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SOME RESULTS ON THE GEOMETRY OF A NON-CONFORMAL DEFORMATION OF A METRIC

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Abstract. Let (M^m, g) be an m-dimensional Riemannian manifold. In this paper, we introduce a new class of metric on (M^m, g), obtained by a non-conformal deformation of the metric g. First we investigate the Levi-Civita connection of this metric. Secondly we characterize the Riemannian curvature, the sectional curvature and the scalar curvature. In the last section we characterizes some class of proper biharmonic maps. Examples of proper biharmonic maps are constructed when (M^m, g) is an Euclidean space.

1. Introduction

Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds. φ is said to be harmonic if it is a critical point of the energy functional

$$E(\varphi; D) = \frac{1}{2}\int_D |\tau(\varphi)|^2 \, v_g$$

for any compact domain $D \subseteq M$. Equivalently, φ is harmonic if it satisfies the associated Euler-Lagrange equations given by the following formula:

$$\tau(\varphi) = Tr_g \nabla d\varphi = 0. \tag{1}$$

Here $\tau(\varphi)$ is the tension field of φ (For more detail on harmonic maps, see [1, 5–7, 9]). As a generalization of harmonic maps, biharmonic maps are defined similarily, as follows:

A map φ is said to be biharmonic if it is a critical point of the bi-energy functional

$$E_2(\varphi; D) = \frac{1}{2}\int_D |\tau(\varphi)|^2 \, v_g$$

over any compact domain D.

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Equivalently, φ is biharmonic if it satisfies the associated Euler-Lagrange equations:

$$\tau_2(\varphi) \equiv -Tr_g R^N(\tau(\varphi), d\varphi)d\varphi - Tr_g\big(\nabla^\varphi \nabla^\varphi \tau(\varphi) - \nabla^\varphi_{\nabla^M} \tau(\varphi)\big) = 0. \tag{2}$$

The operator $\tau_2(\varphi)$ is called the bitension field of φ (see [1–3, 5, 6, 9]). It is obvious to see that any harmonic map is biharmonic, therefore it is interesting to construct proper biharmonic maps (non-harmonic biharmonic maps).

Using the conformal transformation in [1], [2] and [3], the authors give some examples of proper biharmonic maps. In [4] and [8] the authors studied biharmonic maps between warped products where they gave the condition for the biharmonicity of the inclusion of a Riemannian manifold N into the warped product, also they gave some characterizations of non-harmonic biharmonic maps using the product of harmonic maps and warping metric. The main motivation of this work is to give other methods for the construction of new examples of proper biharmonic maps.

In this paper, we introduce a new class of metric on (M^m, g), obtained by a non-conformal deformation of the metric g (Definition 3.1). First we investigate the Levi-Civita connection of this metric (Theorem 3.1). Secondly we characterize the Riemannian curvature, the sectional curvature and the scalar curvature (Theorem 4.1, Theorem 4.2, Theorem 4.5 and Theorem 4.6). In the last section we characterize some class of proper biharmonic maps (Proposition 5.4, Proposition 5.10). Examples of proper biharmonic maps are constructed when (M^m, g) is an Euclidean space.

2. Preliminaries

Let (M^m, g) be an m-dimensional Riemannian manifold, $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and $\Im^r_s(M)$ be the module over $C^\infty(M)$ of C^∞ tensor fields of type (r, s). Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g, this connection is characterized by the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z,[X,Y]) \\ &\quad + g(Y,[Z,X]) - g(X,[Y,Z]) \end{aligned} \tag{3}$$

for all $X, Y, Z \in \Im^1_0(M)$.

By R and $Ricci$ we denote, respectively, the Riemannian curvature tensor and the Ricci tensor of (M^m, g), are defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$
$$Ricci(X) = \sum_{i=1}^m R(X, E_i)E_i$$

for all vector fields $X, Y, Z \in \Im^1_0(M)$, where $(E_1, \dots, E_m)$ is a local orthonormal frame on M.

Let f be a smooth function on M, and the gradient of f, noted $grad\, f$, be defined by

$$g(grad\, f, X) = X(f),$$

and the Hessian of f, noted $Hess_f$, be defined by

$$Hess_f(X,Y) = g(\nabla_X grad\, f, Y) = X(Y(f)) - (\nabla_X Y)(f)$$

for all vector fields $X, Y \in \Im_0^1(M)$.

3. Semi-conformal deformation of metric

Definition 3.1. Let (M^m, g) be a Riemannian manifold and $f : M \to]0, +\infty[$ be a strictly positive smooth function. We define the semi-conformal deformation of the metric g on M noted G by

$$G(X,Y)_x = f(x)g(X,Y)_x + g(\xi, X)_x g(\xi, Y)_x$$

for all $x \in M$, $X, Y \in \Im_0^1(M)$ and $\xi \in \Im_0^1(M)$ such that $g(\xi,\xi) = 1$ and $\xi(f) = 0$.

Note that G is a conformal metric to g on the distribution orthogonal to ξ. In the following, we let ξ be a parallel vector field respect to ∇ (i.e., $\nabla\xi = 0$), where ∇ denote the Levi-Civita connection of (M^m, g).

We shall calculate the Levi-Civita connection $\widetilde{\nabla}$ of (M^m, G) as follows.

Theorem 3.1. *Let (M^m, g) be a Riemannian manifold. Then the Levi-Civita connection $\widetilde{\nabla}$ of (M^m, G) is given by*

$$\begin{aligned}\widetilde{\nabla}_X Y = \nabla_X Y + \frac{X(f)}{2f}Y + \frac{Y(f)}{2f}X - \frac{g(X,Y)}{2f} grad\, f \\ - \Big(\frac{X(f)g(\xi,Y)}{2f(f+1)} + \frac{Y(f)g(\xi,X)}{2f(f+1)}\Big)\xi\end{aligned} \tag{4}$$

for all vector fields $X, Y \in \Im_0^1(M)$.

Proof. From Kozul formula (3), we have

$$\begin{aligned}
2G(\widetilde{\nabla}_X Y, Z) &= XG(Y,Z) + YG(Z,X) - ZG(X,Y) + G(Z,[X,Y]) \\
&\quad + G(Y,[Z,X]) - G(X,[Y,Z]) \\
&= X\big(fg(Y,Z) + g(\xi,Y)g(\xi,Z)\big) \\
&\quad + Y\big(fg(Z,X) + g(\xi,Z)g(\xi,X)\big) \\
&\quad - Z\big(fg(X,Y) + g(\xi,X)g(\xi,Y)\big) + fg(Z,[X,Y]) \\
&\quad + g(\xi,Z)g(\xi,[X,Y]) + fg(Y,[Z,X]) + g(\xi,Y)g(\xi,[Z,X]) \\
&\quad - fg(X,[Y,Z]) - g(\xi,X)g(\xi,[Y,Z]) \\
&= X(f)g(Y,Z) + fXg(Y,Z) + Xg(\xi,Y)g(\xi,Z) \\
&\quad + g(\xi,Y)Xg(\xi,Z) + Y(f)g(Z,X) + fYg(Z,X)
\end{aligned}$$

$$
\begin{aligned}
&+ Yg(\xi,Z)g(\xi,X) + g(\xi,Z)Yg(\xi,X) - Z(f)g(X,Y)\\
&- fZg(X,Y) - Zg(\xi,X)g(\xi,Y) - g(\xi,X)Zg(\xi,Y)\\
&+ fg(Z,[X,Y]) + g(\xi,Z)g(\xi,[X,Y]) + fg(Y,[Z,X])\\
&+ g(\xi,Y)g(\xi,[Z,X]) - fg(X,[Y,Z]) - g(\xi,X)g(\xi,[Y,Z])\\
=\ & 2fg(\nabla_X Y,Z) + X(f)g(Y,Z) + Y(f)g(Z,X) - Z(f)g(X,Y)\\
&+ 2g(\xi,\nabla_X Y)g(\xi,Z) + 2g(\nabla_X\xi,Y)g(\xi,Z)\\
=\ & 2G(\nabla_X Y,Z) + \frac{X(f)}{f}\big(G(Y,Z) - g(\xi,Y)g(\xi,Z)\big)\\
&+ \frac{Y(f)}{f}\big(G(Z,X) - g(\xi,Z)g(\xi,X)\big) - \frac{g(X,Y)}{f}G(grad\, f,Z)\\
=\ & 2G(\nabla_X Y,Z) + \frac{X(f)}{f}G(Y,Z) + \frac{Y(f)}{f}G(X,Z)\\
&- \frac{g(X,Y)}{f}G(grad\, f,Z) - \frac{X(f)g(\xi,Y)}{f(f+1)}G(\xi,Z)\\
&- \frac{Y(f)g(\xi,X)}{f(f+1)}G(\xi,Z)\\
=\ & 2G\big(\nabla_X Y + \frac{X(f)}{2f}Y + \frac{Y(f)}{2f}X - \frac{g(X,Y)}{2f}grad\, f,Z\big)\\
&- 2G\big(\frac{X(f)g(\xi,Y)}{2f(f+1)}\xi + \frac{Y(f)g(\xi,X)}{2f(f+1)}\xi,Z\big).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\widetilde{\nabla}_X Y = \nabla_X Y &+ \frac{X(f)}{2f}Y + \frac{Y(f)}{2f}X - \frac{g(X,Y)}{2f}grad\, f\\
&- \big(\frac{X(f)g(\xi,Y)}{2f(f+1)} + \frac{Y(f)g(\xi,X)}{2f(f+1)}\big)\xi.
\end{aligned}
$$

□

Using Theorem 3.1, we obtain the following lemma.

Lemma 3.2. *Let (M^m, g) be a Riemannian manifold. Then for all vector field $X \in \Im^1_0(M)$, we have*

$$
\begin{aligned}
\widetilde{\nabla}_X grad\, f &= \nabla_X grad\, f + \frac{\|grad\, f\|^2}{2f}X - \frac{g(\xi,X)\|grad\, f\|^2}{2f(f+1)}\xi,\\
\widetilde{\nabla}_X \xi &= \frac{X(f)}{2(f+1)}\xi - \frac{g(\xi,X)}{2f}grad\, f.
\end{aligned}
$$

4. Curvatures of semi-conformal deformation metric

We shall calculate the Riemannian curvature tensor of (M^m, G) as follows.

Theorem 4.1. *Let (M^m, g) be a Riemannian manifold. Then the Riemannian curvature tensor $\widetilde{R}$ of (M^m, G) is given by*

$$\begin{aligned}
\widetilde{R}(X,Y)Z = {} & R(X,Y)Z - \frac{g(Y,Z)}{2f}\nabla_X grad\, f + \frac{g(X,Z)}{2f}\nabla_Y grad\, f \\
& + (\frac{3Y(f)Z(f)}{4f^2} - \frac{Hess_f(Y,Z)}{2f} - \frac{g(Y,Z)\|grad\, f\|^2}{4f^2})X \\
& - (\frac{3X(f)Z(f)}{4f^2} - \frac{Hess_f(X,Z)}{2f} - \frac{g(X,Z)\|grad\, f\|^2}{4f^2})Y \\
& + (\frac{g(\xi,X)g(Y,Z)\|grad\, f\|^2}{4f^2(f+1)} - \frac{g(\xi,Y)g(X,Z)\|grad\, f\|^2}{4f^2(f+1)} \\
& + \frac{(4f+3)X(f)Z(f)g(\xi,Y)}{4f^2(f+1)^2} - \frac{(4f+3)Y(f)Z(f)g(\xi,X)}{4f^2(f+1)^2} \\
& + \frac{g(\xi,X)Hess_f(Y,Z)}{2f(f+1)} - \frac{g(\xi,Y)Hess_f(X,Z)}{2f(f+1)})\xi \\
& + (\frac{Y(f)g(\xi,X)g(\xi,Z)}{4f^2(f+1)} - \frac{X(f)g(\xi,Y)g(\xi,Z)}{4f^2(f+1)} \\
& + \frac{3X(f)g(Y,Z)}{4f^2} - \frac{3Y(f)g(X,Z)}{4f^2})grad\, f \qquad (5)
\end{aligned}$$

for all vector fields $X, Y, Z \in \Im_0^1(M)$, where ∇ and R denote, respectively, the Levi-Civita connection and the curvature tensor of (M^m, g).

Proof. For all $X, Y, Z \in \Im_0^1(M)$.

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z.$$

By virtue of Theorem 3.1, we obtain

$$\begin{aligned}
\text{(i)}\ \widetilde{\nabla}_X\widetilde{\nabla}_Y Z = {} & \widetilde{\nabla}_X\Big(\nabla_Y Z + \frac{Y(f)}{2f}Z + \frac{Z(f)}{2f}Y - \frac{g(Y,Z)}{2f}grad\, f \\
& - (\frac{Y(f)g(\xi,Z)}{2f(f+1)} + \frac{Z(f)g(\xi,Y)}{2f(f+1)})\xi\Big) \\
= {} & \widetilde{\nabla}_X(\nabla_Y Z) + \widetilde{\nabla}_X(\frac{Y(f)}{2f}Z) + \widetilde{\nabla}_X(\frac{Z(f)}{2f}Y) - \widetilde{\nabla}_X(\frac{g(Y,Z)}{2f}grad\, f) \\
& - \widetilde{\nabla}_X(\frac{Y(f)g(\xi,Z)}{2f(f+1)}\xi) - \widetilde{\nabla}_X(\frac{Z(f)g(\xi,Y)}{2f(f+1)}\xi).
\end{aligned}$$

Direct computations give

$$\begin{aligned}
\widetilde{\nabla}_X(\nabla_Y Z) = {} & \nabla_X\nabla_Y Z + \frac{X(f)}{2f}\nabla_Y Z + \frac{(\nabla_Y Z)(f)}{2f}X - \frac{g(X,\nabla_Y Z)}{2f}grad\, f \\
& - (\frac{X(f)g(\xi,\nabla_Y Z)}{2f(f+1)} + \frac{(\nabla_Y Z)(f)g(\xi,X)}{2f(f+1)})\xi,
\end{aligned}$$

$$\widetilde{\nabla}_X(\frac{Y(f)}{2f}Z) = (\frac{X(Y(f))}{2f} - \frac{X(f)Y(f)}{2f^2})Z + \frac{Y(f)}{2f}\nabla_X Z$$

$$+\frac{X(f)Y(f)}{4f^2}Z+\frac{Y(f)Z(f)}{4f^2}X-\frac{Y(f)g(X,Z)}{4f^2}grad\,f$$
$$-\big(\frac{X(f)Y(f)g(\xi,Z)}{4f^2(f+1)}+\frac{Y(f)Z(f)g(\xi,X)}{4f^2(f+1)}\big)\xi,$$

$$\widetilde{\nabla}_X\big(\frac{Z(f)}{2f}Y\big)=\big(\frac{X(Z(f))}{2f}-\frac{X(f)Z(f)}{2f^2}\big)Y+\frac{Z(f)}{2f}\nabla_XY$$
$$+\frac{X(f)Z(f)}{4f^2}Y+\frac{Y(f)Z(f)}{4f^2}X-\frac{Z(f)g(X,Y)}{4f^2}grad\,f$$
$$-\big(\frac{X(f)Z(f)g(\xi,Y)}{4f^2(f+1)}+\frac{Y(f)Z(f)g(\xi,X)}{4f^2(f+1)}\big)\xi,$$

$$\widetilde{\nabla}_X\big(\frac{g(Y,Z)}{2f}grad\,f\big)=\big(\frac{g(\nabla_XY,Z)}{2f}+\frac{g(Y,\nabla_XZ)}{2f}-\frac{X(f)g(Y,Z)}{2f^2}\big)grad\,f$$
$$+\frac{g(Y,Z)}{2f}\nabla_Xgrad\,f+\frac{g(Y,Z)\|grad\,f\|^2}{4f^2}X$$
$$-\frac{g(\xi,X)g(Y,Z)\|grad\,f\|^2}{4f^2(f+1)}\xi,$$

$$\widetilde{\nabla}_X\big(\frac{Y(f)g(\xi,Z)}{2f(f+1)}\xi\big)=\big(\frac{-(3f+2)X(f)Y(f)g(\xi,Z)}{2f^2(f+1)^2}+\frac{X(Y(f))g(\xi,Z)}{2f(f+1)}$$
$$+\frac{Y(f)g(\xi,\nabla_XZ)}{2f(f+1)}\big)\xi-\frac{Y(f)g(\xi,X)g(\xi,Z)}{4f^2(f+1)}grad\,f,$$

$$\widetilde{\nabla}_X\big(\frac{Z(f)g(\xi,Y)}{2f(f+1)}\xi\big)=\big(\frac{-(3f+2)X(f)Z(f)g(\xi,Y)}{2f^2(f+1)^2}+\frac{X(Z(f))g(\xi,Y)}{2f(f+1)}$$
$$+\frac{Z(f)g(\xi,\nabla_XY)}{2f(f+1)}\big)\xi-\frac{Z(f)g(\xi,X)g(\xi,Y)}{4f^2(f+1)}grad\,f.$$

(ii) In fact, by substituting X by Y into the $\widetilde{\nabla}_X\widetilde{\nabla}_YZ$, we get $\widetilde{\nabla}_Y\widetilde{\nabla}_XZ$.

(iii) $$\widetilde{\nabla}_{[X,Y]}Z=\nabla_{[X,Y]}Z+\frac{[X,Y](f)}{2f}Z+\frac{Z(f)}{2f}[X,Y]-\frac{g([X,Y],Z)}{2f}grad\,f$$
$$-\big(\frac{[X,Y](f)g(\xi,Z)}{2f(f+1)}+\frac{Z(f)g(\xi,[X,Y])}{2f(f+1)}\big)\xi.$$ □

For $V,W\in\Im^1_0(M)$ and $x\in M$ such that V_x and W_x are linearly independent, the sectional curvature of the plane spanned by V_x and W_x is given by

$$\widetilde{K}(V,W)=\frac{G(\widetilde{R}(V,W)W,V)}{G(V,V)G(W,W)-G(V,W)^2}.$$

Theorem 4.2. *Let (M^m, g) be a Riemannian manifold. If K (resp., $\widetilde{K}$) denotes the sectional curvature of (M^m, g) (resp., (M^m, G)), then we have*

$$\begin{aligned}\widetilde{K}(X,Y) = \frac{1}{f + g(\xi,X)^2 + g(\xi,Y)^2}\Big(&K(X,Y) + \frac{3X(f)^2}{4f^2} + \frac{3Y(f)^2}{4f^2} \\ &- \frac{\|grad\, f\|^2}{4f^2} - \frac{X(f)^2 g(\xi,Y)^2}{4f^2(f+1)} - \frac{Y(f)^2 g(\xi,X)^2}{4f^2(f+1)} - \frac{Hess_f(X,X)}{2f} \\ &- \frac{Hess_f(Y,Y)}{2f} + \frac{X(f)Y(f)g(\xi,X)g(\xi,Y)}{2f^2(f+1)}\Big) \end{aligned} \tag{6}$$

for any $X, Y \in \Im_0^1(M)$ two vector fields orthonormal with respect to g.

Proof. We have,

$$G(\widetilde{R}(X,Y)Y,X) = fg(\widetilde{R}(X,Y)Y,X) + g(\xi,\widetilde{R}(X,Y)Y)g(\xi,X).$$

From the formula (5) and direct computation we get,

$$\begin{aligned} fg(\widetilde{R}(X,Y)Y,X) = fg(R(X,Y)Y,X) &- \frac{Hess_f(X,X)}{2} + \frac{3Y(f)^2}{4f} \\ &- \frac{Hess_f(Y,Y)}{2} - \frac{\|grad\, f\|^2}{4f} + \frac{g(\xi,X)^2\|grad\, f\|^2}{4f(f+1)} \\ &+ \frac{(5f+4)X(f)Y(f)g(\xi,X)g(\xi,Y)}{4f(f+1)^2} + \frac{3X(f)^2}{4f} \\ &- \frac{(4f+3)Y(f)^2 g(\xi,X)^2}{4f(f+1)^2} + \frac{g(\xi,X)^2 Hess_f(Y,Y)}{2(f+1)} \\ &- \frac{g(\xi,X)g(\xi,Y)Hess_f(X,Y)}{2(f+1)} - \frac{X(f)^2 g(\xi,Y)^2}{4f(f+1)}, \end{aligned} \tag{7}$$

and

$$\begin{aligned} g(\xi,\widetilde{R}(X,Y)Y)g(\xi,X) = &\frac{(3f+2)Y(f)^2 g(\xi,X)^2}{4f(f+1)^2} - \frac{g(\xi,X)^2 Hess_f(Y,Y)}{2(f+1)} \\ &- \frac{g(\xi,X)^2\|grad\, f\|^2}{4f(f+1)} + \frac{g(\xi,X)g(\xi,Y)Hess_f(X,Y)}{2(f+1)} \\ &- \frac{(3f+2)X(f)Y(f)g(\xi,X)g(\xi,Y)}{4f^2(f+1)^2}. \end{aligned} \tag{8}$$

In fact, by adding (7) and (8), we get

$$\begin{aligned} G(\widetilde{R}(X,Y)Y,X) = fK(X,Y) &+ \frac{3X(f)^2}{4f} + \frac{3Y(f)^2}{4f} - \frac{\|grad\, f\|^2}{4f} \\ &- \frac{X(f)^2 g(\xi,Y)^2}{4f(f+1)} - \frac{Y(f)^2 g(\xi,X)^2}{4f(f+1)} - \frac{Hess_f(X,X)}{2} \\ &- \frac{Hess_f(Y,Y)}{2} + \frac{X(f)Y(f)g(\xi,X)g(\xi,Y)}{2f(f+1)}. \end{aligned} \tag{9}$$

On the other hand, we have

$$G(X,X)G(Y,Y)-G(X,Y)^2=f(f+g(\xi,X)^2+g(\xi,Y)^2). \tag{10}$$

From (9) and (10), we get the formula (6). □

Corollary 4.3. *Let (M^m,g) be a Riemannian manifold. If f is constant, the sectional curvature $\widetilde{K}$ of (M^m,G) is given by*

$$\widetilde{K}(X,Y)=\frac{1}{f+g(\xi,X)^2+g(\xi,Y)^2}K(X,Y)$$

for any $X,Y\in\Im^1_0(M)$ two vector fields orthonormal with respect to g.

Remark 4.4. Let $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on (M^m,g) with $E_1=\xi$. We define the orthonormal vector fields

$$\widetilde{E}_1=\frac{1}{\sqrt{f+1}}E_1,\ \widetilde{E}_i=\frac{1}{\sqrt{f}}E_i,\ i=\overline{2,m}. \tag{11}$$

Then $\{\widetilde{E}_i\}_{i=\overline{1,m}}$ is a local orthonormal frame on (M^m,G).

Theorem 4.5. *Let (M^m,g) be a Riemannian manifold. If Ricci (resp. $\widetilde{Ricci}$) denotes the Ricci tensor of (M^m,g) (resp., (M^m,G)), then we have*

$$\begin{aligned}\widetilde{Ricci}(X)=&\frac{1}{f}Ricci(X)+\Big(\frac{((4-m)f+5-m)\|grad\,f\|^2}{4f^3(f+1)}-\frac{\Delta(f)}{2f^2}\Big)X\\&+\frac{(2-m)f+3-m}{2f^2(f+1)}\nabla_X grad\,f-\frac{g(\xi,X)}{2f^2(f+1)}\nabla_\xi grad\,f\\&+\Big(\frac{((m-6)f+m-5)\|grad\,f\|^2}{4f^3(f+1)^2}+\frac{\Delta(f)}{2f^2(f+1)}\Big)g(\xi,X)\xi\\&+\frac{((3m-6)f^2+(6m-16)f+3m-9)X(f)}{4f^3(f+1)^2}grad\,f\end{aligned} \tag{12}$$

for any vector field $X\in\Im^1_0(M)$.

Proof. Let $\{E_i\}_{i=\overline{1,m}}$ such that $E_1=\xi$ be a local orthonormal frame on (M^m,g) and $\{\widetilde{E}_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on (M^m,G) defined by (11). By the definition of Ricci tensor, we have

$$\begin{aligned}\widetilde{Ricci}(X)&=\sum_{i=1}^m\widetilde{R}(X,\widetilde{E}_i)\widetilde{E}_i\\&=\widetilde{R}(X,\frac{1}{\sqrt{f+1}}\xi)\frac{1}{\sqrt{f+1}}\xi+\sum_{i=2}^m\widetilde{R}(X,\frac{1}{\sqrt{f}}E_i)\frac{1}{\sqrt{f}}E_i\\&=\frac{1}{f+1}\widetilde{R}(X,\xi)\xi+\frac{1}{f}\sum_{i=2}^m\widetilde{R}(X,E_i)E_i.\end{aligned}$$

From the formula (5) and direct computation we get,

$$\begin{aligned}\frac{1}{f+1}\widetilde{R}(X,\xi)\xi = & -\frac{1}{2f(f+1)}\nabla_X grad\, f + \frac{g(\xi,X)}{2f(f+1)}\nabla_\xi grad\, f \\ & - \frac{\|grad\, f\|^2}{4f^2(f+1)}X + \frac{(3f+2)X(f)}{4f^2(f+1)^2}grad\, f \\ & + \frac{g(\xi,X)\|grad\, f\|^2}{4f^2(f+1)}\xi, \end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{1}{f}\sum_{i=2}^m \widetilde{R}(X,E_i)E_i = & \frac{1}{f}Ricci(X) + \frac{3-m}{2f^2}\nabla_X grad\, f - \frac{g(\xi,X)}{2f^2}\nabla_\xi grad\, f \\ & + \frac{(-f^2+(m-7)f+m-5)g(\xi,X)\|grad\, f\|^2}{4f^3(f+1)^2}\xi \\ & + \frac{\Delta(f)g(\xi,X)}{2f^2(f+1)}\xi + \Big(\frac{(5-m)\|grad\, f\|^2}{4f^3} - \frac{\Delta(f)}{2f^2}\Big)X \\ & + \frac{(3m-9)X(f)}{4f^3}grad\, f. \end{aligned} \tag{14}$$

In fact, by adding (13) and (14), we get (12). □

Theorem 4.6. *Let (M^m, g) be a Riemannian manifold. If σ (resp., $\widetilde{\sigma}$) denotes the scalar curvature of (M^m, g) (resp., (M^m, G)), then we have*

$$\begin{aligned}\widetilde{\sigma} = & \frac{1}{f}\sigma + \frac{(2-m)f+4-3m}{2f^2(f+1)}\Delta(f) \\ & + \frac{(7m-m^2-6)f^2+(16m-2m^2-22)f+9m-m^2-14)}{4f^3(f+1)^2}\|grad\, f\|^2. \end{aligned} \tag{15}$$

Proof. Let $\{E_i\}_{i=\overline{1,m}}$ such that $E_1 = \xi$ be a local orthonormal frame on (M^m, g) and $\{\widetilde{E}_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on (M^m, G) defined by (11). By the definition of the scalar curvature, we have

$$\begin{aligned}\widetilde{\sigma} &= \sum_{i=1}^m G(\widetilde{Ricci}(\widetilde{E}_i), \widetilde{E}_i) \\ &= \frac{1}{f+1}G(\widetilde{Ricci}(\xi,\xi)) + \frac{1}{f}\sum_{i=2}^m G(\widetilde{Ricci}(E_i), E_i) \\ &= g(\widetilde{Ricci}(\xi,\xi)) + \sum_{i=2}^m g(\widetilde{Ricci}(E_i), E_i).\end{aligned}$$

From the formula (12) and direct computation we get,

$$g(\widetilde{Ricci}(\xi,\xi)) = \frac{(4-m)f+3-m}{4f^2(f+1)^2}\|grad\, f\|^2 - \frac{\Delta(f)}{2f(f+1)}, \tag{16}$$

and

$$\sum_{i=2}^{m} g(\widetilde{Ricci}(E_i), E_i)$$
$$= \frac{1}{f}\sigma + \frac{(3-m)f+4-3m}{2f^2(f+1)}\Delta(f)$$
$$+ \frac{(8m-m^2-10)f^2+(17m-2m^2-25)f+9m-m^2-14}{4f^3(f+1)^2}\|grad\, f\|^2. \tag{17}$$

In fact, by adding (16) and (17), we get (15). □

Corollary 4.7. *Let (M^m, g) be a Riemannian manifold. If f is constant, then the scalar curvature $\widetilde{\sigma}$ of (M^m, G) is given by*

$$\widetilde{\sigma} = \frac{1}{f}\sigma.$$

Corollary 4.8. *Let (M^2, g) be a flat Riemannian manifold. Then (M^m, G) is flat if and only if f is a solution of the following differential equation*

$$2\triangle f = \frac{(2f+1)}{f+1}\|grad(f)\|^2.$$

5. Proper biharmonic maps

Proposition 5.1. *Let $Id : (M^m, g) \to (M^m, G)$ be the identity map. Then we have*

$$\tau(Id) = \frac{2-m}{2} grad(\ln(f)) - \frac{E_1(\ln(f))}{f+1}E_1,$$

where $\{E_i\}_{i=\overline{1,m}}$ is a local orthonormal frame on (M^m, g) such that $E_1 = \xi$.

Proof. From the formula (4), we have

$$\widetilde{\nabla}_{E_1}E_1 - \nabla_{E_1}E_1 = \frac{E_1(f)}{f}E_1 - \frac{1}{2f}grad\, f - \frac{E_1(f)g(\xi, E_1)}{f(f+1)}\xi$$
$$= E_1(\ln(f))E_1 - \frac{1}{2}grad\, \ln(f) - \frac{E_1(\ln(f))}{f+1}\xi, \tag{18}$$

$$\widetilde{\nabla}_{E_i}E_i - \nabla_{E_i}E_i = E_i(\ln(f))E_i - \frac{1}{2}grad\, \ln(f) \quad (i \geq 2). \tag{19}$$

Using the formulae (1), (18) and (19) we obtain

$$\tau(Id) = \sum_i \left(\widetilde{\nabla}_{E_i}E_i - \nabla_{E_i}E_i\right)$$
$$= \sum_i \Big(E_i(\ln(f))E_i - \frac{1}{2}grad\, \ln(f)\Big) - \frac{E_1(\ln(f))}{f+1}\xi$$
$$= \frac{2-m}{2}grad\, \ln(f) - \frac{E_1(\ln(f))}{f+1}\xi.$$

□

Lemma 5.2. *Let* $(M,g)=(\mathbb{R}^m,dx^2)$ *be the real Euclidean manifold. If* $\xi=\partial_1$ *and* $f(x_1,x_2,\dots,x_m)=f(x_2)$*, then we have*:

$$\widetilde{\nabla}_{\partial_1}\widetilde{\nabla}_{\partial_1}\tau(Id)=\frac{m-2}{8}\frac{(f')^3}{f^2(f+1)},$$
$$\widetilde{\nabla}_{\partial_i}\widetilde{\nabla}_{\partial_i}\tau(Id)=\frac{m-2}{8}\Big(\frac{f'}{f}\Big)^3\partial_2=\frac{m-2}{8}\Big(\big(\ln(f)\big)'\Big)^3\partial_2,\ (i\geq 3),$$
$$\widetilde{\nabla}_{\partial_2}\widetilde{\nabla}_{\partial_2}\tau(Id)=\frac{(2-m)}{4}\Big[2\big(\ln(f)\big)'''+3\big(\ln(f)\big)''\big(\ln(f)\big)'+\frac{1}{2}\big((\ln(f))'\big)^3\Big]\partial_2.$$

Proof. If we put $F=\frac{2-m}{2}\frac{f'}{f}$ and $F_2=\frac{1}{2}F\frac{f'}{f+1}$, then from Proposition 5.1, we obtain

$$\tau(Id)=F\,\partial_2$$

using the formula (4), we obtain

$$\begin{aligned}\widetilde{\nabla}_{\partial_1}\tau(Id)&=\frac{1}{2}F\frac{f'}{f}\partial_1-F\frac{f'}{2f(f+1)}\partial_1\\&=\frac{1}{2}F\frac{f'}{f+1}\partial_1=F_2\partial_1,\\ \widetilde{\nabla}_{\partial_1}\widetilde{\nabla}_{\partial_1}\tau(Id)&=\widetilde{\nabla}_{\partial_1}F_2\partial_1\\&=-\frac{g(\partial_1,F_2\partial_1)}{2f}grad(f)\\&=-\frac{1}{2f}F_2f'\partial_2,\end{aligned}$$

$$\begin{aligned}\widetilde{\nabla}_{\partial_i}\tau(Id)&=\frac{1}{2}F\frac{f'}{f}\partial_i,\\ \widetilde{\nabla}_{\partial_i}\widetilde{\nabla}_{\partial_i}\tau(Id)&=\widetilde{\nabla}_{\partial_i}\Big(\frac{1}{2}F\frac{f'}{f}\Big)\partial_i\\&=-\frac{1}{2f}\Big(\frac{1}{2}F\frac{f'}{f}\Big)grad(f)\\&=\frac{m-2}{8}\Big(\frac{f'}{f}\Big)^3\partial_2,\end{aligned}$$

$$\begin{aligned}\widetilde{\nabla}_{\partial_2}\tau(Id)&=\partial_2(F)\partial_2+\frac{1}{2}F\frac{\partial_2(f)}{f}\partial_2\\&=\frac{(2-m)}{2}\Big[\big(\ln(f)\big)''+\frac{1}{2}\big((\ln(f))'\big)^2\Big]\partial_2,\\ \widetilde{\nabla}_{\partial_2}\widetilde{\nabla}_{\partial_2}\tau(Id)&=\partial_2(F_3)\partial_2+\frac{1}{2}F_3\frac{\partial_2(f)}{f}\partial_2,\end{aligned}$$

where $F_3=\frac{(2-m)}{2}\Big[\big(\ln(f)\big)''+\frac{1}{2}\big((\ln(f))'\big)^2\Big]$. □

Lemma 5.3. *Let $(M,g)=(\mathbb{R}^m,dx^2)$ be the real Euclidean manifold. If $\xi=\partial_1$ and $f(x_1,x_2,\dots,x_m)=f(x_2)$, then we have*:

$$Tr\widetilde{R}(\tau(Id),\cdot)\cdot=\frac{(2-m)}{4}\Big[(1-m)\frac{f''f'}{f^2}+\big(m-1-\frac{1}{2(f+1)}\big)\Big(\frac{f'}{f}\Big)^3\Big]\partial_2.$$

Proof. Indeed, from Theorem 4.1, we obtain

$$\begin{aligned}
\widetilde{R}(\partial_2,\partial_2)\partial_2&=0,\\
\widetilde{R}(\partial_2,\partial_i)\partial_i&=-\frac{1}{2f}\nabla_{\partial_2}grad(f)-\frac{\|grad(f)\|^2}{4f^2}\partial_2+\frac{3}{4f^2}\partial_2(f)grad(f)\\
&=-\frac{f''}{2f}\partial_2-\frac{(f')^2}{4f^2}\partial_2+\frac{3(f')^2}{4f^2}\partial_2\\
&=\frac{1}{2}\Big[(\frac{f'}{f})^2-\frac{f''}{f}\Big]\partial_2,\ (3\le i),\\
\widetilde{R}(\partial_2,\partial_1)\partial_1&=-\frac{f''}{2f}\partial_2-\frac{(f')^2}{4f^2}\partial_2-\frac{f'}{4f^2(f+1)}grad(f)+\frac{3f'}{4f^2}grad(f)\\
&=\frac{1}{2}\Big[-\frac{f''}{f}+\frac{(f')^2}{f^2}-\frac{(f')^2}{2f^2(f+1)}\Big]\partial_2\\
&=\frac{1}{2}\Big[-\frac{f''}{f}+\frac{(2f+1)}{(2f+2)}\Big(\frac{f'}{f}\Big)^2\Big]\partial_2,\\
Tr\widetilde{R}(\tau(Id),\cdot)\cdot&=\sum_i\widetilde{R}(\tau(Id),\partial_i)\partial_i=F\sum_i\widetilde{R}(\partial_2,\partial_i)\partial_i\\
&=\frac{m-2}{2}F\Big[(\frac{f'}{f})^2-\frac{f''}{f}\Big]\partial_2+\frac{1}{2}F\Big[-\frac{f''}{f}+\frac{(2f+1)}{(2f+2)}\Big(\frac{f'}{f}\Big)^2\Big]\partial_2\\
&=\frac{1}{2}F\Big[(1-m)\frac{f''}{f}+\big(m-1-\frac{1}{2(f+1)}\big)\Big(\frac{f'}{f}\Big)^2\Big]\partial_2,
\end{aligned}$$

where $F=\frac{2-m}{2}\frac{f'}{f}$. □

Using Lemma 5.2, Lemma 5.3 and formula (2), we obtain the following proposition.

Proposition 5.4. *Let $(M,g)=(\mathbb{R}^m,dx^2)$ be the real Euclidean manifold, $\xi=\partial_1$ and $f(x_1,x_2,\dots,x_m)=f(x_2)$. By putting $K=\frac{f'}{f}$ we obtain $\frac{f''}{f}=K'+K^2$ and the bitension field of $Id:(M,g)\to(M,G)$ is given by*

$$\tau_2(Id)=\frac{(m-2)}{8}\Big[4K''+2(4-m)K'K+(2-m)K^3\Big]\partial_2.$$

Example 5.5 (proper biharmonic map)**.** Let $(M,g)=(\mathbb{R}^m,dx^2)$ be the real Euclidean manifold, $\xi=\partial_1$ and $f(x_1,x_2,\dots,x_m)=f(x_2)$. Then $Id:(M,g)\to(M,G)$ is proper biharmonic if and only if K is a non-constant function solution of the following differential equation

$$4K''+2(4-m)K'K+(2-m)K^3=0,$$

where $K = \big(\ln(f)\big)' = \frac{f'}{f}$.

From Remark 4.4 and formula (4), we obtain the next lemma

Lemma 5.6. *Let (M,g) be an m-dimensional Riemannian manifold and $Id : (M,G) \to (M,g)$ be the identity map. Then we have*:

$$\nabla_{\widetilde{E_1}}\widetilde{E_1} - \widetilde{\nabla}_{\widetilde{E_1}}\widetilde{E_1} = \frac{1}{2f(f+1)}grad(f) - \frac{E_1(f)}{(f+1)^2}E_1,$$
$$\nabla_{\widetilde{E_i}}\widetilde{E_i} - \widetilde{\nabla}_{\widetilde{E_i}}\widetilde{E_i} = \frac{1}{2f^2}\big[grad(f) - 2E_i(f)E_i\big],\ (2 \leq i)$$

where $\xi = E_1$.

Proposition 5.7. *Let (M,g) be an m-dimensional Riemannian manifold. Then the tension field of the identity map $I : (M,G) \to (M,g)$ is given by*

$$\tau(Id) = \frac{(m-2)f + (m-3)}{2f^2(f+1)}grad(f) + \frac{(2f+1)E_1(f)}{f^2(f+1)^2}E_1.$$

From Proposition 5.7, we deduce:

Proposition 5.8. *Let $(M,g) = (\mathbb{R}^m, dx^2)$ be the real Euclidean manifold. If $\xi = \partial_1$ and $f(x_1, x_2, \dots, x_m) = f(x_2)$, then the tension field of $I : (M,G) \to (M,g)$ is given by*

$$\tau(Id) = \frac{(m-2)f + (m-3)}{2f^2(f+1)}f'\partial_2$$

and Id is harmonic if and only $f = const$.

Proposition 5.9. *Let $(M,g) = (\mathbb{R}^m, dx^2)$ be the real Euclidean manifold and $I : (M,G) \to (M,g)$ be the identity map. If $\xi = \partial_1$ and $f(x_1, x_2, \dots, x_m) = f(x_2)$, then we have*:

$$\nabla_{\widetilde{\partial_1}}\nabla_{\widetilde{\partial_1}}\tau(Id) - \nabla_{\widetilde{\nabla}_{\widetilde{\partial_1}}\widetilde{\partial_1}}\tau(Id) = \frac{f'F'}{2f(f+1)}\partial_2,$$
$$\nabla_{\widetilde{\partial_i}}\nabla_{\widetilde{\partial_i}}\tau(Id) - \nabla_{\widetilde{\nabla}_{\widetilde{\partial_i}}\widetilde{\partial_i}}\tau(Id) = \frac{f'F'}{2f^2}\partial_2,\ (i \geq 3),$$
$$\nabla_{\widetilde{\partial_2}}\nabla_{\widetilde{\partial_2}}\tau(Id) - \nabla_{\widetilde{\nabla}_{\widetilde{\partial_2}}\widetilde{\partial_2}}\tau(Id) = \frac{(2f^2F'' - f'F')}{2f^3}\partial_2,$$

where $F = \frac{(m-2)f+(m-3)}{2f^2(f+1)}f'$.

Proof. The proof of Proposition 5.9 follows immediately from Remark 4.4 and formula (4). □

Proposition 5.10. *Let $(M,g) = (\mathbb{R}^m, dx^2)$ be the real Euclidean manifold. If $\xi = \partial_1$ and $f(x_1, x_2, \dots, x_m) = f(x_2)$, then the bitension field of $I : (M,G) \to (M,g)$ is given by the follow formula*:

$$\tau_2(Id) = \Big[\frac{(m-1)f + (m-2)}{2f^2(f+1)}f'F' + \frac{(2f^2F'' - f'F')}{2f^3}\Big]\partial_2,$$

where $F = \frac{(m-2)f+(m-3)}{2f^2(f+1)} f'$.

From Proposition 5.9 and Proposition 5.10, we obtain the following examples of proper biharmonic maps.

Example 5.11. Let $(M, g) = (\mathbb{R}^m, dx^2)$ be the real Euclidean manifold. If $\xi = \partial_1$ and $f(x_1, x_2, \ldots, x_m) = f(x_2)$, then the identity $I : (M, G) \to (M, g)$ is proper biharmonic if and only if f is a non-constant function solution of the following differential equation:

$$\frac{(m-1)f + (m-2)}{2f^2(f+1)} f'F' + \frac{(2f^2F'' - f'F')}{2f^3} = 0,$$

where $F = \frac{(m-2)f+(m-3)}{2f^2(f+1)} f'$.

Example 5.12. Let $(M, g) = (\mathbb{R}\times]-\infty, 0[\times\mathbb{R}, dx^2 + dy^2 + dz^2)$ be a real Euclidean manifold. If $\xi = \partial_1$ and $f(x, y, z) = f(y) = \frac{e^y}{1-e^y} = \frac{1}{e^{-y}-1}$, then

$$F = \frac{1}{2}, \qquad \tau(Id) = \frac{1}{2}\partial_2 \neq 0 \quad \text{and} \quad \tau_2(Id) = 0,$$

therefore, $I : (M, G) \to (M, g)$ is a proper biharmonic map.

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