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A TRANSLATION OF AN ANALOGUE OF WIENER SPACE WITH ITS APPLICATIONS ON THEIR PRODUCT SPACES

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Abstract. Let $C[0,T]$ denote an analogue of Weiner space, the space of real-valued continuous on $[0,T]$. In this paper, we investigate the translation of time interval $[0,T]$ defining the analogue of Winer space $C[0,T]$. As applications of the result, we derive various relationships between the analogue of Wiener space and its product spaces. Finally, we express the analogue of Wiener measures on $C[0,T]$ as the analogue of Wiener measures on $C[0,s]$ and $C[s,T]$ with $0 < s < T$.

1. Introduction and preliminaries

The physical phenomenon described by Robert Brown, known as the Brownian motion, was the complex and erratic motion of grains of pollen suspended in a liquid [4]. Since his description, it has become a significant object of study in pure and applied mathematics. One of the approaches to this motion is the analysis on a function space using an interesting measure. Wiener introduced a Gaussian measure onto the space of continuous functions to describe this measure which is now called the (classical) Wiener measure [11]. Let $C_0[a,b]$ and $C_0[c,d]$ denote the classical Wiener spaces, the spaces of continuous real-valued functions x_1 and x_2 on the intervals $[a,b]$ and $[c,d]$, respectively, with $x_1(a) = 0$ and $x_2(c) = 0$ [6,12]. A translation of the time interval $[a,b]$ onto an arbitrary interval $[c,d]$ has been used in various literatures [1–3,6,12] related to the classical Wiener spaces, that is, there exists an isometric isomorphism between $C_0[a,b]$ and $C_0[c,d]$ so that the two spaces are identified with each other.

Let $C[a,b]$ and $C[c,d]$ denote the analogue of Wiener spaces, the spaces of real-valued continuous functions on $[a,b]$ and $[c,d]$, respectively [5,7–10]. In this paper, we investigate a translation of the time interval $[a,b]$ onto $[c,d]$ on the analogue of Wiener spaces $C[a,b]$ and $C[c,d]$ so that the spaces also can be identified with each other. As applications of the result, we will derive

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relationships between an analogue of (one-dimensional) Wiener space and its product space. In fact, we express the integrals on the product space $C[0,s]\times C[s,T]$ in terms of those on $C[0,T]$ using the relationships with $0<s<T$. Finally, we express the analogue of Wiener measures on $C[0,T]$ as the analogue of Wiener measures on $C[0,s]$ and $C[s,T]$.

Now, we introduce a generalized analogue of Wiener space which is a finite positive measure space as the our underlying space of this work.

Let $\alpha_{a,b}$ be a function on $[a,b]$ and let $\beta_{a,b}$ be a strictly increasing function on $[a,b]$. For $\vec{t}_k=(t_0,t_1,\dots,t_k)$ with $a=t_0<t_1<\cdots<t_k\le b$, let $J^{a,b}_{\vec{t}_k}:C[a,b]\to\mathbb{R}^{k+1}$ be the function given by

$$J^{a,b}_{\vec{t}_k}(x)=(x(t_0),x(t_1),\dots,x(t_k)).$$

For $\prod_{j=0}^k B_j\in\mathcal{B}(\mathbb{R}^{k+1})$, the subset $(J^{a,b}_{\vec{t}_k})^{-1}(\prod_{j=0}^k B_j)$ of $C[a,b]$ is called an interval I and let $\mathcal{I}$ be the set of all such intervals I. Let φ_a be a finite positive measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$. Define a premeasure $m_{a,b;\varphi_a}$ on $\mathcal{I}$ by

$$m_{a,b;\varphi_a}\left[(J^{a,b}_{\vec{t}_k})^{-1}\left(\prod_{j=0}^k B_j\right)\right]=\int_{B_0}\int_{\prod_{j=1}^k B_j}\mathcal{W}^{a,b}_k(\vec{t}_k,\vec{u}_k,u_0)\,dm_L^k(\vec{u}_k)\,d\varphi_a(u_0),$$

where m_L denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R})$, and for $u_0\in\mathbb{R}$, $\vec{u}_k=(u_1,\dots,u_k)\in\mathbb{R}^k$

$$\begin{aligned}(1)\qquad &\mathcal{W}^{a,b}_k(\vec{t}_k,\vec{u}_k,u_0)\\ &=\left[\frac{1}{\prod_{j=1}^k 2\pi[\beta_{a,b}(t_j)-\beta_{a,b}(t_{j-1})]}\right]^{\frac12}\\ &\quad\times\exp\left\{-\frac12\sum_{j=1}^k\frac{[u_j-\alpha_{a,b}(t_j)-u_{j-1}+\alpha_{a,b}(t_{j-1})]^2}{\beta_{a,b}(t_j)-\beta_{a,b}(t_{j-1})}\right\}.\end{aligned}$$

Then, the Borel σ-algebra $\mathcal{B}(C[a,b])$ of $C[a,b]$ with the supremum norm, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique positive finite measure $w_{a,b;\varphi_a}$ on $\mathcal{B}(C[a,b])$ with $w_{a,b;\varphi_a}(I)=m_{a,b;\varphi_a}(I)$ for all $I\in\mathcal{I}$. This measure $w_{a,b;\varphi_a}$ is called a generalized analogue of Wiener measure on $(C[a,b],\mathcal{B}(C[a,b]))$ according to $\alpha_{a,b}$, $\beta_{a,b}$ and φ_a [5, 7–10].

Theorem 1.1. *If $f:\mathbb{R}^{k+1}\to\mathbb{C}$ is a Borel measurable function, then the following relation holds*:

$$\begin{aligned}&\int_{C[a,b]}f(x(t_0),x(t_1),\dots,x(t_k))\,dw_{a,b;\varphi_a}(x)\\ &\overset{*}{=}\int_{\mathbb{R}}\int_{\mathbb{R}^k}f(u_0,u_1,\dots,u_k)\mathcal{W}^{a,b}_k(\vec{t}_k,\vec{u}_k,u_0)\,dm_L^k(\vec{u}_k)\,d\varphi_a(u_0),\end{aligned}$$

where $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal.

2. A translation of the time interval

In this section, we will prove that the analogue of Wiener space is invariant under a translation of the time interval.

Define $\phi : [c, d] \to [a, b]$ by $\phi(t) = \frac{b-a}{d-c}(t - c) + a$ for $t \in [c, d]$ and define $\psi : C[a, b] \to C[c, d]$ by

$$\psi(x)(t) = (x \circ \phi)(t) \text{ for } x \in C[a, b] \text{ and } t \in [c, d].$$

We note that ψ is an isometric isomorphism with the supremum norms. Let φ_c be a positive finite measure on $\mathcal{B}(\mathbb{R})$, let $\alpha_{c,d} = \alpha_{a,b} \circ \phi$, $\beta_{c,d} = \beta_{a,b} \circ \phi$ on $[c, d]$ and let $w_{c,d;\varphi_c}$ denote the generalized analogue of Wiener measure on $C[c, d]$ according to $\alpha_{c,d}$, $\beta_{c,d}$ and φ_c.

We now have the following theorem.

Theorem 2.1. *Suppose that $\varphi_c = \varphi_a$. Then we have for a Borel subset B of $C[c, d]$,*

$$w_{c,d;\varphi_c}(B) = (w_{a,b;\varphi_a} \circ \psi^{-1})(B). \tag{2}$$

Proof. Since all intervals of $C[c, d]$ generate $\mathcal{B}(C[c, d])$, it suffices to prove (2) for which B is an interval of $C[c, d]$. For $\vec{s}_k = (s_0, s_1, \ldots, s_k)$ with $c = s_0 < s_1 < \cdots < s_k \le d$, let $t_j = \phi(s_j)$ for $j = 0, 1, \ldots, k$ and let $\vec{t}_k = (t_0, t_1, \ldots, t_k)$. Let $J^{c,d}_{\vec{s}_k} : C[c, d] \to \mathbb{R}^{k+1}$ be the function given by

$$J^{c,d}_{\vec{s}_k}(y) = (y(s_0), y(s_1), \ldots, y(s_k))$$

for $y \in C[c, d]$. Then we have for $\prod_{j=0}^k B_j \in \mathcal{B}(\mathbb{R}^{k+1})$

$$\begin{aligned}
\psi^{-1}\left[(J^{c,d}_{\vec{s}_k})^{-1}\left(\prod_{j=0}^k B_j\right)\right] &= (J^{c,d}_{\vec{s}_k} \circ \psi)^{-1}\left(\prod_{j=0}^k B_j\right) \\
&= \{x \in C[a, b] : \psi(x)(s_j) \in B_j \text{ for } j = 0, 1, \ldots, k\} \\
&= \{x \in C[a, b] : x(t_j) \in B_j \text{ for } j = 0, 1, \ldots, k\} \\
&= (J^{a,b}_{\vec{t}_k})^{-1}\left(\prod_{j=0}^k B_j\right)
\end{aligned}$$

which is an interval of $C[a, b]$. Since $\alpha_{c,d} = \alpha_{a,b} \circ \phi$, $\beta_{c,d} = \beta_{a,b} \circ \phi$ and $t_j = \phi(s_j)$ for $j = 0, 1, \ldots, k$, it is not difficult to show

$$\mathcal{W}^{a,b}_k(\vec{t}_k, \vec{u}_k, u_0) = \mathcal{W}^{c,d}_k(\vec{s}_k, \vec{u}_k, u_0)$$

by (1), where $\vec{u}_k = (u_1, \ldots, u_k)$. Since $\varphi_c = \varphi_a$, we have by Theorem 1.1

$$\begin{aligned}
&(w_{a,b;\varphi_a} \circ \psi^{-1})\left[(J^{c,d}_{\vec{s}_k})^{-1}\left(\prod_{j=0}^k B_j\right)\right] \\
&= w_{a,b;\varphi_a}\left[(J^{a,b}_{\vec{t}_k})^{-1}\left(\prod_{j=0}^k B_j\right)\right]
\end{aligned}$$

$$= \int_{B_0}\int_{\prod_{j=1}^k B_j} \mathcal{W}_k^{a,b}(\vec{t}_k,\vec{u}_k,u_0)dm_L^k(\vec{u}_k)d\varphi_a(u_0)$$
$$= \int_{B_0}\int_{\prod_{j=1}^k B_j} \mathcal{W}_k^{c,d}(\vec{s}_k,\vec{u}_k,u_0)dm_L^k(\vec{u}_k)d\varphi_c(u_0)$$
$$= w_{c,d;\varphi_c}\bigg[(J_{\vec{s}_k}^{c,d})^{-1}\bigg(\prod_{j=0}^k B_j\bigg)\bigg],$$

which completes the proof. □

By Theorem 2.1, the measures $w_{a,b;\varphi_a}$ and $w_{c,d;\varphi_c}$ are identified with each other if $C[a,b]$ and $C[c,d]$ have the same initial weight, and $\alpha_{a,b}$ and $\beta_{a,b}$ are linearly transformed from $[a,b]$ onto $[c,d]$. Especially, if $\alpha_{a,b}(t)=0$, $\beta_{a,b}(t)=t$ for $t\in[a,b]$ and $\varphi_a=\varphi_c=\delta_0$ which is the Dirac measure at 0, then $C[a,b]$ and $C[c,d]$ are reduced to $C_0[a,b]$ and $C_0[c,d]$, respectively, so that $C_0[a,b]$ and $C_0[c,d]$ are identified with each other as a special case of Theorem 2.1.

Theorem 2.2. *Let $F: C[c,d]\to\mathbb{C}$ be a function and suppose that $\varphi_c=\varphi_a$. Then F is measurable on $C[c,d]$ if and only if $F\circ\psi$ is measurable on $C[a,b]$. In this case, we have*

$$\int_{C[a,b]} F(\psi(x))dw_{a,b;\varphi_a}(x) \overset{*}{=} \int_{C[c,d]} F(x)dw_{c,d;\varphi_c}(x). \tag{3}$$

Proof. Since ψ is an isometric isomorphism, it is obvious that $F\circ\psi$ is measurable on $C[a,b]$ if F is measurable on $C[c,d]$. Conversely, suppose that $F\circ\psi$ is measurable on $C[a,b]$. By the first part, $F=(F\circ\psi)\circ\psi^{-1}$ is measurable on $C[c,d]$. Now, (3) follows from (2) and the change of variable theorem. □

Let h be a real number. Define $\psi_h: C[a,b]\to C[a+h,b+h]$ by

$$\psi_h(x)(t)=x(t-h) \text{ for } x\in C[a,b] \text{ and } t\in[a+h,b+h].$$

Let φ_{a+h} be a positive finite measure on $\mathcal{B}(\mathbb{R})$, let $\alpha_{a+h,b+h}(t)=\alpha_{a,b}(t-h)$, $\beta_{a+h,b+h}(t)=\beta_{a,b}(t-h)$ for $t\in[a+h,b+h]$ and let $w_{a+h,b+h;\varphi_{a+h}}$ be the generalized analogue of Wiener measure on $C[a+h,b+h]$ according to $\alpha_{a+h,b+h}$, $\beta_{a+h,b+h}$ and φ_{a+h}.

Letting $c=a+h$, $d=b+h$ and $\phi(t)=t-h$ for $t\in[a+h,b+h]$, we now have the following corollaries by Theorems 2.1 and 2.2.

Corollary 2.3. *Suppose that $\varphi_{a+h}=\varphi_a$. Then we have for a Borel subset B of $C[a+h,b+h]$*

$$w_{a+h,b+h;\varphi_{a+h}}(B)=(w_{a,b;\varphi_a}\circ\psi_h^{-1})(B).$$

Corollary 2.4. *Let $F: C[a+h,b+h]\to\mathbb{C}$ be a function and suppose that $\varphi_{a+h}=\varphi_a$. Then F is measurable on $C[a+h,b+h]$ if and only if $F\circ\psi_h$ is measurable on $C[a,b]$. In this case, we have*

$$\int_{C[a,b]} F(\psi_h(x))dw_{a,b;\varphi_a}(x) \overset{*}{=} \int_{C[a+h,b+h]} F(x)dw_{a+h,b+h;\varphi_{a+h}}(x).$$

In view of Theorem 2.1, we will consider $C[0,T]$ as the analogue of Wiener space with the initial weight φ_0 rather than $C[a,b]$ with φ_a. Moreover we replace $\alpha_{a,b}$ and $\beta_{a,b}$ by α and β, respectively, which are defined on $[0,T]$ with β being strictly increasing, unless otherwise specified.

Let $0 < s < T$. Define $H : C[0,s] \times C[s,T] \to C[0,T]$ by

$$H(y,z)(t) = \chi_{[0,s]}(t)y(t) + \chi_{(s,T]}(t)[y(s) + z(t) - z(s)]$$

for $(y,z) \in C[0,s] \times C[s,T]$ and $t \in [0,T]$, and define $H_i : C[0,T] \to C[0,s] \times C[s,T] (i = 1,2)$ by

$$H_1(x) = (x|_{[0,s]}, x|_{[s,T]}) \text{ and } H_2(x) = (x|_{[0,s]}, x|_{[s,T]} - x(s))$$

for $x \in C[0,T]$. Then we have the following:

(P1) H and $H_i (i = 1,2)$ are continuous on each domain.
(P2) $H \circ H_1 = I_{C[0,T]} = H \circ H_2$, where $I_{C[0,T]}$ denotes the identity function on $C[0,T]$.
(P3) For $(y,z) \in C[0,s] \times C[s,T]$, $(H_1 \circ H)(y,z) = (y, y(s) + z - z(s))$ and $(H_2 \circ H)(y,z) = (y, z - z(s))$.
(P4) H is surjective and each H_i is injective.

We note that H is not injective and each H_i is not surjective.

The following theorem gives relationships among the analogue of Wiener measures if we connect the time intervals $[0,s]$ and $[s,T]$ onto $[0,T]$. For the relationships, let φ_s be a positive finite measure on $\mathcal{B}(\mathbb{R})$ and let $C_0[s,T]$ denote the space of continuous real-valued functions z on $[s,T]$ with $z(s) = 0$.

Theorem 2.5. *Let $w_{0,s;\varphi_0}$ and $w_{s,T;\varphi_s}$ denote the generalized analogue of Wiener measures on $C[0,s]$ and $C[s,T]$ according to $\chi_{[0,s]}\alpha$, $\chi_{[0,s]}\beta$, φ_0 and $\chi_{[s,T]}\alpha$, $\chi_{[s,T]}\beta$, φ_s, respectively. Then we have for $B \in \mathcal{B}(C[0,T])$*

$$[(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s}) \circ H^{-1}](B) = \varphi_s(\mathbb{R}) w_{0,T;\varphi_0}(B). \tag{4}$$

In particular, if $\varphi_s = \delta_0$, then for all $(y,z) \in C[0,s] \times C_0[s,T]$ (hence for $w_{0,s;\varphi_0} \times w_{s,T;\delta_0}$-a.e. $(y,z) \in C[0,s] \times C[s,T]$), we have

$$(H_2 \circ H)(y,z) = (y,z).$$

In this case, H is bijective on $C[0,s] \times C_0[s,T]$ with $H^{-1} = H_2$, and we have for $B \in \mathcal{B}(C[0,s] \times C[s,T]) = \mathcal{B}(C[0,s]) \times \mathcal{B}(C[s,T])$,

$$\begin{aligned} (w_{0,s;\varphi_0} \times w_{s,T;\delta_0})(B) &= (w_{0,T;\varphi_0} \circ H_2^{-1})(B) \\ &= (w_{0,T;\varphi_0} \circ H)(B \cap (C[0,s] \times C_0[s,T])), \end{aligned} \tag{5}$$

that is, $w_{0,s;\varphi_0} \times w_{s,T;\delta_0} = w_{0,T;\varphi_0} \circ H_2^{-1} = w_{0,T;\varphi_0} \circ H$ on $C[0,s] \times C_0[s,T]$.

Proof. Since all intervals of $C[0,T]$ generate $\mathcal{B}(C[0,T])$, it suffices to prove (4) on the intervals. Without loss of generality, we have for $\vec{t}_{k+n} = (t_0, t_1, \ldots, t_k,$

$t_{k+1}, \ldots, t_{k+n})$ with $0 = t_0 < t_1 < \cdots < t_k = s < t_{k+1} < \cdots < t_{k+n} \le T$ and for $\prod_{j=0}^{k+n} B_j \in \mathcal{B}(\mathbb{R}^{k+n+1})$

$$
\begin{aligned}
&H^{-1}\bigg[(J^{0,T}_{\vec{t}_{k+n}})^{-1}\bigg(\prod_{j=0}^{k+n} B_j\bigg)\bigg] \\
&= (J^{0,T}_{\vec{t}_{k+n}} \circ H)^{-1}\bigg(\prod_{j=0}^{k+n} B_j\bigg) \\
&= \{(y,z) \in C[0,s] \times C[s,T] : \chi_{[0,s]}(t_j)y(t_j) + \chi_{(s,T]}(t_j)[y(s) + z(t_j) - z(s)] \\
&\quad \in B_j \text{ for } j = 0, 1, \ldots, k, k+1, \ldots, k+n\}
\end{aligned}
$$

so that we have for $y \in C[0,s]$ with $y(t_j) \in B_j$ $(j = 0, 1, \ldots, k)$

$$
\begin{aligned}
&\bigg[H^{-1}\bigg[(J^{0,T}_{\vec{t}_{k+n}})^{-1}\bigg(\prod_{j=0}^{k+n} B_j\bigg)\bigg]\bigg]_y \\
&= \{z \in C[s,T] : y(s) + z(t_j) - z(s) \in B_j \text{ for } j = k+1, \ldots, k+n\}.
\end{aligned}
$$

Thus we have by Theorem 1.1

$$
\begin{aligned}
&(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})\bigg[H^{-1}\bigg[(J^{0,T}_{\vec{t}_{k+n}})^{-1}\bigg(\prod_{j=0}^{k+n} B_j\bigg)\bigg]\bigg] \\
&= \int_{C[0,s]} w_{s,T;\varphi_s}\bigg[\bigg[H^{-1}\bigg[(J^{0,T}_{\vec{t}_{k+n}})^{-1}\bigg(\prod_{j=0}^{k+n} B_j\bigg)\bigg]\bigg]_y\bigg] dw_{0,s;\varphi_0}(y) \\
&= \int_{C[0,s]} \chi_{\prod_{j=0}^{k} B_j}(y(t_0), y(t_1), \ldots, y(t_k)) \int_{C[s,T]} \chi_{\prod_{j=k+1}^{k+n} B_j}(z(t_{k+1}) - z(s) \\
&\quad + y(s), \ldots, z(t_{k+n}) - z(s) + y(s)) dw_{s,T;\varphi_s}(z) dw_{0,s;\varphi_0}(y) \\
&= \int_{C[0,s]} \chi_{\prod_{j=0}^{k} B_j}(y(t_0), y(t_1), \ldots, y(t_k)) \int_{\mathbb{R}^{n+1}} \chi_{\prod_{j=k+1}^{k+n} B_j}(u_{k+1} - u_k \\
&\quad + y(s), , \ldots, u_{k+n} - u_k + y(s)) \mathcal{W}^{s,T}_n(\vec{t}_n, \vec{u}_n, u_k) dm^n_L(\vec{u}_n) d\varphi_s(u_k) dw_{0,s;\varphi_0}(y),
\end{aligned}
$$

where $\vec{t}_n = (t_k, t_{k+1}, \ldots, t_n)$, $\vec{u}_n = (u_{k+1}, \ldots, u_{k+n})$ and $\mathcal{W}^{s,T}_n$ is given by (1) with replacing $\alpha_{a,b}$ and $\beta_{a,b}$ by α and β, respectively. For $j = k, k+1, \ldots, k+n$, let $v_j = u_j - u_k + y(s)$. Then $v_k = y(s) = y(t_k)$ so that we have by Theorem 1.1 and the change of variable theorem

$$
\begin{aligned}
&(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})\bigg[H^{-1}\bigg[(J^{0,T}_{\vec{t}_{k+n}})^{-1}\bigg(\prod_{j=0}^{k+n} B_j\bigg)\bigg]\bigg] \\
&= \int_{C[0,s]} \chi_{\prod_{j=0}^{k} B_j}(y(t_0), y(t_1), \ldots, y(t_k)) \int_{\mathbb{R}^{n+1}} \chi_{\prod_{j=k+1}^{k+n} B_j}(\vec{v}_n) \mathcal{W}^{s,T}_n(\vec{t}_n, \vec{v}_n, v_k) \\
&\quad dm^n_L(\vec{v}_n) d\varphi_s(u_k) dw_{0,s;\varphi_0}(y)
\end{aligned}
$$

$$= \varphi_s(\mathbb{R}) \int_{C[0,s]} \chi_{\prod_{j=0}^{k} B_j}(y(t_0), y(t_1), \dots, y(t_k)) \int_{\mathbb{R}^n} \chi_{\prod_{j=k+1}^{k+n} B_j}(\vec{v}_n) \times \mathcal{W}_n^{s,T}(\vec{t}_n, \vec{v}_n, y(t_k)) dm_L^n(\vec{v}_n) dw_{0,s;\varphi_0}(y),$$

where $\vec{v}_n = (v_{k+1}, \dots, v_{k+n})$. Renaming v_k as a real variable and letting $\vec{t}_k = (t_0, t_1, \dots, t_k), \vec{v}_k = (v_1, \dots, v_k)$ and $\vec{v}_{k+n} = (v_1, \dots, v_k, v_{k+1}, \dots, v_{k+n})$, we have by Theorem 1.1

$$(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})\bigg[H^{-1}\bigg[(J_{\vec{t}_{k+n}}^{0,T})^{-1}\bigg(\prod_{j=0}^{k+n} B_j\bigg)\bigg]\bigg]$$
$$= \varphi_s(\mathbb{R}) \int_{\mathbb{R}^{k+n+1}} \chi_{B_0}(v_0) \chi_{\prod_{j=1}^{k} B_j}(\vec{v}_k) \chi_{\prod_{j=k+1}^{k+n} B_j}(\vec{v}_n) \mathcal{W}_k^{0,s}(\vec{t}_k, \vec{v}_k, v_0) \times \mathcal{W}_n^{s,T}(\vec{t}_n, \vec{v}_n, v_k) dm_L^{k+n}(\vec{v}_{k+n}) d\varphi_0(v_0).$$

From (1), it is not difficult to show

$$\mathcal{W}_k^{0,s}(\vec{t}_k, \vec{v}_k, v_0)\mathcal{W}_n^{s,T}(\vec{t}_n, \vec{v}_n, v_k) = \mathcal{W}_{k+n}^{0,T}(\vec{t}_{k+n}, \vec{v}_{k+n}, v_0)$$

so that we have

$$(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})\bigg[H^{-1}\bigg[(J_{\vec{t}_{k+n}}^{0,T})^{-1}\bigg(\prod_{j=0}^{k+n} B_j\bigg)\bigg]\bigg]$$
$$= \varphi_s(\mathbb{R}) \int_{B_0} \int_{\prod_{j=1}^{k+n} B_j} \mathcal{W}_{k+n}^{0,T}(\vec{t}_{k+n}, \vec{v}_{k+n}, v_0) dm_L^{k+n}(\vec{v}_{k+n}) d\varphi_0(v_0)$$
$$= \varphi_s(\mathbb{R}) w_{0,T;\varphi_0}\bigg[(J_{\vec{t}_{k+n}}^{0,T})^{-1}\bigg(\prod_{j=0}^{k+n} B_j\bigg)\bigg],$$

which completes the proof of (4).

To prove (5), suppose that $\varphi_s = \delta_0$. Then we have for all $x \in C[0,T]$ and all $(y,z) \in C[0,s] \times C_0[s,T]$

$$(H \circ H_2)(x) = x \text{ and } (H_2 \circ H)(y,z) = (y, z - z(s)) = (y,z)$$

by **(P2)** so that H is bijective on $C[0,s] \times C_0[s,T]$ and $H^{-1} = H_2$. Moreover, it is not difficult to show

$$(w_{0,s;\varphi_0} \times w_{s,T;\delta_0})(C[0,s] \times C_0[s,T]) = (w_{0,s;\varphi_0} \times w_{s,T;\delta_0})(C[0,s] \times C[s,T]) \tag{6}$$

so that we have for $w_{0,s;\varphi_0} \times w_{s,T;\delta_0}$-a.e. $(y,z) \in C[0,s] \times C[s,T]$

$$(H_2 \circ H)(y,z) = (y,z).$$

Now, by (4) and (6), we have for $B \in \mathcal{B}(C[0,s]) \times \mathcal{B}(C[s,T])$

$$(w_{0,s;\varphi_0} \times w_{s,T;\delta_0})(B)$$
$$= [(w_{0,s;\varphi_0} \times w_{s,T;\delta_0}) \circ H_2 \circ H](B \cap (C[0,s] \times C_0[s,T]))$$

$$= [(w_{0,s;\varphi_0} \times w_{s,T;\delta_0}) \circ H^{-1} \circ H](B \cap (C[0,s] \times C_0[s,T]))$$
$$= (w_{0,T;\varphi_0} \circ H)(B \cap (C[0,s] \times C_0[s,T]))$$
$$= (w_{0,T;\varphi_0} \circ H_2^{-1})(B \cap (C[0,s] \times C_0[s,T])) = (w_{0,T;\varphi_0} \circ H_2^{-1})(B)$$

since $H_2^{-1}(C[0,s] \times (C[s,T] - C_0[s,T])) = \emptyset$. Now, the proof is completed. □

Theorem 2.6. (a) *Let $G_0 : C[0,T] \to \mathbb{C}$ be a function. Then G_0 is measurable on $C[0,T]$ if and only if $G_0 \circ H$ is measurable on $C[0,s] \times C[s,T]$. The measurability of G_0 is also equivalent to the measurability of $G_0 \circ H$ on $C[0,s] \times C_0[s,T]$. In this case, we have*

$$(7) \qquad \int_{C[0,T]} G_0(x) dw_{0,T;\varphi_0}(x) \stackrel{*}{=} \frac{1}{\varphi_s(\mathbb{R})} \int_{C[0,s]\times C[s,T]} G_0(H(y,z)) d(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(y,z).$$

(b) *Let $G_1 : C[0,s] \times C[s,T] \to \mathbb{C}$ be measurable. Then $G_1 \circ H_2$ and G_1 are measurable on $C[0,T]$ and $C[0,s] \times C_0[s,T]$, respectively, and*

$$(8) \qquad \int_{C[0,T]} G_1(H_2(x)) dw_{0,T;\varphi_0}(x) \stackrel{*}{=} \int_{C[0,s]\times C_0[s,T]} G_1(y,z) d(w_{0,s;\varphi_0} \times w_{s,T;\delta_0})(y,z) \stackrel{*}{=} \int_{C[0,s]\times C[s,T]} G_1(y,z) d(w_{0,s;\varphi_0} \times w_{s,T;\delta_0})(y,z).$$

(c) *Let $G_2 : C[0,s] \times C_0[s,T] \to \mathbb{C}$ be a function. Then G_2 is measurable on $C[0,s] \times C_0[s,T]$ if and only if $G_2 \circ H_2$ is measurable on $C[0,T]$. In this case, the first equality of* (8) *holds.*

Proof. Since H is continuous, $G_0 \circ H$ is measurable on $C[0,s] \times C[s,T]$ if G_0 is measurable on $C[0,T]$. In this case, $G_0 \circ H$ is also measurable on $C[0,s] \times C_0[s,T]$ because $C[0,s] \times C_0[s,T]$ is a Borel subset of $C[0,s] \times C[s,T]$. Conversely, suppose that $G_0 \circ H$ is measurable on $C[0,s] \times C[s,T]$ or $C[0,s] \times C_0[s,T]$. By **(P2)**, we have for all $x \in C[0,T]$

$$G_0(x) = (G_0 \circ H \circ H_2)(x)$$

so that G_0 is measurable on $C[0,T]$ because H_2 is continuous. (7) follows from Theorem 2.5 and the change of variable theorem, which proves (a). To prove (b), suppose that G_1 is measurable on $C[0,s] \times C[s,T]$. Since $C[0,s] \times C_0[s,T]$ is a Borel subset of $C[0,s] \times C[s,T]$, the measurability of G_1 on $C[0,s] \times C_0[s,T]$ follows. The measurability of $G_2 \circ H_2$ immediately follows from the continuity of H_2. (8) follows immediately from Theorem 2.5, which completes the proof of (b). By similar argument as the proof of (a), (c) follows from Theorem 2.5, instead of **(P2)**, and the fact that for all $(y,z) \in C[0,s] \times C_0[s,T]$

$$G_2(y,z) = (G_2 \circ H_2 \circ H)(y,z),$$

which completes the proof. □

Remark 2.7. (a) In (5), the measure $w_{0,T;\varphi_0} \circ H$ on $C[0,s] \times C[s,T]$ may not be equivalent to the measure on its subspace $C[0,s] \times C_0[s,T]$ since the space $C[0,T]$ can be wholly covered by $H[C[0,s] \times (C[s,T] - C_0[s,T])]$. For more details, see Example 3.1 in the next section.

(b) If $\varphi_s \neq \delta_0$, the integral in the right-hand side of (7) may not be reduced to the integral on $C[0,s] \times C_0[s,T]$ since it is possible that $w_{s,T;\varphi_s}(C[s,T]) \neq w_{s,T;\varphi_s}(C_0[s,T])$. For an example, see Example 3.2 in the next section.

(c) The converse of Theorem 2.6(b) may not hold, that is, the measurablity of $G_1 \circ H_2$ may not grantee the measurability of G_1 on $C[0,s] \times C[s,T]$ since $w_{0,s;\varphi_0} \times w_{s,T;\delta_0}$ may not be a complete measure. Furthermore, the measurablity of G_1 on $C[0,s] \times C_0[s,T]$ also may not grantee the measurability of G_1 on $C[0,s] \times C[s,T]$. In this case, we can only assure the first equality of (8) by comparing (b) with (c) in Theorem 2.6. For an example, see Example 3.3 in the next section.

3. Applications and examples

In this section, we apply the results in the previous section to evaluate various integrals on the generalized analogue of Wiener spaces.

We begin with this section giving counter examples.

Example 3.1. Let $B = C[0,s] \times (C[s,T] - C_0[s,T])$. Then we have

$$(w_{0,T;\varphi_0} \circ H)(B \cap (C[0,s] \times C_0[s,T])) = w_{0,T;\varphi_0}(\emptyset) = 0$$

and

$$(w_{0,T;\varphi_0} \circ H)(B) = w_{0,T;\varphi_0}(C[0,T]) = \varphi_0(\mathbb{R}) > 0$$

so that by (5), we have

$$w_{0,s;\varphi_0} \times w_{s,T;\delta_0} = w_{0,T;\varphi_0} \circ H_2^{-1} \neq w_{0,T;\varphi_0} \circ H$$

on the whole space $C[0,s] \times C[s,T]$. Compare (5) with Remark 2.7(a).

Example 3.2. Let $\varphi_s = \delta_0 + \delta_1$ on $\mathcal{B}(\mathbb{R})$, where δ_1 is the Dirac measure at 1. Then we have

$$\begin{aligned} w_{s,T;\varphi_s}(C[s,T]) &= \varphi_s(\mathbb{R}) = \delta_0(\{0\}) + \delta_1(\{1\}) = 2 \\ &\neq 1 = \delta_0(\{0\}) = w_{s,T;\varphi_s}(C_0[s,T]), \end{aligned}$$

which is an example of the assertion of Remark 2.7(b).

Example 3.3. Let B be a subset of $\mathbb{R}$ with $0 \in B$ and $B \notin \mathcal{B}(\mathbb{R})$. Define $J_s^{s,T} : C[s,T] \to \mathbb{R}$ and $K : \mathbb{R} \to C[s,T]$ by

$$J_s^{s,T}(x) = x(s), \quad K(x_0)(t) = x_0$$

for $x \in C[s,T]$, $x_0 \in \mathbb{R}$ and $t \in [s,T]$. Then we have for $x_0 \in \mathbb{R}$

$$(J_s^{s,T} \circ K)(x_0) = x_0. \tag{9}$$

Let $B_0 = (J_s^{s,T})^{-1}(B)$. We note that $C_0[s,T] \subseteq B_0$ since $0 \in B$. From (9), we have $K^{-1}(B_0) = B$. Since K is continuous, $B \in \mathcal{B}(\mathbb{R})$ if $B_0 \in \mathcal{B}(C[s,T])$ so that $B_0 \notin \mathcal{B}(C[s,T])$. Now, define $F : C[0,s] \times C[s,T] \to \mathbb{R}$ by

$$F(y,z) = \chi_{C[0,s]\times B_0}(y,z) \text{ for } (y,z) \in C[0,s] \times C[s,T].$$

Then F is not measurable on $C[0,s] \times C[s,T]$ since

$$\{z \in C[s,T] : (0,z) \in C[0,s] \times B_0\} = B_0 \notin \mathcal{B}(C[s,T]).$$

Since $C_0[s,T] \subseteq B_0$, $F \circ H_2 \equiv 1$ on $C[0,T]$ and $F \equiv 1$ on $C[0,s] \times C_0[s,T]$. Now, $F \circ H_2$ and F are measurable on $C[0,T]$ and $C[0,s] \times C_0[s,T]$, respectively. This is an example of the assertion of Remark 2.7(c).

Theorem 3.4. *Let $F : C[0,s] \to \mathbb{C}$ be $w_{0,s;\varphi_0}$-measurable. Then the function $F(x|_{[0,s]})$ is $w_{0,T;\varphi_0}$-measurable on $C[0,T]$ and*

$$\int_{C[0,T]} F(x|_{[0,s]}) dw_{0,T;\varphi_0}(x) \stackrel{*}{=} \int_{C[0,s]} F(y) dw_{0,s;\varphi_0}(y),$$

where $w_{0,T;\varphi_0}$ and $w_{0,s;\varphi_0}$ are as given in Theorem 2.5.

Proof. Let π_1 be the projection from $C[0,s] \times C[s,T]$ onto $C[0,s]$. Then we have for $x \in C[0,T]$

$$F(x|_{[0,s]}) = (F \circ \pi_1 \circ H_1)(x)$$

so that $F(x|_{[0,s]})$ is $w_{0,T;\varphi_0}$-measurable on $C[0,T]$ since both H_1 and π_1 are continuous. Now, we have by Theorem 2.6

$$\begin{aligned}
&\int_{C[0,T]} F(x|_{[0,s]}) dw_{0,T;\varphi_0}(x) \\
&\stackrel{*}{=} \frac{1}{\varphi_s(\mathbb{R})} \int_{C[0,s]\times C[s,T]} (F \circ \pi_1 \circ H_1)(H(y,z)) d(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(y,z) \\
&= \frac{1}{\varphi_s(\mathbb{R})} \int_{C[0,s]\times C[s,T]} F(y) d(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(y,z) \\
&\stackrel{*}{=} \int_{C[0,s]} F(y) dw_{0,s;\varphi_0}(y),
\end{aligned}$$

which completes the proof. □

Theorem 3.5. *Let $F : C[s,T] \to \mathbb{C}$ be $w_{s,T;\varphi_s}$-measurable. Then the function $F(x|_{[s,T]})$ is $w_{0,T;\varphi_0}$-measurable on $C[0,T]$ and*

$$\begin{aligned}
&\int_{C[0,T]} F(x|_{[s,T]}) dw_{0,T;\varphi_0}(x) \\
&\stackrel{*}{=} \frac{1}{\varphi_s(\mathbb{R})} \int_{C[0,s]\times C[s,T]} F(y(s) + z - z(s)) d(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(y,z).
\end{aligned}$$

Proof. Let π_2 be the projection from $C[0,s] \times C[s,T]$ onto $C[s,T]$. Then we have for $x \in C[0,T]$

$$F(x|_{[s,T]}) = (F \circ \pi_2 \circ H_1)(x)$$

so that $F(x|_{[s,T]})$ is $w_{0,T;\varphi_0}$-measurable on $C[0,T]$ since π_2 is continuous. Now, we have by Theorem 2.6

$$\begin{aligned}
&\int_{C[0,T]} F(x|_{[s,T]}) dw_{0,T;\varphi_0}(x) \\
&\overset{*}{=} \frac{1}{\varphi_s(\mathbb{R})} \int_{C[0,s]\times C[s,T]} (F \circ \pi_2 \circ H_1)(H(y,z)) d(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(y,z) \\
&= \frac{1}{\varphi_s(\mathbb{R})} \int_{C[0,s]\times C[s,T]} F(y(s) + z - z(s)) d(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(y,z),
\end{aligned}$$

which completes the proof. □

Theorem 3.6. *Let $F : C_0[s,T] \to \mathbb{C}$ be $w_{s,T;\varphi_s}$-measurable. Then the function $F(x|_{[s,T]} - x(s))$ is $w_{0,T;\varphi_0}$-measurable on $C[0,T]$ and*

$$\int_{C[0,T]} F(x|_{[s,T]} - x(s)) dw_{0,T;\varphi_0}(x) \overset{*}{=} \frac{\varphi_0(\mathbb{R})}{\varphi_s(\mathbb{R})} \int_{C[s,T]} F(z - z(s)) dw_{s,T;\varphi_s}(z).$$

In particular, we have

$$\int_{C[0,T]} F(x|_{[s,T]} - x(s)) dw_{0,T;\varphi_0}(x) \overset{*}{=} \varphi_0(\mathbb{R}) \int_{C_0[s,T]} F(z) dw_{s,T;\delta_0}(z).$$

Proof. For $x \in C[0,T]$, we have

$$F(x|_{[s,T]} - x(s)) = (F \circ \pi_2 \circ H_2)(x)$$

so that $F(x|_{[s,T]} - x(s))$ is $w_{0,T;\varphi_0}$-measurable on $C[0,T]$ by the same argument as the proof of Theorem 3.5. Now, we have by Theorem 2.6

$$\begin{aligned}
&\int_{C[0,T]} F(x|_{[s,T]} - x(s)) dw_{0,T;\varphi_0}(x) \\
&\overset{*}{=} \frac{1}{\varphi_s(\mathbb{R})} \int_{C[0,s]\times C[s,T]} (F \circ \pi_2 \circ H_2)(H(y,z)) d(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(y,z) \\
&= \frac{1}{\varphi_s(\mathbb{R})} \int_{C[0,s]\times C[s,T]} F(z - z(s)) d(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(y,z) \\
&\overset{*}{=} \frac{\varphi_0(\mathbb{R})}{\varphi_s(\mathbb{R})} \int_{C[s,T]} F(z - z(s)) dw_{s,T;\varphi_s}(z).
\end{aligned}$$

Moreover, if $\varphi_s = \delta_0$, then we have by (8)

$$\begin{aligned}
&\int_{C[0,T]} F(x|_{[s,T]} - x(s)) dw_{0,T;\varphi_0}(x) \\
&\overset{*}{=} \frac{1}{\delta_0(\mathbb{R})} \int_{C[0,s]\times C_0[s,T]} F(z - z(s)) d(w_{0,s;\varphi_0} \times w_{s,T;\delta_0})(y,z)
\end{aligned}$$

$$\overset{*}{=} \varphi_0(\mathbb{R}) \int_{C_0[s,T]} F(z-0) dw_{s,T;\delta_0}(z)$$
$$= \varphi_0(\mathbb{R}) \int_{C_0[s,T]} F(z) dw_{s,T;\delta_0}(z)$$

which proves the second equality of this theorem. □

Corollary 3.7. *Let $F : C[0,T] \to \mathbb{C}$ be $w_{0,T;\varphi_0}$-measurable and suppose that $F(x_1) = F(x_2)$ for all $x_1, x_2 \in C[0,T]$ with $x_1|_{[0,s]} = x_2|_{[0,s]}$. Let $F_{0,s}(y) = F(\chi_{[0,s]}y + \chi_{(s,T]}y(s))$ for $y \in C[0,s]$. Then $F_{0,s}$ is $w_{0,s;\varphi_0}$-measurable on $C[0,s]$ and*

$$\int_{C[0,T]} F(x) dw_{0,T;\varphi_0}(x) \overset{*}{=} \int_{C[0,s]} F_{0,s}(y) dw_{0,s;\varphi_0}(y).$$

Proof. Define $\iota_1 : C[0,s] \to C[0,s] \times C[s,T]$ by $\iota_1(y) = (y, y(s))$ for $y \in C[0,s]$. Then we have for $y \in C[0,s]$

$$F_{0,s}(y) = (F \circ H)(y, y(s)) = (F \circ H \circ \iota_1)(y)$$

so that $F_{0,s}$ is $w_{0,s;\varphi_0}$-measurable on $C[0,s]$ since ι_1 is continuous. Now, we have by Theorem 2.6

$$\int_{C[0,T]} F(x) dw_{0,T;\varphi_0}(x)$$
$$\overset{*}{=} \frac{1}{\varphi_s(\mathbb{R})} \int_{C[0,s]\times C[s,T]} F(H(y,z)) d(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(y,z)$$
$$= \frac{1}{\varphi_s(\mathbb{R})} \int_{C[0,s]\times C[s,T]} F(H(y,y(s))) d(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(y,z)$$
$$\overset{*}{=} \int_{C[0,s]} F_{0,s}(y) dw_{0,s;\varphi_0}(y),$$

which completes the proof. □

Corollary 3.8. *Let $F : C[0,T] \to \mathbb{C}$ be $w_{0,T;\varphi_0}$-measurable and suppose that $F(x_1) = F(x_2)$ for all $x_1, x_2 \in C[0,T]$ with $x_1|_{[s,T]} - x_1(s) = x_2|_{[s,T]} - x_2(s)$. Let $F_{s,T}(z) = F(\chi_{[0,s]}z(s) + \chi_{(s,T]}z)$ for $z \in C[s,T]$. Then $F_{s,T}$ is $w_{s,T;\varphi_s}$-measurable on $C[s,T]$ and*

$$\int_{C[0,T]} F(x) dw_{0,T;\varphi_0}(x) \overset{*}{=} \frac{\varphi_0(\mathbb{R})}{\varphi_s(\mathbb{R})} \int_{C[s,T]} F_{s,T}(z) dw_{s,T;\varphi_s}(z).$$

Proof. Define $\iota_2 : C[s,T] \to C[0,s] \times C[s,T]$ by $\iota_2(z) = (z(s), z)$ for $z \in C[s,T]$. Then we have for $z \in C[s,T]$

$$F_{s,T}(z) = (F \circ H)(z(s), z) = (F \circ H \circ \iota_2)(z)$$

so that $F_{s,T}$ is $w_{s,T;\varphi_s}$-measurable since ι_2 is continuous. By Theorem 2.6, we have

$$\begin{aligned}&\int_{C[0,T]} F(x)dw_{0,T;\varphi_0}(x)\\ &\overset{*}{=} \frac{1}{\varphi_s(\mathbb{R})}\int_{C[0,s]\times C[s,T]} F(H(y,z))d(w_{0,s;\varphi_0}\times w_{s,T;\varphi_s})(y,z)\\ &= \frac{1}{\varphi_s(\mathbb{R})}\int_{C[0,s]\times C[s,T]} F(H(z(s),z))d(w_{0,s;\varphi_0}\times w_{s,T;\varphi_s})(y,z)\\ &\overset{*}{=} \frac{\varphi_0(\mathbb{R})}{\varphi_s(\mathbb{R})}\int_{C[s,T]} F_{s,T}(z)dw_{s,T;\varphi_s}(z),\end{aligned}$$

which completes the proof. □

Remark 3.9. If we define $F_{0,s}$ in Corollary 3.7 by $F_{0,s}(y) = F(y^*)$ for $y \in C[0,s]$, where y^* is an arbitrary continuous extension of y on $[0,T]$, we can obtain the same results in the corollary. Similarly, if we define $F_{s,T}$ in Corollary 3.8 by $F_{s,T}(z) = F(z^*)$ for $z \in C[s,T]$, where z^* is an arbitrary continuous extension of z on $[0,T]$, we can obtain the same results in the corollary.

Applying Theorems 3.4, 3.5 and 3.6, we can easily obtain the following examples.

Example 3.10. Let $B_{0,s} \in \mathcal{B}(C[0,s])$ and let

$$B_{0,s}^{0,T} = \{x \in C[0,T] : x|_{[0,s]} \in B_{0,s}\}.$$

Letting $F = \chi_{B_{0,s}}$ in Theorem 3.4, we have for $x \in C[0,T]$

$$F(x|_{[0,s]}) = \chi_{B_{0,s}}(x|_{[0,s]}) = \chi_{B_{0,s}^{0,T}}(x)$$

so that $B_{0,s}^{0,T} \in \mathcal{B}(C[0,T])$ and we have

$$\begin{aligned}w_{0,T;\varphi_0}(B_{0,s}^{0,T}) &= \int_{C[0,T]} \chi_{B_{0,s}}(x|_{[0,s]})dw_{0,T;\varphi_0}(x)\\ &= \int_{C[0,s]} \chi_{B_{0,s}}(y)dw_{0,s;\varphi_0}(y) = w_{0,s;\varphi_0}(B_{0,s}).\end{aligned}$$

Example 3.11. Let $B_{s,T} \in \mathcal{B}(C[s,T])$, let

$$B_{s,T}^{0,T} = \{x \in C[0,T] : x|_{[s,T]} \in B_{s,T}\}$$

and let

$$B_{s,T}^{s,T} = \{(y,z) \in C[0,s]\times C[s,T] : y(s) + z - z(s) \in B_{s,T}\}.$$

Letting $F = \chi_{B_{s,T}}$ in Theorem 3.5, we have for $x \in C[0,T]$

$$F(x|_{[s,T]}) = \chi_{B_{s,T}}(x|_{[s,T]}) = \chi_{B_{s,T}^{0,T}}(x)$$

so that $B_{s,T}^{0,T} \in \mathcal{B}(C[0,T])$ and we have

$$\begin{aligned} & w_{0,T;\varphi_0}(B_{s,T}^{0,T}) \\ &= \int_{C[0,T]} \chi_{B_{s,T}}(x|_{[s,T]}) dw_{0,T;\varphi_0}(x) \\ &= \frac{1}{\varphi_s(\mathbb{R})} \int_{C[0,s]\times C[s,T]} \chi_{B_{s,T}}(y(s)+z-z(s)) d(w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(y,z) \\ &= \frac{1}{\varphi_s(\mathbb{R})} (w_{0,s;\varphi_0} \times w_{s,T;\varphi_s})(B_{s,T}^{s,T}). \end{aligned}$$

Example 3.12. Let $B_{s,T;0} \in \mathcal{B}(C_0[s,T])$, let

$$B_{s,T;0}^{0,T} = \{x \in C[0,T] : x|_{[s,T]} - x(s) \in B_{s,T;0}\}$$

and let

$$B_{s,T;0}^{s,T} = \{z \in C[s,T] : z - z(s) \in B_{s,T;0}\}.$$

Letting $F = \chi_{B_{s,T;0}}$ in Theorem 3.6, we have for $x \in C[0,T]$

$$F(x|_{[s,T]} - x(s)) = \chi_{B_{s,T;0}}(x|_{[s,T]} - x(s)) = \chi_{B_{s,T;0}^{0,T}}(x)$$

so that $B_{s,T;0}^{0,T} \in \mathcal{B}(C[0,T])$ and we have

$$\begin{aligned} w_{0,T;\varphi_0}(B_{s,T;0}^{0,T}) &= \int_{C[0,T]} \chi_{B_{s,T;0}}(x|_{[s,T]} - x(s)) dw_{0,T;\varphi_0}(x) \\ &= \frac{\varphi_0(\mathbb{R})}{\varphi_s(\mathbb{R})} \int_{C[s,T]} \chi_{B_{s,T;0}}(z - z(s)) dw_{s,T;\varphi_s}(z) \\ &= \frac{\varphi_0(\mathbb{R})}{\varphi_s(\mathbb{R})} w_{s,T;\varphi_s}(B_{s,T;0}^{s,T}). \end{aligned}$$

Letting $\varphi_s = \delta_0$, in particular, we have by Theorem 3.6

$$w_{0,T;\varphi_0}(B_{s,T;0}^{0,T}) = \varphi_0(\mathbb{R}) w_{s,T;\delta_0}(B_{s,T;0}).$$

Remark 3.13. (a) In the study of analogue of Wiener space, the initial weight plays a crucial role if it is not a probability measure, in particular, not the Dirac measure at 0. Hence the relationships between $w_{0,T;\varphi_0}$ and $w_{0,s;\varphi_0} \times w_{s,T;\varphi_s}$ are dominated by both φ_0 and φ_s. For more details, see (7), Theorems 2.5, 3.5, 3.6, Corollary 3.8, Examples 3.11 and 3.12.

(b) In (8), Theorem 3.4, Corollary 3.7 and Example 3.10, each integral is affected by the initial weight φ_0 even if it is not appeared in the expression. This is due to the fact that $w_{0,s;\varphi_0}$ and $w_{0,T;\varphi_0}$ may not be probability measures, but they have the same initial weight φ_0.

(c) In Theorem 3.5, the transformation of integral on $C[0,T]$ to the space $C[0,s]\times C[s,T]$ is affected by φ_s so that it can not be reduced to the integral on $C[s,T]$. On the other hand, if $\varphi_s = \delta_0$ in Theorem 3.6, the same transformation can be reduced to the integral on $C[s,T]$ (hence on $C_0[s,T]$) because the initial weights of paths in $C[s,T]$ are concentrated at 0, that is, $w_{s,T;\delta_0}(C[s,T]) = w_{s,T;\delta_0}(C_0[s,T])$.

(d) If $\alpha(t)=0$, $\beta(t)=t$ for $t\in[0,T]$ and $\varphi_0=\delta_0=\varphi_s$, the results of this paper reduce to those on the classical Wiener spaces. We note that most of literatures related to this topic on the classical Wiener space use similar results of this paper without exact proofs.

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