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THE DRAZIN INVERSE OF THE SUM OF TWO PRODUCTS

Safae Alaoui Chrifi and Abdelaziz Tajmouati

Abstract. In this paper, for bounded linear operators A, B, C satisfying $[AB, B] = [BC, B] = [AB, BC] = 0$ we study the Drazin invertibility of the sum of products formed by the three operators A, B and C. In particular, we give an explicit representation of the anti-commutator $\{A, B\} = AB + BA$. Also we give some conditions for which the sum $A + C$ is Drazin invertible.

1. Introduction and preliminaries

Let X and Y be complex Banach spaces. We will denote by $\mathcal{B}(X, Y)$ the Banach space of all bounded linear operators and for simplicity, we write $\mathcal{B}(X)$ rather than $\mathcal{B}(X, X)$ when $X = Y$. The range and the kernel of $T \in \mathcal{B}(X)$ will be denoted by $R(T)$ and $N(T)$, respectively. For $T, S \in \mathcal{B}(X)$ we recall that $[T, S] = TS - ST$ is the commutator of T and S while $\{T, S\} = TS + ST$ is their anti-commutator.

An operator $T \in \mathcal{B}(X)$ is called Drazin invertible if there exists a unique operator $T^D \in \mathcal{B}(X)$ that satisfies

$$TT^D = T^D T, \qquad T^D = T^D T T^D, \qquad \text{and} \qquad T^{k+1} T^D = T^k.$$

The smallest such integer k is called the Drazin index of T and will be denoted by $\operatorname{ind}(T) = k$. It is very common that a bounded linear operator $T \in \mathcal{B}(X)$ is Drazin invertible if and only if 0 is a pole of its resolvent. Accordingly, the spectral idempotent T^π of T corresponding to $\{0\}$ is given by $T^\pi = I - TT^D$ and the Banach space X obeys the decomposition $X = N(T^\pi) \oplus R(T^\pi)$ in which $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent. We should also emphasize that idempotent, as well as nilpotent operators are Drazin invertible, i.e., if $T, S \in \mathcal{B}(X)$ such that $T^2 = T$ and $S^n = 0$, then $T^D = T$ and $S^D = 0$.

Drazin inverses are one of the most important and fruitful kinds of generalized inverses, the work on them gives rise to many problems that constitute a growing body of research, and opens up an entire world of applications (to

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learn more about the usefulness of these inverses we recommend [1, 4] and the references therein).

In particular, much research has been done in attempts to prove the Drazin invertibility of the product and the sum of two operators. Precisely (see [6, 8] and [4]), if T and S are two commuting Drazin invertible operators, then TS is also Drazin invertible thus we have $(TS)^D = T^D S^D = S^D T^D$. If in addition $TS = ST = 0$, then $T + S$ is also Drazin invertible with a Drazin inverse given by $(T + S)^D = T^D + S^D$. For the latter case, it is usually very difficult and sometimes impossible to find an explicit formula for the Drazin inverse of $T+S$ in terms of T^D, S^D without additional restrictive conditions. Hence, finding weaker conditions for which the sum of two Drazin invertible operators is also Drazin invertible has become a worth pursuing issue (cf. [2, 4, 7, 10, 11, 13, 14]).

Among many results, let $A, B, C \in \mathcal{B}(X)$. For the product, X. Wang et al. [12] presented some equivalent conditions concerning the reverse order law. $(AB)^D = B^D A^D$ for Drazin invertible operators A and B under commutative properties $[B, AB] = 0$ or $[A, AB] = 0$ or $[A, ABB^D] = 0$ or $[B, AA^D B] = 0$. In [3], the reverse order law problem for Drazin inverses of the triple product ABC was studied and several equivalent conditions have been given in which the equality $(ABC)^D = C^D B^D A^D$ is verified. The commutative conditions $[B, AB] = 0$, $[B, BC] = 0$ and $[AB, BC] = 0$ played a key role in this problem. For the sum, an explicit formula for $(A + B)^D$ in terms of A, B, A^D and B^D was provided in [5] and [10] and only one of the conditions $AB = 0$ or $AB = BA$ was considered.

In this paper, we will deal with a new structure, which is the sum of products made up of three operators. Precisely, we will give some equivalent conditions for which the sum of products $A \star B + B \star C$ is Drazin invertible and a representation of $(A \star B + B \star C)^D$ is also given, where $A\star B+B\star C \in \{AB+BC, BA+BC, AB+CB, BA+CB, B^D A+B^D C, AB^D + CB^D, AB^D + B^D C, B^D A + CB^D\}$. Throughout this work, we proceed in the same way as in [3, 12] by conserving the same commutative relations. Particularly, we will give an explicit formula of the Drazin inverse of the anticommutator A, B (resp., B, C) when $[B, AB] = 0$ (resp., $[B, BC] = 0$). Finally, under some further conditions the expression of $(A + C)^D$ is also given.

2. Key lemmas

Before going any step further, we recall some useful lemmas which paved the way for the resolution of our main results.

Lemma 2.1 ([12])**.** *Let $A, C, N \in \mathcal{B}(X)$ such that N is nilpotent of index n.*

(1) *If $[N, AN] = 0$, then AN and NA are nilpotent with*

$$\max\{\mathrm{ind}(NA), \mathrm{ind}(AN)\} \leq n.$$

(2) *If $[N, NC] = 0$, then NC and CN are nilpotent with*

$$\max\{\mathrm{ind}(NC), \mathrm{ind}(CN)\} \leq n.$$

Lemma 2.2 ([9]). *Let $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, $C_1 \in \mathcal{B}(Y, X)$ and $C_2 \in \mathcal{B}(X, Y)$. We denote by $M_{C_1} \in \mathcal{B}(X \oplus Y)$ and $M_{C_2} \in \mathcal{B}(Y \oplus X)$ the operator matrices represented as*

$$M_{C_1} = \begin{pmatrix} A & C_1 \\ 0 & B \end{pmatrix}, \quad M_{C_2} = \begin{pmatrix} B & 0 \\ C_2 & A \end{pmatrix}.$$

(1) *If two of M_{C_i}, A and B where $i = 1$ or 2 is Drazin invertible, then the third is also Drazin invertible.*
(2) *If A and B are Drazin invertible of index m and n, respectively, then*

$$M_{C_1}^D = \begin{pmatrix} A^D & X_1 \\ 0 & B^D \end{pmatrix}, \quad M_{C_2}^D = \begin{pmatrix} B^D & 0 \\ X_2 & A^D \end{pmatrix},$$

where

$$X_i = (A^D)^2 [\sum_{k=0}^{m-1} (A^D)^k C_i B^k] B^\pi + A^\pi [\sum_{k=0}^{n-1} A^k C_i (B^D)^k] (B^D)^2 - A^D C_i B^D, \ i \in \{1, 2\}.$$

Lemma 2.3 ([5]). *Let $A, C \in \mathcal{B}(X)$ be Drazin invertible with $\operatorname{ind}(A) = s$ and $\operatorname{ind}(C) = r$. If $AC = CA$, then $A+C$ is Drazin invertible if and only if $I+A^D C$ is Drazin invertible. In this case we have*

$$(A + C)^D = A^D (I + A^D C)^D C C^D + (I - CC^D)[\sum_{k=0}^{r-1} (-C)^k (A^D)^k] A^D + C^D [\sum_{k=0}^{s-1} (C^D)^k (-A)^k](I - AA^D).$$

Lemma 2.4 ([6]). *Let $P, Q \in \mathcal{B}(X)$ be Drazin invertible. If $PQ = QP = 0$, then $P + Q$ is also Drazin invertible and $(P + Q)^D = P^D + Q^D$.*

3. Preparations

In this section, we briefly sketch the forms of operator matrices employed throughout the paper. At first, let recall that every bounded linear operator acting on the Banach space $X \oplus Y$ has the following matrix form

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \tag{3.1}$$

where $T_1 \in \mathcal{B}(X)$, $T_2 \in \mathcal{B}(Y, X)$, $T_3 \in \mathcal{B}(X, Y)$ and $T_4 \in \mathcal{B}(Y)$. It is worth pointing here that under some commutative relations with other operators, the shape of the operator matrix (3.1) is greatly altered.

Now, let $A, B, C \in \mathcal{B}(X)$. By requiring B to be Drazin invertible with $\operatorname{ind}(B) = n$, the Banach space X can be written as

$$X = N(B^\pi) \oplus R(B^\pi). \tag{3.2}$$

In this case, $B = B_1 \oplus N_1$, $B^D = B_1^{-1} \oplus 0$ and $B^n = B_1^n \oplus 0$, where $B_1 = B_{/N(B^\pi)}$ is invertible and $N_1 = B_{/R(B^\pi)}$ is nilpotent.

On the other hand, A and C can be expressed according to the Banach space decomposition (3.2) as follows:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \ C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}.$$

We thus obtain, by a simple calculation as in [3] and [12], that in the case $[B, AB] = 0$ we have $[B^n, AB] = 0$. Hence,

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}, \ AB = \begin{pmatrix} A_1 B_1 & 0 \\ 0 & A_4 N_1 \end{pmatrix}, \ \text{and } BA = \begin{pmatrix} B_1 A_1 & B_1 A_2 \\ 0 & N_1 A_4 \end{pmatrix} \tag{3.3}$$

with,

$$[A_1, B_1] = 0, \ [N_1, A_4 N_1] = 0 \ \text{ and } \ A_2 N_1 = 0. \tag{3.4}$$

Similarly, in the case $[B, BC] = 0$, we get the following matrix form

$$C = \begin{pmatrix} C_1 & 0 \\ C_3 & C_4 \end{pmatrix}, \ BC = \begin{pmatrix} B_1 C_1 & 0 \\ 0 & N_1 C_4 \end{pmatrix}, \ \text{and } CB = \begin{pmatrix} C_1 B_1 & 0 \\ C_3 B_1 & C_4 N_1 \end{pmatrix}, \tag{3.5}$$

where

$$[B_1, C_1] = 0, \ [N_1, N_1 C_4] = 0 \ \text{ and } \ N_1 C_3 = 0. \tag{3.6}$$

At last, when $[AB, BC] = 0$ we obtain that

$$[A_1, C_1] = 0 \ \text{ and } \ [A_4 N_1, N_1 C_4] = 0. \tag{3.7}$$

From all of these conditions, we gain the commutativity of the set $\{A_1, B_1, C_1\}$. Other than that, Lemma 2.1 shows that $A_4 N_1$, $N_1 A_4$, $N_1 C_4$ as well as $C_4 N_1$ are all nilpotent.

Among other results, the proof of [12, Theorem 3.1] shows that, AB (resp., BC) is Drazin invertible if and only if A_1 (resp., C_1) is Drazin invertible. Thus

$$(AB)^D = \begin{pmatrix} A_1^D B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \ (BC)^D = \begin{pmatrix} B_1^{-1} C_1^D & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.8}$$

Furthermore, if A and C are Drazin invertible, then by virtue of Lemma 2.2 we can see that A_4 and C_4 are also Drazin invertible (since A, A_1, C and C_1 are all Drazin invertible). Hence

$$A^D = \begin{pmatrix} A_1^D & X \\ 0 & A_4^D \end{pmatrix}, \ C^D = \begin{pmatrix} C_1^D & 0 \\ Y & C_4^D \end{pmatrix}, \tag{3.9}$$

where

$$X = (A_1^D)^2 [\sum_{n=0}^{t_1 - 1} (A_1^D)^n A_2 A_4^n] A_4^\pi + A_1^\pi [\sum_{n=0}^{s_1 - 1} A_1^n A_2 (A_4^D)^n] (A_4^D)^2 - A_1^D A_2 A_4^D, \tag{3.10}$$

$$Y = (C_4^D)^2 [\sum_{n=0}^{s_2 - 1} (C_4^D)^n C_3 C_1^n] C_1^\pi + C_4^\pi [\sum_{n=0}^{t_2 - 1} C_4^n C_3 (C_1^D)^n] (C_1^D)^2 - C_4^D C_3 C_1^D. \tag{3.11}$$

4. Main results

We begin our main results by studying the Drazin invertibility of the sum of products made up of three operators A, B, C under the assumptions $[B, AB] = 0$, $[B, BC] = 0$ and $[AB, BC] = 0$.

Theorem 4.1. *Let $A, B, C \in \mathcal{B}(X)$ be such that B, AB, BC are Drazin invertible with $\operatorname{ind}(B) = n$ and $[B, AB] = [B, BC] = [AB, BC] = 0$. Denote by*

$$\begin{aligned}\mathcal{A} = \{&AB + BC, I + (AB)^D BC, AB + CB, BA + BC, AB^D + B^D C,\\ &AB^D + CB^D, B^D A + B^D C\}.\end{aligned}$$

If one of the elements of $\mathcal{A}$ is Drazin invertible, then all the element of $\mathcal{A}$ are Drazin invertible. In this case

$$\begin{aligned}(AB + CB)^D &= (AB + BC)^D + R,\\ (BA + BC)^D &= (AB + BC)^D + S,\end{aligned}$$

where

$$\begin{aligned}(AB + BC)^D = B^D[&(AB)^D B(I + (AB)^D BC)^D B(BC)^D C\\ &+ (I + (BC)^D BC)B(\sum_{i=0}^{n-1}((AB)^D)^{i+1}(-BC)^i)\\ &+ (\sum_{i=0}^{n-1}((BC)^D)^{i+1}(-AB)^i)B(I + AB(AB)^D)],\\ R = &\sum_{k=0}^{2n-1}(AB + CB)^k B^\pi CB((AB + BC)^D)^{k+2},\\ S = &\sum_{k=0}^{2n-1}((AB + BC)^D)^{k+2} BAB^\pi (BA + BC)^k.\end{aligned}$$

Proof. If B is Drazin invertible and $[B, AB] = [B, BC] = 0$, then from (3.4) and (3.6) and Lemma 2.1 we have: A_4N_1, N_1A_4, N_1C_4 and C_4N_1 are all nilpotent and

$$\begin{cases}\max\{\operatorname{ind}(A_4N_1), \operatorname{ind}(N_1A_4)\} \le n,\\ \max\{\operatorname{ind}(N_1C_4), \operatorname{ind}(C_4N_1)\} \le n.\end{cases}$$

Therefore, we can clearly state without loss of generality that $A_4N_1 + N_1C_4$, $A_4N_1 + C_4N_1$, $N_1A_4 + N_1C_4$ are all nilpotent of index $2n$.

Now from (3.3) and (3.5) we can write that

$$\begin{aligned}AB + BC &= \begin{pmatrix} A_1B_1 + B_1C_1 & 0\\ 0 & A_4N_1 + N_1C_4\end{pmatrix},\\ AB + CB &= \begin{pmatrix} A_1B_1 + C_1B_1 & 0\\ C_3B_1 & A_4N_1 + C_4N_1\end{pmatrix},\end{aligned}$$

$$BA+BC=\begin{pmatrix} B_1A_1+B_1C_1 & B_1A_2 \\ 0 & N_1A_4+N_1C_4 \end{pmatrix}.$$

On the Banach space decomposition (3.2). Also, we have from (3.4), (3.6) the commutativity of B_1 with A_1 and C_1. So it follows that $[(A_1+C_1)B_1, B_1^{-1}]=0$ and $[(A_1+C_1)B_1^{-1}, B_1]=0$. Consequently, the Drazin invertibility of one of the elements of $\mathcal{A}$ lies in the Drazin invertibility of A_1+C_1. Additionally, A_1, C_1 are Drazin invertible and $[A_1, C_1]=0$. We thus get

$$\begin{aligned} &\text{An element of } \mathcal{A} \text{ is Drazin invertible} \\ \iff\ & A_1+C_1 \text{ is Drazin invertible} \\ \iff\ & I+A_1^DC_1 \text{ is Drazin invertible (see Lemma 2.3).} \end{aligned}$$

As a result,

$$\begin{aligned} (AB+BC)^D &= B_1^{-1}(A_1+C_1)^D \oplus 0 = (A_1+C_1)^D B_1^{-1} \oplus 0 \\ &= (B_1^{-1}\oplus 0)((A_1+C_1)^D \oplus 0), \\ (BA+BC)^D &= \begin{pmatrix} B_1^{-1}(A_1+C_1)^D & X \\ 0 & 0 \end{pmatrix} = (AB+BC)^D + \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \\ (AB+CB)^D &= \begin{pmatrix} B_1^{-1}(A_1+C_1)^D & 0 \\ Y & 0 \end{pmatrix} = (AB+BC)^D + \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}. \end{aligned}$$

By Lemma 2.2 and Lemma 2.3

$$\begin{aligned} (A_1+C_1)^D &= A_1^D(I+A_1^DC_1)^DC_1C_1^D + (I+C_1C_1^D)[\sum_{i=0}^{n-1}(A_1^D)^{i+1}(-C_1)^i] \\ &\quad + [\sum_{i=0}^{n-1}(C_1^D)^{i+1}(-A_1)^i](I+A_1A_1^D), \\ X &= \sum_{k=0}^{2n-1}((A_1B_1+B_1C_1)^D)^{k+2}B_1A_2(N_1A_4+N_1C_4)^k, \\ Y &= \sum_{k=0}^{2n-1}(A_4N_1+C_4N_1)^kC_3B_1((A_1B_1+C_1B_1)^D)^{k+2}. \end{aligned}$$

Finally, it is not difficult to prove that:

$$A_1^D(I+A_1^DC_1)^DC_1C_1^D \oplus 0 = (AB)^DB(I+(AB)^DBC)^DB(BC)^DC,$$

$$(I+C_1C_1^D)[\sum_{i=0}^{n-1}(A_1^D)^{i+1}(-C_1)^i] \oplus 0 = (I+(BC)^DBC)B(\sum_{i=0}^{n-1}((AB)^D)^{i+1}(-BC)^i),$$

$$[\sum_{i=0}^{n-1}(C_1^D)^{i+1}(-A_1)^i](I+A_1A_1^D) \oplus 0 = (\sum_{i=0}^{n-1}((BC)^D)^{i+1}(-AB)^i)B(I+AB(AB)^D),$$

and

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = \sum_{k=0}^{2n-1} ((AB+BC)^D)^{k+2} BAB^\pi (BA+BC)^k,$$

$$\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} = \sum_{k=0}^{2n-1} (AB+CB)^k B^\pi CB((AB+BC)^D)^{k+2}.$$

Gathering all together and denoting by $S = \left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 0 & 0 \\ Y & 0 \end{smallmatrix}\right)$ we arrive at the desired formulas. □

The following two propositions provide the Drazin inverse for other examples of the structure studied in Theorem 4.1.

Proposition 4.1. *Under the same hypothesis of Theorem 4.1, if one of the elements of $\mathcal{A}$ is Drazin invertible and $B^D(A+C)B^\pi CB^D = 0$, then $B^DA + CB^D$ is also Drazin invertible, in this case we have*

$$(B^DA + CB^D)^D = (B^DA + B^DC)^D + B^\pi CB^D((B^DA + B^DC)^D)^2.$$

Proof. By (3.3) and (3.4) we have

$$B^DA + CB^D = \begin{pmatrix} B_1^{-1}(A_1 + C_1) & B_1^{-1}A_2 \\ C_3B_1^{-1} & 0 \end{pmatrix} = B^D(A+C) + B^\pi CB^D,$$

where

$$B^D(A+C) = \begin{pmatrix} B_1^{-1}(A_1 + C_1) & B_1^{-1}A_2 \\ 0 & 0 \end{pmatrix} \quad \text{and}$$

$$B^\pi CB^D = \begin{pmatrix} 0 & 0 \\ C_3B_1^{-1} & 0 \end{pmatrix}. \tag{4.1}$$

Since $A_1 + C_1$ is Drazin invertible, then $B^D(A+C)$ is also Drazin invertible. Also, we have $(B^\pi CB^D)^2 = 0$. Hence, according to [4, Corollary 6.1] $B^DA + CB^D$ is also Drazin invertible with a Drazin inverse given by

$$(B^DA + CB^D)^D = (B^DA + B^DC)^D + B^\pi CB^D((B^DA + B^DC)^D)^2.$$

Note that,

$$(B^DA + B^DC)^D = \begin{pmatrix} B_1(A_1 + C_1)^D & X \\ 0 & 0 \end{pmatrix},$$

where $X = B_1^2((A_1 + C_1)^D)^2 B_1^{-1}A_2 = B_1((A_1 + C_1)^D)^2 A_2$. That is,

$$(B^DA + B^DC)^D = \begin{pmatrix} B_1(A_1 + C_1)^D & B_1((A_1 + C_1)^D)^2A_2 \\ 0 & 0 \end{pmatrix}.$$

Now by simple calculation we obtain

$$((B^DA + B^DC)^D)^2 = \begin{pmatrix} B_1^2((A_1 + C_1)^D)^2 & B_1^2((A_1 + C_1)^D)^3A_2 \\ 0 & 0 \end{pmatrix}$$

and

$$B^{\pi}CB^{D}((B^{D}A+B^{D}C)^{D})^2=\begin{pmatrix} 0 & 0 \\ C_3B_1((A_1+C_1)^D)^2 & C_3B_1((A_1+C_1)^D)^3A_2 \end{pmatrix}.$$

Consequently,

$$(B^DA+CB^D)^D=\begin{pmatrix} B_1(A_1+C_1)^D & B_1((A_1+C_1)^D)^2A_2 \\ C_3B_1((A_1+C_1)^D)^2 & C_3B_1((A_1+C_1)^D)^3A_2 \end{pmatrix}. \qquad \square$$

Proposition 4.2. *Under the same hypothesis of Theorem 4.1 if one element of $\mathcal{A}$ is Drazin invertible and $BB^DAB^{\pi}CB=0$, then $BA+CB$ is also Drazin invertible.*

Proof. From (3.3) and (3.4), it follows that

$$BA+CB=\begin{pmatrix} B_1(A_1+C_1) & B_1A_2 \\ C_3B_1 & N_1A_4+C_4N_1 \end{pmatrix}=P+Q$$

with

$$P=\begin{pmatrix} B_1(A_1+C_1) & B_1A_2 \\ 0 & 0 \end{pmatrix} \text{ and } Q=\begin{pmatrix} 0 & 0 \\ C_3B_1 & N_1A_4+C_4N_1 \end{pmatrix}.$$

Observing that P is Drazin invertible with Drazin inverse given by

$$P^D=\begin{pmatrix} B_1^{-1}(A_1+C_1)^D & B_1^{-1}((A_1+C_1)^D)^2A_2 \\ 0 & 0 \end{pmatrix}$$

and Q is nilpotent of index $\operatorname{ind}(N_1A_4+C_4N_1)+1$, so for the seek of simplicity we take $\operatorname{ind}(N_1A_4+C_4N_1)+1=2n+1$.

The assumption $BB^DAB^{\pi}CB=0$ implies that $A_2C_3=0$ and $A_2C_4N_1=0$, hence $PQ=0$. Then [4, Corollary 6.1] implies that $BA+CB$ is Drazin invertible with a Drazin inverse given by

$$(BA+CB)^D=(P+Q)^D=P^D+Q(P^D)^2+\cdots+Q^{2n}(P^D)^{2n+1}.$$

After a calculation we obtain

$$(P^D)^k=\begin{pmatrix} B_1^{-k}((A_1+C_1)^D)^k & B_1^{-k}((A_1+C_1)^D)^{k+1}A_2 \\ 0 & 0 \end{pmatrix},$$

$$Q^k=\begin{pmatrix} 0 & 0 \\ (N_1A_4+C_4N_1)^{k-1}C_3B_1 & (N_1A_4+C_4N_1)^k \end{pmatrix},$$

and

$$(BA+CB)^D=\begin{pmatrix} B_1^{-1}(A_1+C_1)^D & B_1((A_1+C_1)^D)^2A_2 \\ Y & Z \end{pmatrix},$$

where $Y=C_3B_1^{-1}((A_1+C_1)^D)^2+\cdots+(N_1A_4+C_4N_1)^{2n-1}C_3B_1^{-2n}((A_1+C_1)^D)^{2n+1}$ and $Z=C_3B_1^{-1}((A_1+C_1)^D)^3A_2+\cdots+(N_1A_4+C_4N_1)^{2n-1}C_3B_1^{-2n}((A_1+C_1)^D)^{2n+2}A_2$. $\square$

Some special cases are given in the following result concerning the Drazin invertibility of $AB+BC$.

Corollary 4.1. *Let $A, B, C \in \mathcal{B}(\mathcal{X})$ be such that B, AB, BC are Drazin invertible, and assume that $[AB, B] = [BC, B] = [AB, BC] = 0$ and $I + (AB)^D BC$ is Drazin invertible.*

(1) *If $ABB^D C = 0$, then $(AB + BC)^D = (AB)^D + (BC)^D$.*
(2) *If ABB^D and $B^D BC$ are nilpotent, then $(AB + BC)^D = 0$.*
(3) *If $(B^D BC)^n = 0$, then $(AB + BC)^D = \sum_{k=0}^{n-1} ((AB)^D)^{k+1} (-BC)^k$.*
(4) *If $(ABB^D)^n = 0$, then $(AB + BC)^D = \sum_{k=0}^{n-1} ((BC)^D)^{k+1} (-AB)^k$.*
(5) *If $(ABB^D)^2 = ABB^D$ and $(B^D BC)^2 = B^D BC$, then*

$$(AB + BC)^D = AB^D + B^D C - \frac{3}{2} AB^D C.$$

(6) *If $(B^D BC)^2 = B^D BC$, then*

$$\begin{aligned}(AB + BC)^D = {} & (AB)^D (I + (AB)^D BC)^D B^D BC + (I - B^D BC)(AB)^D \\ & + B^D C (I + ABB^D)^D (I - AB(AB)^D).\end{aligned}$$

(7) *If $(ABB^D)^2 = ABB^D$, then*

$$\begin{aligned}(AB + BC)^D = {} & AB^D (I + (AB)^D BC)^D BC(BC)^D + (BC)^D (I - ABB^D) \\ & + (I - BC(BC)^D)(I + BB^D C)^D AB^D.\end{aligned}$$

If we further assume in Theorem 4.1 that A and C are also Drazin invertible, then we have the following theorem.

Theorem 4.2. *Let $A, B, C \in \mathcal{B}(X)$ be such that A, B, C, AB, BC are Drazin invertible, and assume that* $\operatorname{ind}(AB) = r$, $\operatorname{ind}(BC) = s$ *and* $[B, AB] = [B, BC] = [AB, BC] = 0$. *Then $AB + BC$ is Drazin invertible if and only if $I + A^D BB^D C$ is Drazin invertible, in this case we have*

$$\begin{aligned}(AB + BC)^D = {} & B^D [A^D BB^D (I + A^D BB^D C)^D CBB^D C^D \\ & + (I + B^D CBC^D)(\sum_{i=0}^{r-1} (A^D BB^D)^{i+1} (-B^D BC)^i) \\ & + (\sum_{i=0}^{s-1} (B^D BC^D)^{i+1} (-A^D BB^D)^i)(I + A^D BAB^D)].\end{aligned}$$

Proof. Since, A, B, C, AB, BC are all Drazin invertible and $[B, AB] = [B, BC] = [AB, BC] = 0$, from (3.3), (3.5) and (3.9), we can easily check that

$$B^D BC = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad ABB^D = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B^D BC^D = \begin{pmatrix} C_1^D & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad A^D BB^D = \begin{pmatrix} A_1^D & 0 \\ 0 & 0 \end{pmatrix}.$$

By using the same argumentation as in Theorem 4.1 we obtain that:

$AB + BC$ is Drazin invertible $\Longleftrightarrow$ $A_1 + C_1$ is Drazin invertible

$$\iff I + A_1^D C_1 \text{ is Drazin invertible}$$
$$\iff I + A^D BB^D C \text{ is Drazin invertible.}$$

A straightforward computation now leads to

$$A_1^D (I + A_1^D C_1)^D C_1 C_1^D \oplus 0 = A^D BB^D (I + A^D BB^D C)^D CBB^D C^D,$$

$$(I + C_1 C_1^D)[\sum_{i=0}^{r-1} (A_1^D)^{i+1}(-C_1)^i] \oplus 0 = (I + B^D CBC^D)(\sum_{i=0}^{r-1} (A^D BB^D)^{i+1}(-B^D BC)^i),$$

$$[\sum_{i=0}^{s-1} (C_1^D)^{i+1}(-A_1)^i](I + A_1 A_1^D) \oplus 0 = (\sum_{i=0}^{s-1} (B^D BC^D)^{i+1}(-A^D BB^D)^i)(I + A^D BAB^D)$$

as requested. □

In the next theorem, we take B to be an idempotent.

Theorem 4.3. *Let $A, B, C \in \mathcal{B}(X)$ be such that B is an idempotent, AB and BC are Drazin invertible. Assume that $BC(I - B)AB = 0$ and $BABCB = BCBAB$. Then $AB + BC$ is Drazin invertible if and only if $I + (BAB)^D BCB$ is Drazin invertible.*

Proof. By requiring B to be an idempotent we have

$$B = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \ A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \ \text{ and } \ C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

with respect to the Banach space decomposition $X = N(B) \oplus R(B)$. Hence we can write

$$AB + BC = \begin{pmatrix} A_1 + C_1 & C_2 \\ A_3 & 0 \end{pmatrix} = \begin{pmatrix} A_1 + C_1 & C_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ A_3 & 0 \end{pmatrix} = P + Q,$$

where $Q^2 = 0$ and $PQ = 0$ since $BC(I - B)AB = 0$ which is equivalent to say that $C_2 A_3 = 0$.

On the other side, AB and BC are Drazin invertible. It follows that A_1 and C_1 are Drazin invertible. By $BABCB = BCBAB$ one can show that $[A_1, C_1] = 0$. Now applying Lemma 2.3 and Lemma 2.2: P is Drazin invertible if and only if $A_1 + C_1$ is Drazin invertible if and only if $I + A_1^D C_1$ is Drazin invertible if and only if $I + (BAB)^D BCB$ is Drazin invertible, in this case

$$P^D = \begin{pmatrix} (A_1 + C_1)^D & ((A_1 + C_1)^D)^2 C_2 \\ 0 & 0 \end{pmatrix}.$$

Also, from [4, Corollory 6.1] we drive that

$$\begin{aligned} (AB + BC)^D &= P^D + Q(P^D)^2 \\ &= \begin{pmatrix} (A_1 + C_1)^D & ((A_1 + C_1)^D)^2 C_2 \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ A_3((A_1 + C_1)^D)^2 & A_3((A_1 + C_1)^D)^3 C_2 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} (A_1+C_1)^D & ((A_1+C_1)^D)^2C_2 \\ A_3((A_1+C_1)^D)^2 & A_3((A_1+C_1)^D)^3C_2 \end{pmatrix}. \qquad \square$$

If we take A instead of C in Theorem 4.1 we obtain the Drazin invertibility of the anti-commutator $\{A, B\}$.

Theorem 4.4. *Let $A, B \in \mathcal{B}(X)$ be such that B and AB are Drazin invertible, and assume that $\operatorname{ind}(B) = n$ and $[AB, B] = 0$. Then, $\{A, B\}$ is Drazin invertible with an inverse given by*

$$\{A,B\}^D = \frac{1}{2}(AB)^D + \sum_{k=0}^{2n-1} (\frac{1}{2}(AB)^D)^{k+2} BAB^{\pi}(AB+BA)^k.$$

If in addition A is Drazin invertible, then

$$\{A,B\}^D = \frac{1}{2}A^DB^D + \sum_{k=0}^{2n-1} (\frac{1}{2}A^DB^D)^{k+2} BAB^{\pi}(AB+BA)^k.$$

Proof. Since B is Drazin invertible and $[AB, B] = 0$, by (3.3) and (3.4) we can write

$$\{A,B\} = AB + BA = \begin{pmatrix} 2A_1B_1 & B_1A_2 \\ 0 & A_4N_1 + N_1A_4 \end{pmatrix}$$

with respect to (3.2). Also, from (3.4) we have $[N_1, A_4N_1] = 0$. Then, Lemma 2.1 implies that A_4N_1 and N_1A_4 are nilpotent with

$$\max\{\operatorname{ind}(N_1A_4), \operatorname{ind}(A_4N_1)\} \le n.$$

As a result, $A_4N_1 + N_1A_4$ is also nilpotent and $\operatorname{ind}(A_4N_1 + N_1A_4) \le 2n$. Therefore, according to Lemma 2.2 we conclude that

$$\{A,B\}^D = \begin{pmatrix} \frac{1}{2}A_1^DB_1^{-1} & X \\ 0 & 0 \end{pmatrix},$$

where

$$X = \sum_{k=0}^{2n-1} (\frac{1}{2}A_1^DB_1^{-1})^{k+2} B_1A_2(A_4N_1 + N_1A_4)^k.$$

Finally, a trivial verification shows that

$$\{A,B\}^D = \frac{1}{2}(AB)^D + M, \tag{4.2}$$

with

$$M = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = \sum_{k=0}^{2n-1} (\frac{1}{2}(AB)^D)^{k+2} BAB^{\pi}(AB+BA)^k.$$

When A is Drazin invertible, we have

$$A^DB^D = (AB)^D = \begin{pmatrix} A_1^DB_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.3}$$

By substituting (4.3) in (4.2) we obtain the desired conclusion. $\square$

With the same spirit as in the previous theorem we obtain the following result.

Corollary 4.2. *Let $A, C \in \mathcal{B}(X)$ be such that B, BC are Drazin invertible, and assume that* $\operatorname{ind}(B) = n$ *and* $[BC, B] = 0$. *Then,* $\{B, C\}$ *is Drazin invertible with an inverse given by*

$$\{B,C\}^D = \frac{1}{2}(BC)^D + \sum_{k=0}^{2n-1}(BC+CB)^k B^\pi CB(\frac{1}{2}(BC)^D)^{k+2}.$$

If in addition C is Drazin invertible, then

$$\{B,C\}^D = \frac{1}{2}B^D C^D + \sum_{k=0}^{2n-1}(BC+CB)^k B^\pi CB(\frac{1}{2}(B^D C^D)^{k+2}.$$

In the following result, we give the Drazin inverse of the block operator matrix $A + C$.

Theorem 4.5. *Let $A, B, C \in \mathcal{B}(X)$ be such that A, B, C, AB, BC are Drazin invertible with* $\operatorname{ind}(AB) = s$ *and* $\operatorname{ind}(BC) = r$. *If* $[AB, B] = [BC, B] = [AB, BC] = [B^\pi A, CB^\pi] = [(A+C), BB^D(A+C)] = 0$. *Then $A+C$ is Drazin invertible if and only if $AB + BC$ and $B^\pi A + CB^\pi$ are Drazin invertible. In this case*

$$\begin{aligned}(A+C)^D = &(AB+BC)^D B + ((AB+BC)^D B)^2 AB^\pi + (B^\pi A + CB^\pi)^D \\ &+ ((B^\pi A + CB^\pi)^D)^2 CB^D B.\end{aligned}$$

Proof. Since B is Drazin invertible and $[B, AB] = [B, BC] = 0$, we have

$$A + C = \begin{pmatrix} A_1 + C_1 & A_2 \\ C_3 & A_4 + C_4 \end{pmatrix},$$

on the Banach space decomposition (3.2). In the light of the commutative assumption $[(A+C), BB^D(A+C)] = 0$, we get:

$$\begin{pmatrix} A_1 + C_1 & A_2 \\ C_3 & A_4 + C_4 \end{pmatrix}\begin{pmatrix} A_1 + C_1 & A_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1 + C_1 & A_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_1 + C_1 & A_2 \\ C_3 & A_4 + C_4 \end{pmatrix}$$

thus,

$$\begin{pmatrix} (A_1+C_1)^2 & (A_1+C_1)A_2 \\ C_3(A_1+C_1) & C_3A_2 \end{pmatrix} = \begin{pmatrix} (A_1+C_1)^2 + A_2C_3 & (A_1+C_1)A_2 + A_2(A_4+C_4) \\ 0 & 0 \end{pmatrix}.$$

The comparison of the last equality yields

$$A_1C_3 = 0,\ A_2(A_4+C_4) = 0,\ C_3(A_1+C_1) = 0,\ C_3A_2 = 0. \tag{4.4}$$

Subsequently,

$$A + C = \begin{pmatrix} A_1 + C_1 & A_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ C_3 & A_4 + C_4 \end{pmatrix} = P + Q.$$

Then, by means of (4.4) one can see that $PQ = QP = 0$.

Now since AB, A (resp., BC, C) are Drazin invertible, A_1, A_4 (resp., C_1, C_4) are also Drazin invertible. Using the conditions $[B^\pi A, CB^\pi] = [AB, BC] = 0$, we find that $[A_1, C_1] = [A_4, C_4] = 0$. Thus, it follows by Lemma 2.2, Lemma 2.3 and Lemma 2.4 that P, Q are Drazin invertible and

$$(A + C)^D = P^D + Q^D,$$

where

$$P^D = \begin{pmatrix} (A_1 + C_1)^D & ((A_1 + C_1)^D)^2 A_2 \\ 0 & 0 \end{pmatrix},$$

$$Q^D = \begin{pmatrix} 0 & 0 \\ ((A_4 + C_4)^D)^2 C_3 & (A_4 + C_4)^D \end{pmatrix}.$$

In the end, we conclude by a quick computation that

$$(A_1 + C_1)^D \oplus 0 = B(AB + BC)^D = (AB + BC)^D B,$$

$$\begin{pmatrix} 0 & ((A_1 + C_1)^D)^2 A_2 \\ 0 & 0 \end{pmatrix} = (B(AB + BC)^D)^2 AB^\pi,$$

$$0 \oplus (A_4 + C_4)^D = (B^\pi A + CB^\pi)^D,$$

$$\begin{pmatrix} 0 & 0 \\ ((A_4 + C_4)^D)^2 C_3 & 0 \end{pmatrix} = ((B^\pi A + CB^\pi)^D)^2 CBB^D.$$

Which completes the proof. □

Concluding remarks

The problem of finding the generalized inverse of a product or a sum has attracted a lot of attention in recent years. This subject was treated many times in different ways by changing: the structure (e.g. product of two elements or more, sum, sum of product, commutator, anticommutator, ...), the kind of inverse (e.g. Moore-Penrose inverse, Drazin inverse, generalized Drazin inverse, ... or the setting (e.g. matrices, operators, elements in algebras or rings, ...). In this paper, we confined our attention to the study of the Drazin invertibility of the sum of products for bounded linear operators. Namely, the proof of Theorem 4.1 and Theorem 4.4 was based on the Drazin invertibility of triangular operator matrices while in Proposition 4.1, Proposition 4.2, Theorem 4.3 as well as Theorem 4.5 we used the formula of the Drazin inverse of the sum in order to deal with the Drazin inverse of block operator matrices.

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Safae Alaoui Chrifi
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar Al Mahraz
Laboratory of Mathematical Sciences and Applications (LASMA)
Fez, Morocco
Email address: safae.alaouichrifi@usmba.ac.ma

Abdelaziz Tajmouati
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar Al Mahraz
Laboratory of Mathematical Sciences and Applications (LASMA)
Fez, Morocco
Email address: abdelaziz.tajmouati@usmba.ac.ma