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STUDY OF GRADIENT SOLITONS IN THREE DIMENSIONAL RIEMANNIAN MANIFOLDS

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Abstract. We characterize a three-dimensional Riemannian manifold endowed with a type of semi-symmetric metric P-connection. At first, it is proven that if the metric of such a manifold is a gradient m-quasi-Einstein metric, then either the gradient of the potential function ψ is collinear with the vector field P or, $\lambda = -(m+2)$ and the manifold is of constant sectional curvature -1, provided $P\psi \neq m$. Next, it is shown that if the metric of the manifold under consideration is a gradient ρ-Einstein soliton, then the gradient of the potential function is collinear with the vector field P. Also, we prove that if the metric of a 3-dimensional manifold with semi-symmetric metric P-connection is a gradient ω-Ricci soliton, then the manifold is of constant sectional curvature -1 and $\lambda + \mu = -2$. Finally, we consider an example to verify our results.

1. Introduction

The examination of Ricci solitons on Riemannian and semi-Riemannian manifolds is a significant topic in the area of differential geometry and in material science too. Throughout the most recent couple of years, Ricci solitons and their generalizations are getting of fast development by giving new procedures in understanding the geometry and topology of Riemannian manifolds. One more interest of concentrating on Ricci solitons and their generalizations in various mathematical settings have impressively expanded, because of their association with general relativity.

In reality, solitons are physically the waves that propagate with little loss of energy and holds its shape and speed after colliding with another such wave. Solitons are performed significant role in initial-value problems for nonlinear partial differential equations describing wave propagation. It moreover explained the recurrence in the Fermi-Pasta-Ulam system.

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In 1982, Hamilton [29] introduced the *Ricci flow* ($\frac{\partial g}{\partial t} = -2Ric$, Ric denotes the Ricci tensor). Later, Perelman used Ricci flow to prove century long problem named 'Poincaré conjecture'. A metric g of a complete Riemannian manifold (N, g) is a *Ricci soliton* (see [8], [9], [12] and [38]) if there are a vector field Z and a real constant λ such that

$$Ric + \frac{1}{2}\pounds_Z g = \lambda g,$$

where $\pounds_Z$ is the Lie differentiation along Z. If Z is a gradient of some smooth function ψ, that is, $Z = D\psi$, where D denotes the gradient operator, then the Ricci soliton is called a gradient Ricci soliton [13] and satisfies the equation

$$Ric + H^\psi = \lambda g, \tag{1.1}$$

where H^ψ is the Hessian operator of ψ. For constant function ψ, the equation (1.1) reduces to Einstein equation ($Ric = \lambda g$) and the metric becomes an Einstein metric. On any compact manifold every Ricci soliton is a gradient Ricci soliton [34].

A metric g of a Riemannian manifold (N, g) is called a *gradient m-quasi-Einstein metric* (in short form, gradient m-QE metric) [15], which is a generalization of a gradient Ricci soliton, if there are a smooth function $\psi : N \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that

$$Ric + H^\psi - \frac{1}{m} d\psi \otimes d\psi = \lambda g, \tag{1.2}$$

where $0 < m \leq \infty$ is an integer. For $m = \infty$, the metric becomes a gradient Ricci soliton. The left hand side of (1.2) is known as *m-Bakry-Emery Ricci tensor.* The gradient m-QE metric is said to be *shrinking, steady or, expanding* according as $\lambda > 0$, $\lambda = 0$ or, $\lambda < 0$. The metric of the warped product manifold $(N, g) \times (M^m, h)$ with warping function $\exp(-\frac{\psi}{m})$ is an Einstein metric if and only if h is an Einstein metric and g is a gradient m-QE metric. For details about a gradient m-QE metric see [3], [14], [22] and [28].

In the similar way of Ricci flow, Catino and Mazzieri [17] introduced the new type of geometric flow, known as *gradient flow* ($\frac{\partial g}{\partial t} = -2Ric + \tau g$, τ is the scalar curvature). The gradient Einstein soliton is the self-similar solution of the gradient flow, which is defined by

$$Ric + H^\psi = \left(\lambda + \frac{1}{2}\tau\right) g. \tag{1.3}$$

In [11], Bourguignon introduced the *Ricci-Bourguignon flow* ($\frac{\partial g}{\partial t} = -2(Ric - \rho\tau g)$, $\rho \in \mathbb{R} - \{0\}$). A metric g of a Riemannian manifold N^n, $\dim N = n \geq 3$, is said to be a *gradient ρ-Einstein soliton* (see [16] and [17]), which is a generalization of a gradient Einstein soliton and self-similar solution of Ricci-Bourguignon flow, if there are a $\psi \in C^\infty(N^n)$ and $\lambda \in \mathbb{R}$ such that

$$Ric + H^\psi = (\rho\tau + \lambda) g = \beta g. \tag{1.4}$$

Gradient ρ-Einstein solitons have been investigated by several authors such as [18, 31, 37] and many others. For different values of ρ, gradient ρ-Einstein solitons have different names. They are

(i) *a gradient Einstein soliton* for $\rho = \frac{1}{2}$.
(ii) *a gradient traceless Ricci soliton* for $\rho = \frac{1}{n}$.
(iii) *a gradient Schouten soliton* for $\rho = \frac{1}{2(n-1)}$.

In [23], Cho and Kimura introduced the notion of an η-Ricci soliton, which is a generalization of a Ricci soliton. An *η-Ricci soliton* on a Riemannian manifold (N, g), denoted by (g, Z, λ, μ), is defined by

$$Ric + \frac{1}{2}\pounds_Z g - \mu\eta \otimes \eta = \lambda g, \tag{1.5}$$

where η is a 1-form, λ and μ are real constants. Many authors have worked on η-Ricci solitons ([2, 5, 6, 10, 25, 32, 35, 36]). If $Z = D\psi$ for some $\psi \in C^\infty(N)$, then the η-Ricci soliton is called a *gradient η-Ricci soliton* [7]. For a gradient η-Ricci soliton, from (1.5) it follows that

$$Ric + H^\psi - \mu\eta \otimes \eta = \lambda g. \tag{1.6}$$

Let (N^n, g) be an n-dimensional Riemannian manifold equipped with Riemannian metric g and $\breve{\nabla}$ be a linear connection on N^n. The *torsion tensor* $\breve{T}$ of the linear connection $\breve{\nabla}$ is defined by

$$\breve{T}(E, F) = \breve{\nabla}_E F - \breve{\nabla}_F E - [E, F] \tag{1.7}$$

for all $E, F \in \chi(N^n)$, the set of all smooth vector fields on N^n. The connection $\breve{\nabla}$ is called *symmetric* if the torsion tensor $\breve{T}$ vanishes identically, otherwise, it is *non-symmetric*. If $\breve{\nabla}g = 0$, then the connection is named as *metric connection* and if $\breve{\nabla}g \neq 0$, then it is *non-metric* [21]. The *Levi-Civita* connection on a Riemannian manifold N^n is a linear connection which is both symmetric and metric connection. By fundamental theorem of Riemannian geometry every Riemannian manifold admits a unique Levi-Civita connection.

In 1924, Frindmann and Schounten [27] introduced the notion of semi-symmetric connection. A *semi-symmetric connection* $\breve{\nabla}$ on a Riemannian manifold (N^n, g) is a particular type of non-symmetric connection whose torsion tensor $\breve{T}$ is of the form

$$\breve{T}(E, F) = \omega(F)E - \omega(E)F \tag{1.8}$$

for all $E, F \in \chi(N^n)$, where ω is a 1-form associated with the vector field P on N^n by $\omega(E) = g(E, P)$ for all $E \in \chi(N^n)$. A *semi-symmetric metric connection* (shortly, SSM connection) $\breve{\nabla}$ on a Riemannian manifold (N^n, g) is a semi-symmetric connection with $\breve{\nabla}g = 0$. The idea of SSM connection was given by Hayden [30] in 1932. For instance about SSM connection, we cite [1, 24, 26, 33, 40, 41]. In addition, if $\breve{\nabla}P = 0$, then the connection $\breve{\nabla}$ is called *semi-symmetric metric P-connection* (briefly, SSM P-connection) [20].

In this paper we characterize a 3-dimensional Riemannian manifold with SSM P-connection admitting gradient m-QE metric, gradient ρ-Einstein solitons, or gradient η-Ricci solitons. We provide the following results:

Theorem 1.1. *If the metric of a* 3*-dimensional manifold* (N^3, g) *with SSM* P*-connection is a gradient* m*-QE metric, then either*

(i) *the gradient of the potential function is collinear with the vector field* P*, or*
(ii) $\lambda = -(m+2)$ *and the metric is expanding. In this case, the manifold is of constant sectional curvature* -1*, provided* $P\psi \neq m$*.*

Theorem 1.2. *If the metric of a* 3*-dimensional manifold* (N^3, g) *with SSM* P*-connection is a gradient* ρ*-Einstein soliton, then the gradient of the potential function is collinear with the vector field* P*.*

Theorem 1.3. *If the metric of a* 3*-dimensional manifold* (N^3, g) *with SSM* P*-connection is a gradient* ω*-Ricci soliton, then the manifold is of constant sectional curvature* -1 *and* $\lambda + \mu = -2$*.*

2. SSM *P*-connection on Riemannian manifolds

Let (N^n, g) be a Riemannian manifold and ∇ be the Levi-Civita connection corresponding to the metric g. In 1970, Yano [39] obtained the relation between SSM connection $\breve{\nabla}$ defined in (1.8) and Levi-Civita connection ∇, which is

$$\breve{\nabla}_E F = \nabla_E F + \omega(F)E - g(E,F)P \tag{2.1}$$

for all $E, F \in \chi(N^n)$. From (2.1), it follows that the condition $\breve{\nabla} P = 0$ is equivalent to

$$\nabla_E P = \omega(E)P - \omega(P)E, \quad \forall E \in \chi(N^n). \tag{2.2}$$

We derive: $E(\omega(P)) = E(g(P,P)) = 2g(\nabla_E P, P) = 0$ for all $E \in \chi(N^n)$. This implies that the length of the vector field P is constant. Throughout the paper we consider P as a unit vector field. In this situation, the equation (2.2) takes the form

$$\nabla_E P = \omega(E)P - E \tag{2.3}$$

which implies

$$(\nabla_E \omega)F = \omega(E)\omega(F) - g(E,F) \tag{2.4}$$

for all $E, F \in \chi(N^n)$. For a SSM P-connection, the following relations hold [20]:

$$K(E,F)P = \omega(E)F - \omega(F)E, \tag{2.5}$$

$$K(P,E)F = \omega(F)E - g(E,F)P, \tag{2.6}$$

$$\omega(K(E,F)W) = \omega(F)g(E,W) - \omega(E)g(F,W), \tag{2.7}$$

$$Ric(E, P) = -(n-1)\omega(E), \text{ that is, } QP = -(n-1)P \tag{2.8}$$

for all $E, F, W \in \chi(N^n)$, K is the Riemannian curvature tensor corresponding to the Levi-Civita connection and Q is the Ricci operator. The curvature tensor K on a 3-dimensional manifold (N^3, g) is given by

$$\begin{aligned} K(E,F)W = &\ Ric(F,W)E - Ric(E,W)F + g(F,W)QE - g(E,W)QF \\ & - \frac{\tau}{2}\{g(F,W)E - g(E,W)F\} \end{aligned} \tag{2.9}$$

for all $E, F, W \in \chi(N^3)$. Replacing F and W by P in (2.9) and making use of (2.5) and (2.8), we have

$$QE = \left(\frac{\tau}{2} + 1\right) E - \left(\frac{\tau}{2} + 3\right) \omega(E)P \tag{2.10}$$

for all $E \in \chi(N^3)$. From above, it follows that if a 3-dimensional manifold (N^3, g) with SSM P-connection is Einstein if and only if $\tau = -6$. In this case, the manifold is of constant sectional curvature -1.

Lemma 2.1 ([19])**.** *In a 3-dimensional manifold (N^3, g) with SSM P-connection, we have*

$$P\tau = 2(\tau + 6). \tag{2.11}$$

From (2.11), we see if the scalar curvature of (N^3, g) with SSM P-connection is constant, then $\tau = -6$.

3. Proof of the main results

Proof of Theorem 1.1. Let the metric of a 3-dimensional Riemannian manifold (N^3, g) with SSM P-connection be a gradient m-QE metric.

By the equation (2.10), the equation (1.2) takes the form

$$\nabla_E D\psi = -\left(\frac{\tau}{2} + 1 - \lambda\right) E + \left(\frac{\tau}{2} + 3\right) \omega(E)P + \frac{1}{m} g(E, D\psi)D\psi. \tag{3.1}$$

By (2.3) and (2.4), we obtain from (3.1)

$$\begin{aligned} \nabla_F \nabla_E D\psi = &\ -\frac{1}{2}(F\tau)\{E - \omega(E)P\} - \left(\frac{\tau}{2} + 1 - \lambda\right) \nabla_F E \\ & + \left(\frac{\tau}{2} + 3\right) \{\omega(\nabla_F E)P + 2\omega(E)\omega(F)P - g(E,F)P - \omega(E)F\} \\ & + \frac{1}{m}\{g(\nabla_F E, D\psi) + g(E, \nabla_F D\psi)\}D\psi \\ & + \frac{1}{m} g(E, D\psi)\left\{-\left(\frac{\tau}{2} + 1 - \lambda\right) F + \left(\frac{\tau}{2} + 3\right) \omega(F)P\right\} \\ & + \frac{1}{m^2} g(E, D\psi) g(F, D\psi) D\psi. \end{aligned} \tag{3.2}$$

Interchanging E and F in (3.2), we have

$$\begin{aligned}\nabla_E\nabla_F D\psi = & -\frac{1}{2}(E\tau)\{F-\omega(F)P\} - \Big(\frac{\tau}{2}+1-\lambda\Big)\nabla_E F \\ & + \Big(\frac{\tau}{2}+3\Big)\{\omega(\nabla_E F)P + 2\omega(E)\omega(F)P - g(E,F)P - \omega(F)E\} \\ & + \frac{1}{m}\{g(\nabla_E F, D\psi) + g(F,\nabla_E D\psi)\}D\psi \\ & + \frac{1}{m}g(F,D\psi)\Big\{-\Big(\frac{\tau}{2}+1-\lambda\Big)E + \Big(\frac{\tau}{2}+3\Big)\omega(E)P\Big\} \\ & + \frac{1}{m^2}g(E,D\psi)g(F,D\psi)D\psi. \end{aligned} \tag{3.3}$$

Substituting the values (3.1)-(3.3) into $K(E,F) = [\nabla_E,\nabla_F] - \nabla_{[E,F]}$, we get

$$\begin{aligned} K(E,F)D\psi = & \frac{1}{2}(F\tau)\{E-\omega(E)P\} - \frac{1}{2}(E\tau)\{F-\omega(F)P\} \\ & + \Big(\frac{\tau}{2}+3\Big)\{\omega(E)F - \omega(F)E\} \\ & + \frac{1}{m}\Big(\frac{\tau}{2}+1-\lambda\Big)\{(E\psi)F - (F\psi)E\} \\ & - \frac{1}{m}\Big(\frac{\tau}{2}+3\Big)\{(E\psi)\omega(F) - (F\psi)\omega(E)\}P. \end{aligned} \tag{3.4}$$

Contracting the above equation by (2.11) yields

$$Ric(F,D\psi) = \frac{1}{2}(F\tau) + \frac{1}{m}\Big(-\frac{\tau}{2}+1+2\lambda\Big)(F\psi) - \frac{1}{m}\Big(\frac{\tau}{2}+3\Big)(P\psi)\omega(F). \tag{3.5}$$

Taking inner product (3.4) with P, we have

$$g(K(E,F)D\psi, P) = \frac{\lambda+2}{m}\{(F\psi)\omega(E) - (E\psi)\omega(F)\}. \tag{3.6}$$

Using (2.5) in (3.6), it follows that

$$\frac{\lambda+m+2}{m}\{(E\psi)\omega(F) - (F\psi)\omega(E)\} = 0,$$

which gives either, $(E\psi)\omega(F) - (F\psi)\omega(E) = 0$ or, $\lambda + m = -2$.

Case (i) In this case $(E\psi)\omega(F) - (F\psi)\omega(E) = 0$, that is, $D\psi = (P\psi)P$. Therefore the gradient of the potential function ψ is collinear with the vector field P.

Case (ii) In this case $\lambda + m = -2$, that is, $\lambda = -(m+2) < 0$. Therefore, the metric is expanding. Taking $F = P$ in the equation (3.5) and in the view of (2.8) and (2.11), it follows that

$$(\tau+6)\left(\frac{1}{m}(P\psi) - 1\right) = 0,$$

which implies $\tau = -6$, provided $P\psi \neq m$. Hence, the manifold is of constant sectional curvature -1, provided $P\psi \neq m$.

This finishes the proof. □

Proof of Theorem 1.2. Let the metric of a 3-dimensional Riemannian manifold (N^3, g) with SSM P-connection be a gradient ρ-Einstein soliton.

In view of (1.4) and (2.10), it follows that

$$\nabla_E D\psi = -\left(\frac{\tau}{2}+1\right)E + \left(\frac{\tau}{2}+3\right)\omega(E)P + \beta E. \tag{3.7}$$

By (2.3) and (2.4), we obtain from (3.7)

$$\begin{aligned}\nabla_F\nabla_E D\psi = & -\frac{1}{2}(F\tau)\{E-\omega(E)P\} - \left(\frac{\tau}{2}+1\right)\nabla_F E \\ & + \left(\frac{\tau}{2}+3\right)\{\omega(\nabla_F E)P + 2\omega(E)\omega(F)P - g(E,F)P - \omega(E)F\} \\ & + (F\beta)E + \beta\nabla_F E.\end{aligned} \tag{3.8}$$

Interchanging E and F in (3.8), we derive

$$\begin{aligned}\nabla_E\nabla_F D\psi = & -\frac{1}{2}(E\tau)\{F-\omega(F)P\} - \left(\frac{\tau}{2}+1\right)\nabla_E F \\ & + \left(\frac{\tau}{2}+3\right)\{\omega(\nabla_E F)P + 2\omega(E)\omega(F)P - g(E,F)P - \omega(F)E\} \\ & + (E\beta)F + \beta\nabla_E F.\end{aligned} \tag{3.9}$$

Putting the values (3.7)-(3.9) in $K(E,F) = [\nabla_E, \nabla_F] - \nabla_{[E,F]}$, we have

$$\begin{aligned}K(E,F)D\psi = & \frac{1}{2}(F\tau)\{E-\omega(E)P\} - \frac{1}{2}(E\tau)\{F-\omega(F)P\} \\ & + \left(\frac{\tau}{2}+3\right)\{\omega(E)F - \omega(F)E\} + (E\beta)F - (F\beta)E.\end{aligned} \tag{3.10}$$

Contracting the equation (3.10) and using (2.11), we infer

$$Ric(F, D\psi) = \frac{1}{2}(F\tau) - 2(F\beta). \tag{3.11}$$

Taking inner product of (3.10) with P, we get

$$g(K(E,F)D\psi, P) = (E\beta)\omega(F) - (F\beta)\omega(E). \tag{3.12}$$

Using (2.5) in the equation (3.12), we obtain

$$(E\psi)\omega(F) - (F\psi)\omega(E) = (E\beta)\omega(F) - (F\beta)\omega(E). \tag{3.13}$$

Replacing F by P in (3.11) and making use of (2.8) and (2.11), we infer

$$P\psi - P\beta = -\frac{1}{2}(\tau+6).$$

Setting $F = P$ in (3.13) and using the above equation, we obtain

$$D\beta = D\psi + \frac{1}{2}(\tau+6)P. \tag{3.14}$$

It is known that

$$df(E) = g(Df, E) = (Ef), \tag{3.15}$$

where d and D denote the exterior derivative and gradient operator, respectively. Therefore we have

$$
\begin{aligned}
d^2 f(E,F) &= \frac{1}{2}[E(df(F)) - F(df(E)) - df([E,F])] \\
&= \frac{1}{2}[g(\nabla_E Df, F) - g(\nabla_F Df, E)]. \qquad (3.16)
\end{aligned}
$$

By Poincare lemma ($d^2 = 0$), the above equation implies

$$g(\nabla_E Df, F) = g(\nabla_F Df, E). \qquad (3.17)$$

Taking covariant derivative of (3.14) along the vector field E and using (2.3), we have

$$\nabla_E D\beta = \nabla_E D\psi + \frac{1}{2}(E\tau)P + \frac{1}{2}(\tau+6)(\omega(E)P - E), \qquad (3.18)$$

which implies

$$
\begin{aligned}
g(\nabla_E D\beta, F) = g(\nabla_E D\psi, F) &+ \frac{1}{2}(E\tau)\omega(F) \\
&+ \frac{1}{2}(\tau+6)[\omega(E)\omega(F) - g(E,F)]. \qquad (3.19)
\end{aligned}
$$

Interchanging E and F in (3.19) entails that

$$
\begin{aligned}
g(\nabla_F D\beta, E) = g(\nabla_F D\psi, E) &+ \frac{1}{2}(F\tau)\omega(E) \\
&+ \frac{1}{2}(\tau+6)[\omega(E)\omega(F) - g(E,F)]. \qquad (3.20)
\end{aligned}
$$

Subtracting (3.20) from (3.19) and using (3.17), we provide

$$(E\tau)\omega(F) - (F\tau)\omega(E) = 0.$$

Since $\beta = \rho\tau + \lambda$, $(E\beta)\omega(F) - (F\beta)\omega(E) = \rho\{(E\tau)\omega(F) - (F\tau)\omega(E)\} = 0$. In the view of (3.13), it follows that $(E\psi)\omega(F) - (F\psi)\omega(E) = 0$, that is, $D\psi = (P\psi)P$.

Hence the theorem follows. □

Proof of Theorem 1.3. Here we replace the 1-form η in the gradient η-Ricci soliton by the 1-form ω which is the associated 1-form of the SSM P-connection. Let the metric of a 3-dimensional manifold (N^3, g) with SSM P-connection is a gradient ω-Ricci soliton.

By (2.10), the equation (1.6) can be written as

$$\nabla_E D\psi = -\left(\frac{\tau}{2} + 1 - \lambda\right) E + \left(\frac{\tau}{2} + 3 + \mu\right) \omega(E)P. \qquad (3.21)$$

By direct computation, we have

$$
\begin{aligned}
K(E,F)D\psi = \frac{1}{2}(F\tau)\{E - \omega(E)P\} &- \frac{1}{2}(E\tau)\{F - \omega(F)P\} \\
&+ \left(\frac{\tau}{2} + 3 + \mu\right)\{\omega(E)F - \omega(F)E\}. \qquad (3.22)
\end{aligned}
$$

Contracting the equation (3.22), we obtain

$$Ric(E, D\psi) = \frac{1}{2}(E\tau) - 2\mu\omega(E). \tag{3.23}$$

Taking inner product (3.22) with P, we get

$$g(K(E,F)D\psi, P) = 0 \implies (E\psi)\omega(F) - (F\psi)\omega(E) = 0.$$

For $F = P$ the above equation becomes

$$E\psi = (P\psi)\omega(E), \text{ that is, } D\psi = (P\psi)P. \tag{3.24}$$

Using (3.24) and (2.8) in (3.23), we obtain

$$-2(P\psi)\omega(E) = \frac{1}{2}(E\tau) - 2\mu\omega(E). \tag{3.25}$$

Setting $E = P$ in (3.25) and making use of (2.11), we get

$$P\psi = \mu - \frac{1}{2}(\tau + 6). \tag{3.26}$$

Using (3.26) in the equations (3.24) and (3.25), we infer

$$D\psi = \left(\mu - \frac{1}{2}(\tau + 6)\right) P \tag{3.27}$$

and

$$E\tau = 2(\tau + 6)\omega(E). \tag{3.28}$$

By (2.3) and (3.28), we obtain from (3.27)

$$\nabla_E D\psi = -(\tau + 6)\omega(E)P + \left(\mu - \frac{1}{2}(\tau + 6)\right)(\omega(E)P - E).$$

Consider $E = P$ in the above equation, it follows that

$$\nabla_P D\psi = -(\tau + 6)P.$$

On the other hand, from (3.21), we have

$$\nabla_P D\psi = (\lambda + \mu + 2)P.$$

From the above two equations, we have $-(\tau + 6)P = (\lambda + \mu + 2)P$, that is, $\tau = -\lambda - \mu - 8$. This implies, the scalar curvature is constant. Therefore, from (2.11), it follows that $\tau = -6$ and $\lambda + \mu = -2$. Hence, the manifold is of constant sectional curvature -1.

This completes the proof. □

For $\mu = 0$, the gradient η-Ricci soliton becomes a gradient Ricci soliton. Putting $\mu = 0$ and $\tau = -6$ in (3.26), we have $P\psi = 0$. From (3.24), $D\psi = 0$, that is, $\psi =$ constant. Thus, we can state the following corollary:

Corollary 3.1. *If the metric of a 3-dimensional manifold with SSM P-connection is a gradient Ricci soliton, then the manifold is of constant sectional curvature -1 and the soliton is expanding.*

4. Example

We consider 3-dimensional manifold $N^3 = \{(x, y, z) \in \mathbb{R}^3 : x > 0,\ z > 0\}$. Let us define a Riemannian metric g on N^3 as

$$g = \frac{1}{z^2}(dx \otimes dx + dy \otimes dy + dz \otimes dz).$$

Let $V_1 = z\frac{\partial}{\partial x}$, $V_2 = z\frac{\partial}{\partial y}$, $V_3 = -z\frac{\partial}{\partial z}$. Then $\{V_1, V_2, V_3\}$ is an orthonormal basis of (N^3, g). We have

$$[V_1, V_2] = 0, \quad [V_1, V_3] = V_1, \quad [V_2, V_3] = V_2.$$

The Riemannian connection ∇ is given by

$$\begin{aligned} &\nabla_{V_1} V_1 = -V_3, \quad \nabla_{V_1} V_2 = 0, \quad \nabla_{V_1} V_3 = V_1, \\ &\nabla_{V_2} V_1 = 0, \quad \nabla_{V_2} V_2 = -V_3, \quad \nabla_{V_2} V_3 = V_2, \\ &\nabla_{V_3} V_1 = 0, \quad \nabla_{V_3} V_2 = 0, \quad \nabla_{V_3} V_3 = 0. \end{aligned} \tag{4.1}$$

Putting these values in $K(E, F)W = [\nabla_E, \nabla_F] - \nabla_{[E,F]}$, we get

$$\begin{aligned} &K(V_1, V_2)V_1 = V_2, \quad K(V_1, V_2)V_2 = -V_1, \quad K(V_1, V_2)V_3 = 0, \\ &K(V_1, V_3)V_1 = V_3, \quad K(V_1, V_3)V_2 = 0, \quad K(V_1, V_3)V_3 = -V_1, \\ &K(V_2, V_3)V_1 = 0, \quad K(V_2, V_3)V_2 = V_3, \quad K(V_2, V_3)V_3 = -V_2. \end{aligned}$$

From the above expression we obtain

$$Ric(V_i, V_j) = \begin{cases} -2 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for $i, j = 1, 2, 3$, which gives $Ric = -2g$ and $\tau = -6$. Hence, N^3 is a manifold of constant sectional curvature -1.

We define the semi-symmetric metric connection $\breve{\nabla}$ by

$$\breve{\nabla}_E F = \nabla_E F + \omega(F)E - g(E, F)P,$$

where $P = -V_3$ and $\omega(E) = g(E, P)$ for all $E, F \in \chi(N^3)$.

Now,

$$\begin{aligned} \breve{\nabla}_{V_1} P &= -\breve{\nabla}_{V_1} V_3 \\ &= -(\nabla_{V_1} V_3 + \omega(V_3)V_1 + g(V_1, V_3)V_3) \\ &= -(V_1 - V_1) = 0. \end{aligned}$$

Similarly, $\breve{\nabla}_{V_2} P = 0$ and $\breve{\nabla}_{V_3} P = 0$. So, $\breve{\nabla}_E P = 0$ for all $E \in \chi(N^3)$. Hence, $\breve{\nabla}$ is a SSM P-connection.

Suppose let $\psi = -\ln x + \ln z$. Then $D\psi = -\frac{z}{x}V_1 - V_3$ with respect to the metric g. With the help of (4.1), we get

$$\begin{cases} \nabla_{V_1} D\psi = \left(\frac{z^2}{x^2} - 1\right) V_1 + \frac{z}{x} V_3, \\ \nabla_{V_2} D\psi = -V_2, \\ \nabla_{V_3} D\psi = \frac{z}{x} V_1. \end{cases}$$

We can easily verify that

$$Ric(E,F) + H^{\psi}(E,F) - g(E,D\psi)g(F,D\psi) = -3g(E,F)$$

for all $E,F \in \chi(N^3)$. Hence, g is a gradient m-quasi-Einstein metric for $m = 1$ and $\lambda = -3$. Since $\lambda < 0$, the metric g is expanding. Also, $\lambda = -(m+2)$ and $D\psi \neq (P\psi)P$. Thus, Theorem 1.1 is verified.

Now consider $\psi = -\ln z$. Then $D\psi = V_3$. By direct computation

$$H^{\psi}(E,F) = g(E,F) - \omega(E)\omega(F)$$

for all $E,F \in \chi(N^3)$. It is easy to verify that

$$Ric(E,F) + H^{\psi}(E,F) + \omega(E)\omega(F) = -g(E,F)$$

for all $E,F \in \chi(N^3)$. Hence, g is a gradient ω-Ricci soliton for $\lambda = -1$ and $\mu = -1$. Thus, Theorem 1.3 is verified.

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