overline{Z}$ and $h \circ f(x_0) = y$. This implies that $h \circ f \in \cap\gamma$ but $h \circ f \neq \bar{0}_X$. We arrive at a contradiction. □

Corollary 4.6. *Let $C_{ph}(X, Y)$ be submetrizable. Then X is separable.*

Proof. Since each submetrizable space has a countable pseudocharacter, separability of X follows from Theorem 4.5. □

Corollary 4.7. *Let $C_h(X, Y)$ be submetrizable. Then X is separable.*

Proof. Since $C_h(X, Y) \leq C_{ph}(X, Y)$, $C_{ph}(X, Y)$ is also submetrizable. So by Corollary 4.6, X is separable. □

Corollary 4.8. *Let $C_p(X, Y)$ be submetrizable. Then X is separable and Y has a countable pseudocharacter at $\bar{0}$.*

Proof. Since $C_p(X, Y) \leq C_{ph}(X, Y)$, $C_{ph}(X, Y)$ is also submetrizable. So by Corollary 4.6, X is separable. Also, since each submetrizable space has a countable pseudocharacter, so by Theorem 4.4, Y has a countable pseudocharacter at $\bar{0}$. □

Corollary 4.9. *Let X be any space and Y be a second countable space. Then $d(X) = iw(C_p(X,Y)) = \psi(C_p(X,Y))$.*

Proof. Since $iw(C_p(X,Y)) \geq \psi(C_p(X,Y))$, it is enough to show that

$$iw(C_p(X,Y)) \leq d(X) \leq \psi(C_p(X,Y)).$$

Since Y is second countable, $iw(C_p(X,Y)) \leq d(X)$ follows from Theorem 4.1. $d(X) \leq \psi(C_p(X,Y))$ follows from Theorem 4.5. □

Recall that, if the set $\{(x,x) : x \in X\}$ is a G_δ-set in the product space $X \times X$, that is, the space X has a G_δ-diagonal, then every point in X is a G_δ-set. Note that every metrizable space has a G_δ-diagonal. Consequently every submetrizable space also has a G_δ-diagonal. Also, compact subsets, countably compact subsets and the singletons in a submetrizable space are G_δ-sets. So far in this section, we have discussed about necessary conditions on X and Y for submetrizability of $C_p(X,Y)$, $C_h(X,Y)$ and $C_{ph}(X,Y)$ and we are looking forward to studying sufficient conditions on X and Y for the same purpose in future. But for now, we characterize submetrizability of $C_p(X,Y)$, $C_h(X,Y)$ and $C_{ph}(X,Y)$ with the condition of second countability on Y.

Theorem 4.10. *Let X be space and Y be a second countable space. Then the following statements are equivalent*:

(a) *$C_{ph}(X,Y)$ is submetrizable.*
(b) *$C_{ph}(X,Y)$ has a G_δ-diagonal.*
(c) *$C_{ph}(X,Y)$ has a countable pseudocharacter.*
(d) *$\{\bar{0}_X\}$ is a G_δ-set in $C_{ph}(X,Y)$.*
(e) *X is separable.*
(f) *$C_p(X,Y)$ is submetrizable.*

Proof. (a)⇒(b)⇒(c)⇒(d) It follows from the above discussion.

(d)⇒(e) If $\bar{0}_X$ is a G_δ-set in $C_{ph}(X,Y)$, then there exists a countable family α of open sets such that $\cap\alpha = \{\bar{0}_X\}$. We can assume that each member $A \in \alpha$ is of the form $[x_1, U_1]^+ \cap \cdots \cap [x_n, U_n]^+ \cap [V_1, \bar{0}]^- \cap \cdots \cap [V_m, \bar{0}]^-$, where $x_i \in X$ and U_i is open in Y for each $i \in \{1, \ldots, n\}$; V_j is open in X for each $j \in \{1, \ldots, m\}$ and $\bar{0} \in Y$. Fix $y_i \in V_j$ for each $j \in \{1, \ldots, m\}$. For each A in α, let $K(A) = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$, let $W = \bigcup_{A \in \alpha} K(A)$. We show that $\overline{W} = X$. Suppose $x_0 \in X \setminus \overline{W}$, then there exists a continuous function f such that $f(\overline{W}) = \{0\}$ and $f(x_0) = 1$. Take $y \in Y$, where $p(y) = 1$ for some $p \in \mathcal{P}$. Since Y is a locally convex space, there exists a continuous function $h : \mathbb{R} \to Y$ defined by $h(r) = ry$. So the composition $h \circ f : X \to Y$ is also continuous, where $h \circ f(a) = \bar{0}$ for all a in $\overline{W}$ and $h \circ f(x_0) = y$. This implies that $h \circ f \in \cap\alpha$ but $h \circ f \neq \bar{0}_X$. We arrive at a contradiction.

(e)⇔(f) It follows from Corollary 4.9.

(f)⇒(a) It is immediate since $C_p(X,Y) \leq C_{ph}(X,Y)$. □

Theorem 4.11. *If X^0 is dense in X and Y is a second countable space, then the following statements are equivalent*:

(a) $C_h(X,Y)$ *is submetrizable.*
(b) $C_h(X,Y)$ *has a* G_δ*-diagonal.*
(c) $C_h(X,Y)$ *has a countable pseudocharacter.*
(d) $\{\bar{0}_X\}$ *in* $C(X,Y)$ *is a* G_δ*-set in* $C_h(X,Y)$*.*
(e) $C_{ph}(X,Y)$ *is submetrizable.*
(f) $C_p(X,Y)$ *is submetrizable.*
(g) X^0 *is countable.*

Proof. (a)⇒(b)⇒(c)⇒(d) It follows from the above discussion.

(d)⇒(e) Since $C_h(X,Y) \le C_{ph}(X,Y)$, $\{\bar{0}_X\}$ is also a G_δ-set in $C_{ph}(X,Y)$. Then Theorem 4.10 implies that $C_{ph}(X,Y)$ is submetrizable.

(e)⇒(f) It is immediate from Theorem 4.10.

(f)⇒(g) If $C_p(X,Y)$ is submetrizable, then X is separable. Hence X^0 is countable.

(g)⇒(a) Consider the restriction map $\pi_{X^0} : C_h(X,Y) \to C_h(X^0,Y)$ defined by $\pi_{X^0}(f) = f|_{X^0}$. Let $[V,y]^-$ be any subbasic open set in $C_h(X^0,Y)$, where V is open in X^0 and $y \in Y$. Since V is open in X^0 and X^0 is open in X, V is open in X. So $\pi_{X^0}([V,y]^-) \subset [V,y]^-$. Hence π_{X^0} is continuous. Let $f,g \in C_p(X,Y)$. Then the continuity of f,g and $\overline{X^0} = X$ imply that $f|_{X^0} \ne g|_{X^0}$. So $\pi_{X^0}(f) \ne \pi_{X^0}(g)$. It follows that map π_{X^0} is one to one. Since X^0 is a countable and discrete subspace of X, the space $C_h(X^0,Y)$ is metrizable (using Theorem 3.4). Hence $C_h(X,Y)$ is submetrizable. □

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