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MULTI-DERIVATIONS AND SOME APPROXIMATIONS

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Abstract. In this paper, we introduce the multi-derivations on rings and present some examples of such derivations. Then, we unify the system of functional equations defining a multi-derivation to a single formula. Applying a fixed point theorem, we will establish the generalized Hyers–Ulam stability of multi-derivations in Banach module whose upper bounds are controlled by a general function. Moreover, we give some important applications of this result to obtain the known stability outcomes.

1. Introduction

The first stability problem concerning of group homomorphisms was introduced by Ulam [24] in 1940 as follows: Let G be a group and H be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a homomorphism $\phi : G \to H$ satisfies the inequality $d(\phi(xy), \phi(x)\phi(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $\psi : G \to H$ with $d(\phi(x), \psi(x)) < \epsilon$ for all $x \in G$? In case of a positive answer to the previous problem, we usually say that the homomorphisms from G to H are *stable* or that the Cauchy functional equation $\varphi(xy) = \varphi(x)\varphi(y)$ is stable. In other words, for a functional equation

$$\mathcal{F}_1(\phi) = \mathcal{F}_2(\phi) \tag{1.1}$$

and a mapping ψ which is an approximate solution of (1.1), that is, $\mathcal{F}_1(\psi)$ and $\mathcal{F}_2(\psi)$ are close in some sense, we may ask whether a solution ϕ of (1.1) exists near to ψ.

The famous Ulam stability problem was partially solved by Hyers [15] for linear functional equation of Banach spaces. The Hyers' theorem was generalized by Aoki [2] for additive mappings, by Th. M. Rassias [21] and by J. M. Rassias [20] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruţa [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. The terminology Hyers–Ulam stability originates from

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these historical backgrounds and this terminology is also applied to the cases of other functional equations.

Throughout, for the set X, we denote $\overbrace{X \times X \times \cdots \times X}^{n\text{-times}}$ by X^n. Let V be an abelian group, W be a linear space, and $n \geq 2$ be an integer. A mapping $f : V^n \to W$ is called *multi-additive* if it is additive (satisfies Cauchy's functional equation $C(x+y) = C(x) + C(y)$) in each variable. A lot of information about the structure of multi-additive mappings, their Ulam stabilities and related topics are available in [5], [7], [8], [10], [11], [12], [16, Sections 13.4 and 17.2] and [25].

The study of derivations in rings though initiated long back, but got impetus only after Posner [19] who in 1957 presented some significant results on derivations in prime rings. The notion of derivation has also been generalized in various directions such as Jordan derivation, left derivation, (θ, ϕ)-derivation, generalized derivation, generalized Jordan derivation, higher derivations and etc. The stability of derivations between operator algebras was first obtained by Šemrl [22]. Badora [3] and Miura et al. [17] proved the Hyers–Ulam stability of ring derivations on Banach algebras. For some results on the stability of generalized derivations and (θ, ψ)-derivation, ternary quadratic derivations and cubic derivation, we refer to [1], [4], [6], [13] and [18].

In this paper, we define the multi-derivations and indicate some examples. We also describe and characterize the structure of such mappings. In other words, we reduce the system of n equations defining the multi-derivations to obtain a single equation. Furthermore, we prove the generalized Hyers-Ulam stability for multi-derivations by using a fixed point result (Theorem 3.1) which was proved in [9, Theorem 1].

2. Characterization of multi-derivations

Let $\mathcal{R}$ be a commutative ring and $\mathcal{M}$ be a bimodule over $\mathcal{R}$. The operation of ring $\mathcal{R}$ on $\mathcal{M}$ is called *scalar multiplication*, and here it is denoted as $r \cdot x$ and $x \cdot r$ ($r \in \mathcal{R}, x \in \mathcal{M}$) to distinguish it from the ring multiplication operation which is usually written by juxtaposition rs, where $r, s \in \mathcal{R}$.

A derivation from $\mathcal{R}$ into an $\mathcal{R}$-bimodule $\mathcal{M}$ is an additive mapping $D : \mathcal{R} \to \mathcal{M}$ that satisfies

$$D(rs) = r \cdot D(s) + D(r) \cdot s \tag{2.1}$$

for all $r, s \in \mathcal{R}$. For each $x \in \mathcal{M}$, we define a derivation ad_x via

$$ad_x(r) = r \cdot x - x \cdot r, \qquad (r \in \mathcal{R}).$$

Such derivations are called *inner*.

Let $\mathcal{R}$ be a ring and $\mathcal{M}$ be a bimodule over $\mathcal{R}$. Then, $\mathcal{R}^n$ is also a ring with pointwise addition and multiplication. Moreover, $\mathcal{M}^n$ is an abelian group with pointwise addition. It is easy to check that $\mathcal{M}^n$ is an $\mathcal{R}^n$-module with the scalar multiplication

$$(r_1, \ldots, r_n) \bullet (x_1, \ldots, x_n) = (r_1 \cdot x_1, \ldots, r_n \cdot x_n)$$

for all $r_1, \ldots, r_n \in \mathcal{R}$ and $x_1, \ldots, x_n \in \mathcal{M}$. If $D : \mathcal{R} \to \mathcal{M}$ is a derivation, then $\mathfrak{D}_n : \mathcal{R}^n \to \mathcal{M}^n$ defined by

$$\mathfrak{D}_n(r_1, \ldots, r_n) := (D(r_1), \ldots, D(r_n)) \tag{2.2}$$

is also a derivation.

From now on, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$, $n \in \mathbb{N}$. Let V and W be vector spaces over the rational numbers, $n \in \mathbb{N}$. In [10], Ciepliński proved that a mapping $f : V^n \to W$ is multi-additive if and only if the equation

$$f(x_1 + x_2) = \sum_{j_1, j_2, \ldots, j_n \in \{1,2\}} f(x_{j_1 1}, x_{j_2 2}, \ldots, x_{j_n n}), \tag{2.3}$$

holds, where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$ for $i = 1, 2$.

Definition 2.1. Let $\mathcal{R}$ and $\mathcal{M}$ be as in the above. A mapping $\mathfrak{D}_n : \mathcal{R}^n \to \mathcal{M}$ is called an *n-derivation* or a *multi-derivation* if $\mathfrak{D}_n$ is a derivation in each variable, that is

$$\begin{aligned}\mathfrak{D}_n(r_1, \ldots, r_{i-1}, r_i r_i', r_{i+1}, \ldots, r_n) &= r_i \cdot \mathfrak{D}_n(r_1, \ldots, r_{i-1}, r_i', r_{i+1}, \ldots, r_n) \\ &\quad + \mathfrak{D}_n(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_n) \cdot r_i'\end{aligned}$$

for all $i \in \{1, \ldots, n\}$.

By Definition 2.1, every multi-derivation $\mathfrak{D}_n : \mathcal{R}^n \to \mathcal{M}$ is also a multi-additive mapping and so it satisfies (2.3). Here, we present some examples of multi-derivations.

Example 2.2. Let $\mathcal{A}$ be an algebra. Then, it is an $\mathcal{A}$-module where the scalar multiplication and algebraic product coincide. Suppose that for each $j \in \{1, \ldots, n\}$, $D_j : \mathcal{A} \to \mathcal{A}$ is a derivation. It is easy to check that the mapping $\mathcal{D} : \mathcal{A}^n \to \mathcal{A}$ defined via

$$\mathcal{D}(a_1, \ldots, a_n) := \prod_{j=1}^{n} D_j(a_j)$$

is a multi-derivation. Given now $x_1, \ldots, x_n \in \mathcal{M}$ are fixed. We see that the mapping $\mathcal{D}_{x_1,\ldots,x_n} : \mathcal{A}^n \to \mathcal{M}$ defined by $\mathcal{D}_{x_1,\ldots,x_n}(a_1, \ldots, a_n) := \prod_{j=1}^{n} ad_{x_j}(a_j)$ is a multi-derivation, where $\mathcal{M}$ is an $\mathcal{A}$-bimodule. For a typical case, let $C(\mathbb{R})$ and $C^\infty(\mathbb{R})$ denote the algebra of all continuous and infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$, respectively. Then, the mapping $\mathcal{D} : (C^\infty(\mathbb{R}))^n \to C(\mathbb{R})$ defined through $\mathcal{D}(g_1, \ldots, g_n) := \prod_{j=1}^{n} g_j'$ is a multi-derivation, where g_j' is the derivative of g_j.

Example 2.3. (i) Let M be an m-smooth manifold, $C^\infty(M)$ be the commutative algebra of smooth real-valued functions $f : M \to \mathbb{R}$ over $\mathbb{R}$, and $\mathfrak{X}(M)$ be the set of all smooth vector fields on M. It is known that $\mathfrak{X}(M)$ is a $C^\infty(M)$-module and moreover $\mathrm{Der}(C^\infty(M))$, the derivations of $C^\infty(M)$, can be identified with $\mathfrak{X}(M)$ [23]. Then, for arbitrary smooth vector fields $X_1, \dots, X_n$ on M, the mapping $D_{X_1,\dots,X_n} : (C^\infty(M))^n \to C^\infty(M)$ defined by

$$D_{X_1,\dots,X_n}(f_1, \dots, f_n) := \prod_{i=1}^{n} X_i(f_i)$$

is a multi-derivation.

(ii) Let (M, ω) be a symplectic manifold, where ω is a non-degenerated 2-form on M. For any smooth function $f \in C^\infty(M)$, we define the Hamiltonian vector field of f to be the smooth vector field X_f defined by $X_f = \hat{\omega}^{-1}(df)$, where $\widehat{\omega} : TM \to T^*M$ is the bundle isomorphism determined by ω [23]. Equivalently, $X_f \lrcorner\, \omega = df$ (called interior multiplication by X_f) or for any vector field Y, $\omega(X_f, Y) = df(Y) = Y(f)$. In fact, $X_f \lrcorner\, \omega$ can be obtained from ω by inserting X_f into the first slot. Another common notation is

$$\nu \lrcorner\, \omega = i_\nu \omega.$$

This is often read "ν into ω". Then, the mapping

$$\begin{aligned} D : C^\infty(M) &\to \mathfrak{X}(M) \\ f &\longmapsto X_f \end{aligned}$$

is a derivation. In fact, for each $f, g \in C^\infty(M)$ and $Y \in C^\infty(M)$, we have

$$\begin{aligned} \omega(X_{fg}, Y) &= Y(fg) = Y(f)g + fY(g) \\ &= \omega(X_f, Y)g + f\omega(X_g, Y) \\ &= \omega(X_f g + fX_g, Y). \end{aligned}$$

Now, the non-degeneracy of ω implies that $X_{fg} = X_f g + fX_g$. Consequently, $D(fg) = X_{fg} = X_f g + fX_g = D(f)g + fD(g)$. Therefore, D is a derivation.

Suppose $\mathcal{R}$ is a ring and $1_{\mathcal{R}}$ (or briefly, 1) is its multiplicative identity and also $\mathcal{M}$ is a bimodule over $\mathcal{R}$. We have $1 \cdot x = x \cdot 1 = x$ for all $x \in \mathcal{M}$. With the previous assumptions, if $D : \mathcal{R} \to \mathcal{M}$ is a derivation, then $D(1) = 0$. This is also valid for multi-derivations. In other words, for the multi-derivation $\mathfrak{D}_n : \mathcal{R}^n \to \mathcal{M}$, we have $\mathfrak{D}_n(r) = 0$ for any $r \in \mathcal{R}^n$ with at least one component which is equal to 1. Indeed,

$$\begin{aligned} \mathfrak{D}_n(r_1, \dots, r_{j-1}, 1, r_{j+1}, \dots, r_n) = {} & 1 \cdot \mathfrak{D}_n(r_1, \dots, r_{j-1}, 1, r_{j+1}, \dots, r_n) \\ & + \mathfrak{D}_n(r_1, \dots, r_{j-1}, 1, r_{j+1}, \dots, r_n) \cdot 1, \end{aligned}$$

and so $\mathfrak{D}_n(r_1, \dots, r_{j-1}, 1, r_{j+1}, \dots, r_n) = 0$.

Let $n \in \mathbb{N}$ with $n \geq 2$ and $r_i^n = (r_{i1}, r_{i2}, \dots, r_{in}) \in \mathcal{R}^n$, where $i \in \{1, 2\}$. We will write r_i^n simply r_i when no confusion can arise.

Let $\mathcal{R}$ be a commutative ring and $\mathcal{M}$ be a bimodule over $\mathcal{R}$, and $r_1, r_2 \in \mathcal{R}^n$. For the multi-additive mapping $\mathfrak{D}_n : \mathcal{R}^n \to \mathcal{M}$, we consider the equation

$$\begin{aligned}\mathfrak{D}_n(r_1 r_2) = \sum_{k=0}^{n} \sum_{1\leq j_1<\cdots<j_k\leq n} & r_{1j_1} \cdots r_{1j_k} \cdot \mathfrak{D}_n^{j_k}(r_{1j_1}, \dots, r_{1j_k}) \\ & \cdot r_{21} \cdots \hat{r}_{2j_1} \cdots \hat{r}_{2j_k} \cdots r_{2n},\end{aligned} \tag{2.4}$$

where the hats indicate omitted arguments, and

$$\begin{aligned}&\mathfrak{D}_n^{j_k}(r_{1j_1}, \dots, r_{1j_k}) \\ &:= \mathfrak{D}_n\left(r_{11}, \dots, r_{1,j_1-1}, r_{2j_1}, r_{1,j_1+1}, \dots, r_{1,j_k-1}, r_{2j_k}, r_{1,j_k+1}, \dots, r_{1n}\right).\end{aligned} \tag{2.5}$$

Here, we adopt the convention that $\mathfrak{D}_n^{j_0}(r_{1j_1}, \dots, r_{1j_k}) := \mathfrak{D}_{\mathfrak{n}}(r_{11}, \dots, r_{1n})$, and note that for $k = n$, we have

$$\mathfrak{D}_n^{j_n}(r_{1j_1}, \dots, r_{1j_k}) := \mathfrak{D}_{\mathfrak{n}}(r_{21}, \dots, r_{2n}).$$

Therefore, the terms in sum (2.4) for $k = 0$ and $k = n$ have the following form, respectively:

$$\mathfrak{D}_{\mathfrak{n}}(r_{11}, \dots, r_{1n}) \cdot r_{21} \cdots r_{2n}$$

and

$$r_{11} \cdots r_{1n} \cdot \mathfrak{D}_{\mathfrak{n}}(r_{21}, \dots, r_{2n}).$$

Put $\mathbf{n} := \{1, \dots, n\}$, $n \in \mathbb{N}$. For a subset $T = \{j_1, \dots, j_i\}$ of $\mathbf{n}$ with $1 \leq j_1 < \cdots < j_i \leq n$ and $r = (r_1, \dots, r_n) \in \mathcal{R}^n$,

$${}_T r := (1, \dots, 1, r_{j_1}, 1, \dots, 1, r_{j_i}, 1, \dots, 1) \in \mathcal{R}^n$$

denotes the vector which coincides with r in exactly those components, which are indexed by the elements of T and whose other components are set equal 1.

We wish to show that a mapping $\mathfrak{D}_n : \mathcal{R}^n \to \mathcal{M}$ is a multi-derivation if and only if it satisfies (2.3) and (2.4). In order to do this, we bring the following lemma.

Lemma 2.4. *Let $\mathcal{R}$ be a unital commutative ring and $\mathcal{M}$ be a bimodule over $\mathcal{R}$. If a multi-additive mapping $\mathfrak{D}_n : \mathcal{R}^n \to \mathcal{M}$ satisfies* (2.4)*, then $\mathfrak{D}_n(r) = 0$ for any $r \in \mathcal{R}^n$, at least with one component equal to* 1.

Proof. We argue by induction on q that $f\left({}_q r\right) = 0$ for $0 \leq q \leq n-1$. For $q = 0$, by putting $r_1 = r_2 = (1, \dots, 1)$ in (2.4), we have

$$\mathfrak{D}_n(1, \dots, 1) = 2^n \mathfrak{D}_n(1, \dots, 1). \tag{2.6}$$

It follows from (2.6) that $\mathfrak{D}_n(1, \dots, 1) = 0$. Assume that for each ${}_{q-1}r$, we have $f\left({}_{q-1}r\right) = 0$. We show that $f\left({}_q r\right) = 0$. Without loss of generality, we assume that ${}_q r = (r_{j_1}, \dots, r_{j_q}, 1, \dots, 1)$. Putting $r_1 = {}_q r$ and $r_2 = (1, \dots, 1)$ in (2.4) and then using our assumption, we have

$$\mathfrak{D}_n\left({}_q r\right) = 2^{n-q} \mathfrak{D}_n\left({}_q r\right).$$

Hence, $\mathfrak{D}_n\left({}_q r\right) = 0$. This shows that $\mathfrak{D}_n(r) = 0$ for any $r \in \mathcal{R}^n$ with at least one component which is equal to 1. $\square$

We now prove the main result of this section.

Theorem 2.5. *Let $\mathcal{R}$ be a unital commutative ring and $\mathcal{M}$ be an $\mathcal{R}$-module. A mapping $\mathfrak{D}_n : \mathcal{R}^n \to \mathcal{M}$ is a multi-derivation if and only if it satisfies* (2.3) *and* (2.4).

Proof. Assume that $\mathfrak{D}_n$ is a multi-derivation. It is shown in [10, Theorem 2] that $\mathfrak{D}_n$ satisfies equation (2.3). We prove that it satisfies equation (2.4) by induction on n. For $n = 1$, it is trivial that $\mathfrak{D}_n$ satisfies (2.1). If (2.4) is valid for some positive integer $n > 1$, then

$$\begin{aligned}
\mathfrak{D}_{n+1}(r_1 r_2) &= r_{1,n+1} \cdot \mathfrak{D}_n(r_1 r_2) + \mathfrak{D}_n(r_1 r_2) \cdot r_{2,n+1} \\
&= r_{1,n+1} \cdot \sum_{k=0}^{n} \sum_{1 \le j_1 < \cdots < j_k \le n} r_{1j_1} \cdots r_{1j_k} \cdot \mathfrak{D}_n^{j_k}(r_{1j_1}, \ldots, r_{1j_k}) \\
&\quad \cdot r_{21} \cdots \hat{r}_{2j_1} \cdots \hat{r}_{2j_k} \cdots r_{2n} \\
&\quad + \sum_{k=0}^{n} \sum_{1 \le j_1 < \cdots < j_k \le n} r_{1j_1} \cdots r_{1j_k} \cdot \mathfrak{D}_n^{j_k}(r_{1j_1}, \ldots, r_{1j_k}) \\
&\quad \cdot r_{21} \cdots \hat{r}_{2j_1} \cdots \hat{r}_{2j_k} \cdots r_{2n} \cdot r_{2,n+1} \\
&= \sum_{k=0}^{n+1} \sum_{1 \le j_1 < \cdots < j_k \le n+1} r_{1j_1} \cdots r_{1j_k} \cdot \mathfrak{D}_{n+1}^{j_k}(r_{1j_1}, \ldots, r_{1j_k}) \\
&\quad \cdot r_{21} \cdots \hat{r}_{2j_1} \cdots \hat{r}_{2j_k} \cdots r_{2,n+1},
\end{aligned}$$

where $\mathfrak{D}_n^{j_k}(r_{1j_1}, \ldots, r_{1j_k})$ is defined by (2.5). This means that (2.4) holds for $n + 1$.

Conversely, suppose that $\mathfrak{D}_n$ satisfies equations (2.3) and (2.4). By Theorem 2 from [10], $\mathfrak{D}_n$ is multi-additive. Fix $j \in \{1, \ldots, n\}$. Putting $r_{2k} = 1$ for all $k \in \{1, \ldots, n\} \backslash \{j\}$ in (2.4) and using Lemma 2.4, we get

$$\begin{aligned}
&\mathfrak{D}_n\left(r_{11}, \ldots, r_{1,j-1}, r_{1j} r_{2j}, r_{1,j+1}, \ldots, r_{1n}\right) \\
&= \mathfrak{D}_n\left(r_{11} 1, \ldots, r_{1,j-1} 1, r_{1j} r_{2j}, r_{1,j+1} 1, \ldots, r_{1n} 1\right) \\
&= r_{1j} \cdot \mathfrak{D}_n\left(r_{11}, \ldots, r_{1,j-1}, r_{1j}, r_{1,j+1}, \ldots, r_{1n}\right) \\
&\quad + \mathfrak{D}_n\left(r_{11}, \ldots, r_{1,j-1}, r_{1j}, r_{1,j+1}, \ldots, r_{1n}\right) \cdot r_{2j}.
\end{aligned}$$

Therefore, the above relation implies that $\mathfrak{D}_n$ is derivation in the jth variable. Since j is arbitrary, we obtain the desired result. $\square$

3. Stability results for multi-derivations

In this section, we prove the generalized Hyers–Ulam stability of multi-derivations by a fixed point result (Theorem 3.1) in Banach spaces. Throughout, for two sets X and Y, the set of all mappings from X to Y is denoted by Y^X.

In the following, we state a result in fixed point theory [9, Theorem 1] which plays an important role in our work.

Theorem 3.1. *Suppose the hypotheses*

(A1) *Y is a Banach space, $\mathcal{S}$ is a nonempty set, $j \in \mathbb{N}$, $g_1, \dots, g_j : \mathcal{S} \to \mathcal{S}$ and $L_1, \dots, L_j : \mathcal{S} \to \mathbb{R}_+$,*

(A2) *$\mathcal{T} : Y^{\mathcal{S}} \to Y^{\mathcal{S}}$ is an operator satisfying the inequality*

$$\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{j} L_i(x) \left\|\lambda(g_i(x)) - \mu(g_i(x))\right\|, \quad \lambda, \mu \in Y^{\mathcal{S}}, \ x \in \mathcal{S},$$

(A3) *$\Lambda : \mathbb{R}_+^{\mathcal{S}} \to \mathbb{R}_+^{\mathcal{S}}$ is an operator defined through*

$$\Lambda\delta(x) := \sum_{i=1}^{j} L_i(x)\delta(g_i(x)), \qquad \delta \in \mathbb{R}_+^{\mathcal{S}}, \ x \in \mathcal{S},$$

hold, and let a function $\theta : \mathcal{S} \to \mathbb{R}_+$ and a mapping $\phi : \mathcal{S} \to Y$ satisfy the following conditions:

$$\|\mathcal{T}\phi(x) - \phi(x)\| \leq \theta(x), \quad \theta^*(x) := \sum_{l=0}^{\infty} \Lambda^l \theta(x) < \infty \qquad (x \in \mathcal{S}).$$

Then, there exists a unique fixed point ψ of $\mathcal{T}$ such that

$$\|\phi(x) - \psi(x)\| \leq \theta^*(x) \qquad (x \in \mathcal{S}).$$

Moreover, $\psi(x) = \lim_{l \to \infty} \mathcal{T}^l \phi(x)$ for all $x \in \mathcal{S}$.

Let $\mathcal{A}$ be a Banach algebra. A Banach space X which is also a left $\mathcal{A}$-module is said to be a *left Banach $\mathcal{A}$-module* if there is $k > 0$ such that

$$\|a \cdot x\| \leq k\|a\|\|x\|, \qquad (a \in \mathcal{A}, x \in X).$$

A right Banach $\mathcal{A}$-module and a Banach $\mathcal{A}$-module can be defined similarly. Recall that X is a commutative Banach $\mathcal{A}$-module if $a \cdot x = x \cdot a$ for all $a \in \mathcal{A}$, $x \in X$.

Here and subsequently, it is assumed that $\mathcal{A}$ is a unital commutative Banach algebra and X is a commutative Banach $\mathcal{A}$-module. In addition, for a mapping $\mathfrak{D} : \mathcal{A}^n \to X$, we consider the difference operators $\Gamma\mathfrak{D}, \Lambda\mathfrak{D} : \mathcal{A}^n \times \mathcal{A}^n \to X$ by

$$\Lambda\mathfrak{D}(a_1, a_2) := \mathfrak{D}(a_1 + a_2) - \sum_{j_1, j_2, \dots, j_n \in \{1,2\}} \mathfrak{D}(a_{j_1 1}, a_{j_2 2}, \dots, a_{j_n n}),$$

and

$$\Gamma\mathfrak{D}(a_1,a_2):=\mathfrak{D}(a_1a_2)-\sum_{k=1}^{n}\sum_{1\leq j_1\leq\cdots\leq j_k\leq n} a_{1j_1}\cdots a_{1j_k}\cdot\mathfrak{D}_n^{j_k}(a_{1j_1},\ldots,a_{1j_k})$$
$$\cdot\, a_{21}\cdots\hat{a}_{2j_1}\cdots\hat{a}_{2j_k}\cdots a_{2n},$$

where $\mathfrak{D}_n^{j_k}(a_{1j_1},\ldots,a_{1j_k})$ is defined in (2.5) and $a_i=(a_{i1},\ldots,a_{in})$.

We have the next stability result for multi-derivations.

Theorem 3.2. *Let $\beta\in\{-1,1\}$ be fixed. Suppose that $\psi:\mathcal{A}^n\times\mathcal{A}^n\times\mathcal{A}^n\times\mathcal{A}^n\to\mathbb{R}_+$ is a function satisfying the equality*

$$\lim_{l\to\infty}\left(\frac{1}{2^{n\beta}}\right)^l\psi\left(2^{\beta l}a_1,2^{\beta l}a_2,2^{\beta l}a_3,2^{\beta l}a_4\right)=0 \tag{3.1}$$

for all $a_1,a_2,a_3,a_4\in\mathcal{A}^n$ and

$$\Psi(a)=:\frac{1}{2^{\frac{\beta+1}{2}n}}\sum_{l=0}^{\infty}\left(\frac{1}{2^{n\beta}}\right)^l\psi\left(2^{\beta l+\frac{\beta-1}{2}}a,2^{\beta l+\frac{\beta-1}{2}}a,0,0\right)<\infty \tag{3.2}$$

for all $a\in\mathcal{A}^n$. Assume also $\mathcal{D}:\mathcal{A}^n\to X$ is a mapping satisfying the inequalities

$$\|\Lambda\mathcal{D}(a_1,a_2)\|\leqslant\psi(a_1,a_2,0,0), \tag{3.3}$$

and

$$\|\Gamma\mathcal{D}(a_3,a_4)\|\leqslant\psi(0,0,a_3,a_4) \tag{3.4}$$

for all a_i's in $\mathcal{A}^n$. Then, there exists a multi-derivation $\mathfrak{D}_n:\mathcal{A}^n\to X$ such that

$$\|\mathcal{D}(a)-\mathfrak{D}_n(a)\|\leq\Psi(a) \tag{3.5}$$

for all $a\in\mathcal{A}^n$.

Proof. Putting $a=a_1=a_2$ in (3.3), we have

$$\|\mathcal{D}(2a)-2^n\mathcal{D}(a)\|\leq\psi(a,a,0,0) \tag{3.6}$$

for all $a\in\mathcal{A}^n$. Set

$$\theta(a):=\frac{1}{2^{\frac{\beta+1}{2}n}}\psi\left(2^{\frac{\beta-1}{2}}a,2^{\frac{\beta-1}{2}}a,0,0\right),\ \text{and}\ \mathcal{T}\theta(a):=\frac{1}{2^{n\beta}}\theta\left(2^{\beta}a\right)\quad\left(\theta\in X^{\mathcal{A}^n}\right).$$

Then, inequality (3.6) can be written as

$$\|\mathcal{D}(a)-\mathcal{T}\mathcal{D}(a)\|\leq\theta(a)\quad(a\in\mathcal{A}^n). \tag{3.7}$$

Define $\Lambda\eta(x):=\frac{1}{2^{n\beta}}\eta(2^{\beta}a)$ for all $\eta\in\mathbb{R}_+^{\mathcal{A}^n}$, $a\in\mathcal{A}^n$. We now see that Λ has the form described in (A3) with $\mathcal{S}=\mathcal{A}^n$, $g_1(a)=2^{\beta}a$ and $L_1(a)=\frac{1}{2^{n\beta}}$ for all $a\in\mathcal{A}^n$. Furthermore, for each $\lambda,\mu\in X^{\mathcal{A}^n}$ and $a\in\mathcal{A}^n$, we get

$$\|\mathcal{T}\lambda(a)-\mathcal{T}\mu(a)\|=\left\|\frac{1}{2^{n\beta}}\left[\lambda\left(2^{\beta}a\right)-\mu\left(2^{\beta}a\right)\right]\right\|\leq L_1(a)\left\|\lambda(g_1(a))-\mu(g_1(a))\right\|.$$

The last relation shows that the hypothesis (A2) holds. By induction on l, one can check that for any $l \in \mathbb{N}_0$ and $a \in \mathcal{A}^n$, we have

$$\Lambda^l\theta(a) := \left(\frac{1}{2^{n\beta}}\right)^l \theta(2^{\beta l}a) = \left(\frac{1}{2^{n\beta}}\right)^l \psi\left(2^{\beta l+\frac{\beta-1}{2}}a, 2^{\beta l+\frac{\beta-1}{2}}a, 0, 0\right) \tag{3.8}$$

for all $a \in \mathcal{A}^n$. Now, (3.2) and (3.8) necessitate that all assumptions of Theorem 3.1 are satisfied. Hence, there exists a mapping $\mathfrak{D}_n : \mathcal{A}^n \to X$ such that

$$\mathfrak{D}_n(a) = \lim_{l\to\infty}(\mathcal{T}^l\mathcal{D})(a) = \frac{1}{2^{n\beta}}\mathfrak{D}_n(2^\beta a) \qquad (a \in \mathcal{A}^n),$$

and (3.5) holds. One can by induction on l show that

$$\left\|\Lambda\left(\mathcal{T}^l\mathcal{D}\right)(a_1, a_2)\right\| \le \left(\frac{1}{2^{n\beta}}\right)^l \psi(2^{\beta l}a_1, 2^{\beta l}a_2, 0, 0) \tag{3.9}$$

for all $a_1, a_2 \in \mathcal{A}^n$ and $l \in \mathbb{N}_0$. It is clear that inequality (3.9) is valid for $l = 0$ by (3.3). Assume that (3.9) is true for an $l \in \mathbb{N}_0$. Then

$$\begin{aligned}
&\left\|\Lambda\left(\mathcal{T}^{l+1}\mathcal{D}\right)(a_1, a_2)\right\| \\
&= \left\|\Lambda\left(\mathcal{T}^{l+1}\mathcal{D}\right)(a_1 + a_2) - \sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} \Lambda\left(\mathcal{T}^{l+1}\mathcal{D}\right)(a_{j_1 1}, a_{j_2 2}, \ldots, a_{j_n n})\right\| \\
&= \frac{1}{2^{n\beta}}\left\|\Lambda\left(\mathcal{T}^l\mathcal{D}\right)(2^\beta a_1 + 2^\beta a_2) - \sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} \Lambda\left(\mathcal{T}^l\mathcal{D}\right)(2^\beta a_{j_1 1}, 2^\beta a_{j_2 2}, \ldots, 2^\beta a_{j_n n})\right\| \\
&= \frac{1}{2^{n\beta}}\left\|\Lambda\left(\mathcal{T}^l\mathcal{D}\right)(a_1, a_2)\right\| \le \left(\frac{1}{2^{n\beta}}\right)^{l+1} \psi\left(2^{\beta(l+1)}a_1, 2^{\beta(l+1)}a_2, 0, 0\right)
\end{aligned}$$

for all $a_1, a_2 \in \mathcal{A}^n$. Letting $l \to \infty$ in (3.9) and applying (3.1), we obtain $\Lambda\mathfrak{D}_n(a_1, a_2) = 0$ for all $a_1, a_2 \in \mathcal{A}^n$. This means that the mapping $\mathfrak{D}_n$ satisfies (2.3). Finally, assume that $\mathfrak{D}'_n : \mathcal{A}^n \to X$ is another mapping satisfying equation (2.3) and inequality (3.5), and fix $a \in \mathcal{A}^n$, $j \in \mathbb{N}$. Then, by Lemma 2.4 and (3.2), we have

$$\begin{aligned}
&\|\mathfrak{D}_n(a) - \mathfrak{D}'_n(a)\| \\
&= \left\|\left(\frac{1}{2^{n\beta}}\right)^j \mathfrak{D}_n\left(2^{\beta j}a\right) - \left(\frac{1}{2^{n\beta}}\right)^j \mathfrak{D}'_n\left(2^{\beta j}a\right)\right\| \\
&\le \left(\frac{1}{2^{n\beta}}\right)^j \left(\left\|\mathfrak{D}_n\left(2^{\beta j}a\right) - \mathcal{D}\left(2^{\beta j}a\right)\right\| + \left\|\mathfrak{D}'_n\left(2^{\beta j}a\right) - \mathcal{D}\left(2^{\beta j}a\right)\right\|\right) \\
&\le 2\left(\frac{1}{2^{n\beta}}\right)^j \Phi\left(2^{\beta j}a\right) \\
&\le 2\left(\frac{1}{2^{n\beta}}\right)^j \sum_{l=j}^{\infty}\left(\frac{1}{2^{n\beta}}\right)^l \psi\left(2^{\beta l+\frac{\beta-1}{2}}a, 2^{\beta l+\frac{\beta-1}{2}}a, 0, 0\right).
\end{aligned}$$

Consequently, letting $j \to \infty$ and using the fact that series (3.2) is convergent for all $a \in \mathcal{A}^n$, we obtain $\mathfrak{D}_n(a) = \mathfrak{D}'_n(a)$ for all $a \in \mathcal{A}^n$. Similar to (3.9), we have

$$\left\|\Gamma\left(\mathcal{T}^l\mathcal{D}\right)(a_3, a_4)\right\| \leq \left(\frac{1}{2^{n\beta}}\right)^l \psi\left(0, 0, 2^{\beta l}a_3, 2^{\beta l}a_4\right) \tag{3.10}$$

for all $a_3, a_4 \in \mathcal{A}^n$ and $l \in \mathbb{N}_0$. Taking the limit as $n \to \infty$, we see that $\mathfrak{D}_n$ is a multi-derivation and hence the proof is now complete. $\square$

The following corollaries are abrupt effects relevant to the stability of multi-derivations by using Theorem 3.2.

Corollary 3.3. *Let $\delta > 0$. Suppose that $\mathcal{D} : \mathcal{A}^n \to X$ is a mapping satisfying the inequalities*

$$\|\Lambda\mathcal{D}(a_1, a_2)\| \leq \delta \ \text{ and } \ \|\Gamma\mathcal{D}(a_1, a_2)\| \leq \delta$$

for all $a_1, a_2 \in \mathcal{A}^n$. Then, there exists a multi-derivation $\mathfrak{D}_n : \mathcal{A}^n \to X$ such that

$$\|\mathcal{D}(a) - \mathfrak{D}_n(a)\| \leq \frac{\delta}{2^n - 1}$$

for all $a \in \mathcal{A}^n$.

Proof. It is sufficient to set $\psi(a_1, a_2, a_3, a_4) = \delta$ in Theorem 3.2 when $\beta = 1$. $\square$

Corollary 3.4. *Given $\theta > 0$ and $\alpha > 0$ such that $\alpha \neq n$. If $\mathcal{D} : \mathcal{A}^n \to X$ is a mapping satisfying the inequalities*

$$\|\Lambda\mathcal{D}(a_1, a_2)\| \leq \sum_{k=1}^{2}\sum_{j=1}^{n} \|a_{kj}\|^\alpha \ \text{ and } \ \|\Gamma\mathcal{D}(a_1, a_2)\| \leq \sum_{k=1}^{2}\sum_{j=1}^{n} \|a_{kj}\|^\alpha$$

for all $a_1, a_2 \in \mathcal{A}^n$, then there exists a multi-derivation $\mathfrak{D}_n : \mathcal{A}^n \to X$ such that

$$\|\mathcal{D}(a) - \mathfrak{D}_n(a)\| \leq \begin{cases} \frac{2}{2^n - 2^\alpha}\sum_{j=1}^{n} \|a_{1j}\|^\alpha, & \alpha \in (0, n) \\ \frac{2}{2^\alpha - 2^n}\sum_{j=1}^{n} \|a_{1j}\|^\alpha, & \alpha \in (n, \infty) \end{cases}$$

for all $a = a_1 \in \mathcal{A}^n$.

Proof. Putting $\psi(a_1, a_2, a_3, a_4) = \sum_{k=1}^{4}\sum_{j=1}^{n} \|a_{kj}\|^\alpha$ in Theorem 3.2, one can obtain the first and second inequalities for $\beta = 1$ and $\beta = -1$, respectively. $\square$

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