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MULTIPLE SOLUTIONS OF A PERTURBED YAMABE-TYPE EQUATION ON GRAPH

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Abstract. Let u be a function on a locally finite graph $G=(V,E)$ and Ω be a bounded subset of V. Let $\varepsilon>0$, $p>2$ and $0\leq\lambda<\lambda_1(\Omega)$ be constants, where $\lambda_1(\Omega)$ is the first eigenvalue of the discrete Laplacian, and $h:V\to\mathbb{R}$ be a function satisfying $h\geq 0$ and $h\not\equiv 0$. We consider a perturbed Yamabe equation, say

$$\begin{cases} -\Delta u-\lambda u=|u|^{p-2}u+\varepsilon h, & \text{in} \quad \Omega, \\ u=0, & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω and $\partial\Omega$ denote the interior and the boundary of Ω, respectively. Using variational methods, we prove that there exists some positive constant $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$, the above equation has two distinct solutions. Moreover, we consider a more general nonlinear equation

$$\begin{cases} -\Delta u=f(u)+\varepsilon h, & \text{in} \quad \Omega, \\ u=0, & \text{on} \quad \partial\Omega, \end{cases}$$

and prove similar result for certain nonlinear term $f(u)$.

1. Introduction

In a series of works [8–10], Grigor'yan, Lin and Yang solved several discrete differential equations, say the Yamabe equation, the Kazdan-Warner equation and the Schrödinger equation, by finding critical points for various functionals. Their main contribution is to establish a functional framework on graphs, through which a direct method of variation, the mountain-pass theorem, and the principle of upper-lower solutions are applied. Recently, the research in this field has received great interests. Inspired by [10], Zhang-Zhao [22] obtained nontrivial solutions to certain nonlinear Schrödinger equation on locally finite graphs. Similar problem on infinite metric graphs were discussed by Akduman-Pankov [2]. The Kazdan-Warner equation was generalized by Keller-Schwarz [15] to canonically compactifiable graphs, and by Ge-Jiang [6] to certain infinite

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graph. For other related works, we refer the reader to [7, 11, 12, 14, 17–19, 21] and the references therein.

In this paper, we concern multiplicity of solutions to a perturbed Yamabe equation on graphs. Similar topic was studied by Grigor'yan, Lin and Yang [10] on the Schrödinger equation, by Liu and Yang [16] on the Kazdan-Warner equation, and by Hou [13] on the bi-Laplacian equation. Earlier results on Euclidean space are referred to [1, 4, 5, 20]. To state our results, we recall some definitions on graphs. Let $G=(V, E)$ be a graph, where V denotes the vertex set and E denotes the edge set. Ω is said to be a domain if it is a connected subset of V. Throughout this paper, we always assume that G satisfies the following conditions (a)-(d) and Ω is a domain satisfying the condition (e).

(a) (Locally finite) For any $x \in V$, there exist only finite vertices $y \in V$ such that $xy \in E$.
(b) (Connected) For any $x, y \in V$, there exist finite edges connecting x and y.
(c) (Symmetric weight) For any $x, y \in V$, let $\omega : V \times V \to \mathbb{R}$ be a positive symmetric weight, i.e., $\omega_{xy} > 0$ and $\omega_{xy} = \omega_{yx}$.
(d) (Positive finite measure) $\mu : V \to \mathbb{R}^+$ defines a positive finite measure on a graph G.
(e) (Bounded domain) For any two vertices $x, y \in \Omega$, the distance $d(x, y)$ is uniformly bounded from above, where $d(x, y)$ is defined as the minimal number of edges which connect x and y.

For any function $u : V \to \mathbb{R}$, the μ-Laplacian of u at any vertex x is defined by

$$\triangle u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x)),$$

where $y \sim x$ means $xy \in E$. The associated gradient form of two functions u and v at any vertex x reads

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))(v(y) - v(x)).$$

For our convenience, we write $\Gamma(u)(x) = \Gamma(u, u)(x)$, and denote the length of the gradient

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))^2 \right)^{\frac{1}{2}}.$$

For any function $h : V \to \mathbb{R}$, the integral of h on a bounded domain Ω is denoted by

$$\int_\Omega h d\mu = \sum_{x \in \Omega} \mu(x) h(x).$$

We consider the following perturbed Yamabe equation

$$(1)\qquad \begin{cases} -\Delta u - \lambda u = |u|^{p-2}u + \varepsilon h, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

where $\partial\Omega = \{y \in V, y \notin \Omega : \exists x \in \Omega \text{ such that } xy \in E\}$ denotes the boundary of Ω, ε is a positive real number, $h : V \to \mathbb{R}$ is a function, and $\lambda_1(\Omega)$ is the first eigenvalue of the μ-Laplacian with respect to Dirichlet boundary condition, which is defined by

$$(2)\qquad \lambda_1(\Omega) = \inf_{u \not\equiv 0, u|_{\partial\Omega}=0} \frac{\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu}{\int_\Omega u^2 d\mu}.$$

According to [16], $L^p(\Omega)$ on graphs is defined by

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R},\ \|u\|_{L^p(\Omega)} < +\infty\},\ 1 \le p \le \infty,$$

where the norm of $u \in L^p(\Omega)$ is defined by

$$\|u\|_{L^p(\Omega)} = \begin{cases} (\sum_{x\in\Omega} \mu(x)|u(x)|^p)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max_{x\in\Omega} |u(x)|, & p = \infty. \end{cases}$$

Grigor'yan et al. [8] defined the Sobolev space $W_0^{1,2}(\Omega)$ and its norm on graphs by

$$(3)\qquad W_0^{1,2}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ \middle|\ u|_{\partial\Omega} = 0, \int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu < +\infty \right\}$$

and

$$(4)\qquad \|u\|_{W_0^{1,2}(\Omega)} = \left(\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu \right)^{\frac{1}{2}}.$$

In fact, $W_0^{1,2}(\Omega)$ is exactly a finite dimensional linear space since the bounded domain Ω only contains finite vertexes. Without loss of generality, we may assume $\Omega = \{x_1, x_2, \ldots, x_n\}$, then $W_0^{1,2}(\Omega) = \mathbb{R}^n$, and the norm (4) is equivalent to $(\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu + \int_\Omega u^2 d\mu)^{\frac{1}{2}}$, which is the traditional norm of Sobolev space $W_0^{1,2}(\Omega)$.

Obviously, $W_0^{1,2}(\Omega)$ is a Hilbert space with the inner product

$$\langle u,\ v\rangle = \int_{\Omega\cup\partial\Omega} \Gamma(u, v) d\mu, \quad \forall u,\ u \in W_0^{1,2}(\Omega).$$

Set $\mathcal{H}(\Omega)$ as the dual space of the $W_0^{1,2}(\Omega)$ with the norm

$$\|h\|_{\mathcal{H}(\Omega)} = \sup_{\|u\|_{W_0^{1,2}(\Omega)} \le 1} \left| \int_\Omega h u d\mu \right|, \quad \forall h \in \mathcal{H}(\Omega).$$

Because the Hilbert space is reflexive, by Riesz representation theorem, $W_0^{1,2}(\Omega)$ is equivalent to $\mathcal{H}(\Omega)$ in the sense of isomorphism. That is to say, ignoring their concrete contents, $W_0^{1,2}(\Omega)$ and $\mathcal{H}(\Omega)$ can be regarded as the same abstract space without distinction. What's interesting about the graph is that they are all Euclidean space $\mathbb{R}^n$. Now we are ready to state our first result.

Theorem 1.1. *Let $G = (V, E)$ be a graph satisfies conditions* (a)-(d)*, Ω be a connected domain satisfies* (e) *with $\Omega \neq \emptyset$, $\lambda_1(\Omega)$ is defined by* (2)*. Suppose that $h \in \mathcal{H}(\Omega)$ satisfies $0 \leq h(x) \not\equiv 0$ for all $x \in \Omega$. Then for any $0 \leq \lambda < \lambda_1(\Omega)$ and $p > 2$, there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,* (1) *has two distinct strictly positive solutions.*

We also consider a more general case of (1), which is the following perturbed Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f(u) + \varepsilon h, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{5}$$

As we make the equation more general, our theorem condition becomes correspondingly more stronger, which is the following theorem that we get here.

Theorem 1.2. *Let $G = (V, E)$ be a graph satisfies* (a)-(d)*, Ω be a connected domain satisfies* (e) *with $\Omega \neq \emptyset$, $\lambda_1(\Omega)$ is defined by* (2)*. Suppose that $h \in \mathcal{H}(\Omega)$ satisfies $0 \leq h(x) \not\equiv 0$ for all $x \in \Omega$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying the following conditions*:

- (H_1) *$f(s)$ is continuous with respect to s. For any $s \in [0, +\infty)$ there always holds $f(s) \geq 0$ and $f(0) = 0$;*
- (H_2) *for any fixed $L > 0$, there exists a constant M_L such that* $\max_{s\in[0,L]} f(s) \leq M_L$;
- (H_3) *there exists a constant $q > 2$ such that for any $s \geq 0$, it holds*

$$0 < F(s) = \int_0^s f(t)dt \leq \frac{sf(s)}{q};$$

- (H_4) $\limsup_{s\to 0^+} 2F(s)/s^2 < \lambda_1(\Omega)$.

Then there exists a constant $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$, (5) *has two distinct strictly positive solutions.*

Following the lines of [10, 13], we prove Theorems 1.1 and 1.2 by employing the mountain-pass theorem, which is studied by Ambrosetti-Rabinowitz [3], and a direct method of variation. Compared with [10, 13], where the Schrödinger equation and the bi-Laplacian equation were studied respectively, our results are nontrivial extensions. In this paper, we only care about the multiple solutions of the perturbed Yamabe equation on graphs, and the general solvable problem of the Yamabe equation on graphs is not discussed. Furthermore, we do not distinguish sequence and its subsequence.

The remaining parts of this paper are organized as follow: In Section 2, we give the Sobolev embedding theorem on a locally finite graph and the variational form of the equation under the functional framework. Then we introduce the mountain-pass theorem and the weak solution of the equation, then prove that the solution is strictly positive. In Section 3, we use the variational methods to prove Theorem 1.1 by two steps. And then Theorem 1.2 is proved in Section 4.

2. Preliminary analysis

In this section, we review some important facts on graphs.

2.1. Sobolev embedding theorem

For any function $u : V \to \mathbb{R}$ on a graph, we assert that it maps to an exact finite real number for every vertex on the graph. Thus, if $G = (V, E)$ is a graph satisfies (a)-(d) and $\Omega \subset V$ is a bounded connected domain, then there are only finite vertexes in domain Ω and we can simplify (3) by

$$W_0^{1,2}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \mid u|_{\partial\Omega} = 0 \right\}$$

since $\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu < +\infty$ is held for the bounded domain Ω. To better illustrate the properties of $W_0^{1,2}(\Omega)$ and to be used by variational methods, we recall the following Sobolev embedding theorem. It is not difficult for reader to prove the under lemma. For the details of the proof, we refer the reader to [8].

Lemma 2.1 (Sobolev embedding theorem)**.** *Let $G = (V, E)$ be a graph satisfies* (a)-(d)*, Ω be a connected domain satisfies* (e) *with $\Omega \neq \emptyset$. Then $W_0^{1,2}(\Omega)$ is compactly embedded into $L^q(\Omega)$ for all $1 \leq q \leq +\infty$. Namely, there exists a constant C only depending on q and Ω such that for all $1 \leq q \leq +\infty$ and for all $u \in W_0^{1,2}(\Omega)$*

$$\|u\|_{L^q(\Omega)} \leq C(q, \Omega)\|u\|_{W_0^{1,2}(\Omega)}. \tag{6}$$

Moreover, $W_0^{1,2}(\Omega)$ is a pre-compact and reflexive Hilbert space, so for any bounded sequence $\{u_k\} \subset W_0^{1,2}(\Omega)$, there exists some $u \in W_0^{1,2}(\Omega)$ such that up to a subsequence,

$$\begin{cases} u_k \to u & \text{in } W_0^{1,2}(\Omega), \\ u_k \to u & \text{in } L^q(\Omega),\ \forall\ q \in [1, +\infty], \\ \int_\Omega h u_k d\mu \to \int_\Omega h u d\mu,\ \forall\ h \in \mathcal{H}(\Omega). \end{cases}$$

2.2. Mountain-pass theorem

Define

$$J_\epsilon(u) = \frac{1}{2}\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu - \frac{\lambda}{2}\int_\Omega u^2 d\mu - \frac{1}{p}\int_\Omega |u|^p d\mu - \varepsilon\int_\Omega h u d\mu, \tag{7}$$

$$J_f(u) = \frac{1}{2}\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu - \int_\Omega F(u) d\mu - \varepsilon\int_\Omega h u d\mu \tag{8}$$

as the variational forms of (1) and (5), respectively. It's clear that $W_0^{1,2}(\Omega)$ under the form (4) is a Banach space, thus $J_\epsilon(u)$ and $J_f(u)$ are the functionals of $(W_0^{1,2}(\Omega), \|\cdot\|)$. Next we give the mountain-pass theorem, which comes from [8] directly.

Lemma 2.2 (Mountain-pass theorem [3])**.** *Let $(X, \|\cdot\|)$ be a Banach space, $J \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$, which satisfy $\|e\| > r$ and*

$$b = \inf_{\|u\|=r} J(u) > J(0) \geq J(e).$$

If J satisfies the $(PS)_c$ condition with $c = \min_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$, where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 1, \gamma(1) = e\},$$

then c is a critical value of J.

The mountain-pass theorem is a method to find the critical point of the functional satisfying the $(PS)_c$ condition. As for the $(PS)_c$ condition, we refer the reader to [10], namely, if $\{u_k\} \subset X$ and satisfy $J(u_k) \to c$ and $J'(u_k) \to 0$, then there exists some $u \in X$ such that up to a subsequence, $u_k \to u$ in X. In Section 3 and Section 4, we will see $J_\epsilon(u)$ and $J_f(u)$ satisfy the $(PS)_c$ condition. Hence we can use Lemma 2.2 to find the critical points of $J_\epsilon(u)$ and $J_f(u)$ when we find the multiple solutions of perturbed Yamabe equation.

2.3. Weak solution

Now we define the weak solution of the equation (1).

Definition 2.3. If $u \in W_0^{1,2}(\Omega)$ holds that

$$(9)\quad \int_{\Omega\cup\partial\Omega} \Gamma(u,\phi)d\mu - \lambda \int_\Omega u\phi d\mu = \int_\Omega |u|^{p-2}u\phi d\mu + \varepsilon \int_\Omega h\phi d\mu,\ \forall\phi \in W_0^{1,2}(\Omega),$$

then u is called a weak solution of the equation (1).

Next we introduce an important fact.

Proposition 2.4. *If $u \in W_0^{1,2}(\Omega)$ is a weak solution of the equation* (1)*, then u is also a point-wise solution of the equation* (1)*.*

Proof. Since $u \in W_0^{1,2}(\Omega)$ is a weak solution of the equation (1), from (9) we can get

$$(10)\quad -\int_\Omega \Delta u\phi d\mu - \lambda \int_\Omega u\phi d\mu = \int_\Omega |u|^{p-2}u\phi d\mu + \varepsilon \int_\Omega h\phi d\mu,\ \forall\phi \in W_0^{1,2}(\Omega).$$

For any fixed $x_0 \in \Omega$, take a test function $\phi : \Omega \to \mathbb{R}$ in (10) as

$$\phi(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases}$$

We have

$$-\Delta u(x_0) - \lambda u(x_0) = |u(x_0)|^{p-2}u(x_0) + \varepsilon h(x_0).$$

Since x_0 is arbitrary, then the proposition is proved. $\square$

As the same way, we can also define the weak solution of the equation (5) by analogy. And choosing the same test function can prove that Proposition 2.4 holds for equation (5). From now on, we only need to find the weak solution in the functional framework other than the point-wise solution, which only needs to find critical point of the functional.

2.4. Strictly positive solution

In this part, we will prove that if $u \in W_0^{1,2}(\Omega)$ is a nontrivial weak solution, then it is a strictly positive weak solution.

For the equation (1), inspirited by [1, 5, 20], we let

$$u^+ = \max\{u, 0\}, \quad u^- = \min\{u, 0\}.$$

Suppose $u \in W_0^{1,2}(\Omega)$ is a nontrivial weak solution of the following equation

$$-\Delta u - \lambda u = (u^+)^{p-1} + \varepsilon h. \tag{11}$$

Then we take u^- as the test function of (11) and have

$$-\int_\Omega u^- \Delta u d\mu - \lambda \int_\Omega u u^- d\mu = \int_\Omega (u^+)^{p-1} u^- d\mu + \varepsilon \int_\Omega h u^- d\mu.$$

Noting that $u^+ u^- = 0$, $u^- u = (u^-)^2$ and $u = u^+ + u^-$, we get

$$-\int_\Omega u^- \Delta u^+ d\mu - \varepsilon \int_\Omega h u^- d\mu \le \frac{\lambda - \lambda_1(\Omega)}{\lambda_1(\Omega)} \int_{\Omega \cup \partial\Omega} |\nabla u^-|^2 d\mu. \tag{12}$$

However,

$$\begin{aligned} -\int_\Omega u^- \Delta u^+ d\mu &= -\sum_{x \in \Omega} \mu(x) u^-(x) \Big(\frac{1}{\mu(x)} \sum_{y \sim x} w_{xy} (u^+(y) - u^+(x)) \Big) \\ &= -\sum_{x \in \Omega} \sum_{y \sim x} w_{xy} u^-(x) u^+(y) \ge 0. \end{aligned}$$

In view of $h(x) \ge 0$ and $u^-(x) \le 0$ for any $x \in \Omega$, together with $0 \le \lambda < \lambda_1(\Omega)$ we have $\int_{\Omega \cup \partial\Omega} |\nabla u^-|^2 d\mu = 0$ in (12), which implies that $u^- \equiv 0$. Thus we obtain $u(x) \ge 0$ for all $x \in \Omega$ and the equation (11) is equivalent to

$$\begin{cases} -\Delta u - \lambda u = |u|^{p-2} u + \varepsilon h, & \text{in} \quad \Omega, \\ u \ge 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{13}$$

That is to say, if $u \in W_0^{1,2}(\Omega)$ is a nontrivial weak solution of (11), then it's also the nontrivial weak solution of (13). Now we claim that $u(x) > 0$ for all $x \in \Omega$. To prove this, by contradiction, suppose there exists some x_0 such that $u(x_0) = \min_{x \in \Omega} u(x) = 0$, then insert it into the equation (13). On the one hand, we have $-\Delta u(x_0) = \varepsilon h(x_0) \ge 0$. On the other hand, it's obvious that $\Delta u(x_0) \ge 0$ since x_0 is the minimum vertex. From these it follows that $\Delta u(x_0) = 0$. Noting that Ω is a connected bounded domain, it implies that $u(y) = u(x_0) = 0$ for all $y \sim x_0$, and $u(x) = 0$ for all $x \in \Omega$. Finally, this leads to $u \equiv 0$ in (13). That would be contradict with the fact that u is a nontrivial weak solution. Thus we have proved that if $u \in W_0^{1,2}(\Omega)$ is a nontrivial weak solution of (1), then it is a strictly positive weak solution.

As for the equation (5), we have the same conclusion. Assume that $f(s) \equiv 0$ if $s < 0$, hence we let

$$\tilde{f}(s) = \begin{cases} 0, & f(s) < 0, \\ f(s), & f(s) \geq 0. \end{cases}$$

Suppose $u \in W_0^{1,2}(\Omega)$ is the weak solution of the following equation

$$-\Delta u = \tilde{f}(u) + \varepsilon h. \tag{14}$$

Take $u^- = \min\{u, 0\}$ as the test function of (14), and then

$$-\int_\Omega u^- \Delta u d\mu = \int_\Omega \tilde{f} u^- d\mu + \varepsilon \int_\Omega h u^- d\mu.$$

Noting that $u = u^+ + u^-$, we have

$$0 \leq \int_{\Omega \cup \partial\Omega} |\nabla u^-|^2 d\mu = \int_\Omega \tilde{f}(u) u^- d\mu + \varepsilon \int_\Omega h u^- d\mu + \int_\Omega u^- \Delta u^+ d\mu \leq 0.$$

Then we have $\int_{\Omega \cup \partial\Omega} |\nabla u^-|^2 d\mu = 0$. It follows that $u^- \equiv 0$. Thus we obtain $u(x) \geq 0$ for all $x \in \Omega$. Then we have $f(u) \geq 0$ since f satisfies (H_1). Thus $\tilde{f}(u) = f(u)$, which implies our assumption is true. Therefore, the function (14) is equivalent to

$$\begin{cases} -\Delta u = f(u) + \varepsilon h, & \text{in} \quad \Omega, \\ u \geq 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

Through the same analysis as above, with (H_1), we get the same contradiction. It is not difficult for the reader to prove the contradiction, and we omit this part of the proof.

3. Proof of Theorem 1.1

In this section, we use Lemma 2.1 and Lemma 2.2 to prove Theorem 1.1 by two steps.

Step 1. There exists a strictly positive weak solution u_P such that $J_\varepsilon(u_P) = c_P > 0$, where c_P is the critical value of $J_\varepsilon(u)$.

Lemma 3.1. *There exist positive constants ρ_ε and δ_ε such that $J_\varepsilon(u) \geq \delta_\varepsilon$ for all $u \in W_0^{1,2}(\Omega)$ with $\rho_\varepsilon/2 \leq \|u\|_{W_0^{1,2}(\Omega)} \leq 2\rho_\varepsilon$ if $0 < \varepsilon < \varepsilon_0$ for a sufficiently small $\varepsilon_0 > 0$.*

Proof. Inserting (6) into functional (7) and noting that $0 \leq \lambda < \lambda_1(\Omega)$, we have

$$\begin{aligned} J_\varepsilon(u) &\geq \frac{1}{2}\|u\|^2_{W_0^{1,2}(\Omega)} - \frac{\lambda}{2}\int_\Omega u^2 d\mu - \frac{C}{p}\|u\|^p_{W_0^{1,2}(\Omega)} - \varepsilon\|h\|_{\mathcal{H}(\Omega)}\|u\|_{W_0^{1,2}(\Omega)} \\ &\geq \frac{\eta}{2}\|u\|^2_{W_0^{1,2}(\Omega)} - \frac{C}{p}\|u\|^p_{W_0^{1,2}(\Omega)} - \varepsilon\|h\|_{\mathcal{H}(\Omega)}\|u\|_{W_0^{1,2}(\Omega)} \\ &\geq \|u\|_{W_0^{1,2}(\Omega)}\Big(\frac{\eta}{2}\|u\|_{W_0^{1,2}(\Omega)} - \frac{C}{2}\|u\|^{p-1}_{W_0^{1,2}(\Omega)} - \varepsilon\|h\|_{\mathcal{H}(\Omega)}\Big), \end{aligned}$$

where $\eta = (\lambda_1(\Omega) - \lambda)/\lambda_1(\Omega) > 0$. Let $\rho_\varepsilon = \sqrt{\varepsilon}$. By $\rho_\varepsilon/2 \leq \|u\|_{W_0^{1,2}(\Omega)} \leq 2\rho_\varepsilon$, we have

$$\lim_{\varepsilon\to 0^+} \frac{\frac{\eta}{4}\varepsilon^{\frac{1}{2}} - C2^{p-2}\varepsilon^{\frac{p-1}{2}} - \varepsilon\|h\|_{\mathcal{H}(\Omega)}}{\frac{\eta}{4}\varepsilon^{\frac{1}{2}}} = 1.$$

By the properties of the limit, there exists some $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{\eta}{4}\varepsilon^{\frac{1}{2}} - C2^{p-2}\varepsilon^{\frac{p-1}{2}} - \varepsilon\|h\|_{\mathcal{H}(\Omega)} \geq \frac{\eta}{8}\varepsilon^{\frac{1}{2}}.$$

Let $\delta_\varepsilon = \eta\varepsilon/16$. Then there holds $J_\varepsilon(u) \geq \delta_\varepsilon$ if $\varepsilon \in (0, \varepsilon_0)$. □

Lemma 3.2. *J_ε satisfies the $(PS)_c$ condition.*

Proof. For any $c \in \mathbb{R}$, take $\{u_k\} \subset W_0^{1,2}(\Omega)$ such that $J_\varepsilon(u_k) \to c$ and $dJ_\varepsilon(u_k)(\phi) \to 0$ for all $\phi \in W_0^{1,2}(\Omega)$ as $k \to +\infty$. Namely, for all $\phi \in W_0^{1,2}(\Omega)$, there hold

$$\frac{1}{2}\int_{\Omega\cup\partial\Omega} |\nabla u_k|^2 d\mu - \frac{\lambda}{2}\int_\Omega u_k^2 d\mu - \frac{1}{p}\int_\Omega |u_k|^p d\mu - \varepsilon\int_\Omega h u_k d\mu = c + o_k(1), \tag{15}$$

$$\begin{aligned} &\Big|\int_{\Omega\cup\partial\Omega} \Gamma(u_k, \phi) d\mu - \lambda\int_\Omega u_k\phi d\mu - \int_\Omega |u_k|^{p-2}u_k\phi d\mu - \varepsilon\int_\Omega h\phi d\mu\Big| \\ &= \|\phi\| o_k(1). \end{aligned} \tag{16}$$

Taking $\{u_k\}$ as the test function ϕ in (16), we get

$$\int_\Omega |u_k|^p d\mu = \|u_k\|^2_{W_0^{1,2}(\Omega)} - \lambda\int_\Omega u_k^2 d\mu - \varepsilon\int_\Omega h u_k d\mu + \|u_k\|_{W_0^{1,2}(\Omega)} o_k(1). \tag{17}$$

Inserting (17) into (15), we have

$$\begin{aligned} \|u_k\|^2_{W_0^{1,2}(\Omega)} \leq\ & 2c + \frac{2}{p}\|u_k\|^2_{W_0^{1,2}(\Omega)} + \lambda(1 - \frac{2}{p})\int_\Omega u_k^2 d\mu \\ &+ 2\varepsilon(1 - \frac{1}{p})\|h\|_{\mathcal{H}(\Omega)}\|u_k\|_{W_0^{1,2}(\Omega)} + \|u_k\|_{W_0^{1,2}(\Omega)} o_k(1) + o_k(1) \\ \leq\ & 2c + [\frac{2}{p} + \frac{\lambda}{\lambda_1(\Omega)}(1 - \frac{2}{p})]\|u_k\|^2_{W_0^{1,2}(\Omega)} + \frac{4\varepsilon^2(p-1)^2}{p(p-2)\eta}\|h\|^2_{\mathcal{H}(\Omega)} \\ &+ \frac{\eta(p-2)}{4p}\|u_k\|^2_{W_0^{1,2}(\Omega)} + \frac{\eta(p-2)}{4p}\|u_k\|^2_{W_0^{1,2}(\Omega)} + o_k(1), \end{aligned}$$

which implies that

$$\|u_k\|^2_{W_0^{1,2}(\Omega)} \le \frac{4pc}{(p-2)\eta} + \frac{8\varepsilon^2(p-1)^2}{\eta^2(p-2)^2}\|h\|^2_{\mathcal{H}(\Omega)} + o_k(1).$$

Thus $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$ due to $p > 2$. By Lemma 2.1, there exists some $u \in W_0^{1,2}(\Omega)$ such that up to a subsequence, $u_k \to u$ in $W_0^{1,2}(\Omega)$. $\square$

Lemma 3.3. *There exists some strictly positive function $\tilde{u} \in W_0^{1,2}(\Omega)$ such that $J_\varepsilon(\tilde{u}) < 0$ with $\|\tilde{u}\|_{W_0^{1,2}(\Omega)} > \rho_\varepsilon$.*

Proof. Lemma 3.3 is equivalent to the fact that there exists $u \in W_0^{1,2}(\Omega)$ such that $J_\varepsilon(tu) \to -\infty$ as $t \to +\infty$ for any $\varepsilon > 0$. For any fixed $x_0 \in \Omega$, let

$$u(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases}$$

Thus,

$$\begin{aligned} J_\varepsilon(tu) = -\frac{\mu(x_0)}{p}t^p + \frac{1}{2}\Big(\sum_{\substack{x\sim x_0 \\ x\in\Omega\cup\partial\Omega}} \mu(x)|\nabla u|^2(x) - \lambda\mu(x_0)\Big)t^2 - \varepsilon h(x_0)\mu(x_0)t \\ \to -\infty \end{aligned}$$

as $t \to +\infty$ since Ω contains finite vertexes and $p > 2$. $\square$

Now we can use Lemma 2.2. Obviously, $J_\varepsilon(u) \in C^1(W_0^{1,2}(\Omega), \|\cdot\|)$, $J_\varepsilon(0) = 0$; $J_\varepsilon(\tilde{u}) < 0$ with $\|\tilde{u}\|_{W_0^{1,2}(\Omega)} > \rho_\varepsilon$; $J_\varepsilon(u) \ge \delta_\varepsilon > 0$ with $\|u\|_{W_0^{1,2}(\Omega)} = \rho_\varepsilon$. Besides, J_ε satisfies the $(PS)_c$ condition with $c_P = \min_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$, where

$$\Gamma = \Big\{\gamma \in C\big([0,1], W_0^{1,2}(\Omega)\big) : \gamma(0) = 1, \gamma(1) = \tilde{u}\Big\}.$$

Then c_P is the critical value of $J_\varepsilon(u)$. Correspondingly, we can find the critical point $u_P \in W_0^{1,2}(\Omega)$ such that $J_\varepsilon(u_P) = c_P \ge \delta_\varepsilon > 0$. Thus the critical point u_P of $J_\varepsilon(u)$ is a strictly positive weak solution of the equation (1).

Step 2. There exists another strictly positive weak solution u_N such that $J_\varepsilon(u_N) = c_N < 0$, where c_N is another critical value of $J_\varepsilon(u)$.

Lemma 3.4. *There exist $\nu_0 > 0$ and $u \in W_0^{1,2}(\Omega)$ with $\|u\|_{W_0^{1,2}(\Omega)} = 1$ such that $J_\varepsilon(tu) < 0$ if $0 < t < \nu_0$.*

Proof. We consider the equation

$$\begin{cases} -\Delta u - \lambda u = h, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{18}$$

Next we will find the weak solution of (18). From the functional we have

$$\begin{aligned} J_h(u) &= \frac{1}{2}\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu - \frac{\lambda}{2}\int_\Omega u^2 d\mu - \int_\Omega hud\mu \\ &\ge \frac{\eta}{4}\|u\|^2_{W_0^{1,2}(\Omega)} - \frac{1}{\eta}\|h\|^2_{\mathcal{H}(\Omega)}, \end{aligned} \tag{19}$$

where $\eta = (\lambda_1(\Omega) - \lambda)/\lambda_1(\Omega)$ is the same as Lemma 3.1. Hence $J_h(u)$ has a lower bound in $W_0^{1,2}(\Omega)$. Let $m_h = \inf_{u \in W_0^{1,2}(\Omega)} J_h(u)$. Then take a sequence $\{u_k\}$ such that $J_h(u_k) \to m_h$ as $k \to +\infty$. By (19), we obtain $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$. Since the pre-compact and reflexivity of $W_0^{1,2}(\Omega)$, by Lemma 2.1, there exists some $u^* \in W_0^{1,2}(\Omega)$ such that up to a subsequence, $u_k \to u^*$ strongly in $W_0^{1,2}(\Omega)$ and in $L^q(\Omega)$, the convergence is also weakly in $W_0^{1,2}(\Omega)$. It follows that

$$J_h(u^*) = \lim_{k \to +\infty} J_h(u_k) = m_h,$$

and u^* is the weak solution of (18). It yields that

$$\int_\Omega hu^* d\mu = \int_{\Omega \cup \partial\Omega} |\nabla u^*|^2 d\mu - \lambda \int_\Omega (u^*)^2 d\mu \geq \eta \|u^*\|^2_{W_0^{1,2}(\Omega)} > 0. \tag{20}$$

Now we consider the derivative of $J_\varepsilon(tu^*)$, namely

$$\frac{d}{dt} J_\varepsilon(tu^*) = t \int_{\Omega \cup \partial\Omega} |\nabla u^*|^2 d\mu - \lambda t \int_\Omega (u^*)^2 d\mu - t^{p-1} \int_\Omega |u^*|^p d\mu - \varepsilon \int_\Omega hu^* d\mu.$$

By (20), we have

$$\left.\frac{d}{dt}\right|_{t=0} J_\varepsilon(tu^*) < 0.$$

Then there exists some ν such that $J_\varepsilon(tu^*) < 0$ if $0 < t < \nu$. Set $u = u^*/\|u^*\|_{W_0^{1,2}(\Omega)}$, let $\nu_0 = \|u^*\|_{W_0^{1,2}(\Omega)} \nu$, this finishes the proof. In particularly, we can get an equivalent conclusion that there exists some $\nu_0 > 0$ such that $J_\varepsilon(u) < 0$ if $0 < \|u\|_{W_0^{1,2}(\Omega)} < \nu_0$. □

Now we prove that there exists another weak solution $u_N \in W_0^{1,2}(\Omega)$ with $\|u_N\|_{W_0^{1,2}(\Omega)} < \rho_\varepsilon/2$ such that $J_\varepsilon(u_N) = c_N = \inf_{\|u\| \leq 2\rho_\varepsilon} J_\varepsilon(u) < 0$ for any $\varepsilon \in (0, \varepsilon_0)$, where $\rho_\varepsilon = \sqrt{\varepsilon}$. Let ε_0 be given as in Lemma 3.1, $\varepsilon \in (0, \varepsilon_0)$ be fixed. By Lemma 3.1, we see that $J_\varepsilon(u)$ has a lower bound on $B_{2\rho_\varepsilon} = \{u \in W_0^{1,2}(\Omega) : \|u\|_{W_0^{1,2}(\Omega)} \leq 2\rho_\varepsilon\}$. Together with Lemma 3.4, we have $c_N = \inf_{\|u\| \leq 2\rho_\varepsilon} J_\varepsilon(u) < 0$.

Take a function sequence $\{u_k\} \subset W_0^{1,2}(\Omega)$ with $\|u_k\|_{W_0^{1,2}(\Omega)} \leq 2\rho_\varepsilon$ such that $J_\varepsilon(u_k) \to c_N$ as $k \to +\infty$. Since $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$, by Lemma 2.1, we can find some $u_N \in W_0^{1,2}(\Omega)$ such that up to a subsequence $\{u_k\}$ and have

$$\|u_N\|_{W_0^{1,2}(\Omega)} = \lim_{k \to +\infty} \|u_k\|_{W_0^{1,2}(\Omega)}, \qquad \int_\Omega hu_N d\mu = \lim_{k \to +\infty} \int_\Omega hu_k d\mu,$$

$$\lambda \int_\Omega u_N^2 d\mu = \lim_{k \to +\infty} \lambda \int_\Omega u_k^2 d\mu, \qquad \int_\Omega |u_N|^p d\mu = \lim_{k \to +\infty} \int_\Omega |u_k|^p d\mu.$$

Then

$$J_\varepsilon(u_N) = \lim_{k \to +\infty} J_\varepsilon(u_k) = c_N < 0$$

and u_N is the minimizer of $J_\varepsilon(u)$ on $B_{2\rho_\varepsilon}$. Moreover, Lemma 3.1 implies that $\|u_N\|_{W_0^{1,2}(\Omega)} < \rho_\varepsilon/2$. By a straightforward calculation, we get the Euler-Lagrange equation as follows

$$-\Delta u_N - \lambda u_N = |u_N|^{p-2}u_N + \varepsilon h.$$

Hence, u_N is another weak solution of (1). This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

In this section, we use the same method to prove Theorem 1.2. In order to avoid unnecessary repetition and simplify the proof, we give details of the difference between two theorems. By analogy with Section 3, we directly give the proof of the four lemmas in the general case.

Lemma 4.1. *Let f satisfy conditions (H_1)-(H_4). Then there exist positive constants ρ_ε and δ_ε such that $J_f(u) \geq \delta_\varepsilon$ for all $u \in W_0^{1,2}(\Omega)$ with $\rho_\varepsilon/2 \leq \|u\|_{W_0^{1,2}(\Omega)} \leq 2\rho_\varepsilon$ if $0 < \varepsilon < \varepsilon_1$ for a sufficiently small $\varepsilon_1 > 0$.*

Proof. By (H_4), there exist constants $\eta > 0, \tau > 0$ such that $F(s) \leq (\lambda_1(\Omega) - \eta)s^2/2$ for any $0 < s < \tau$. By (H_1) and (H_3), for any $s \geq \tau$ and $p > 2$ there holds $0 < F(s) \leq (s^pF(s))/\tau^p$. Thus for all $s > 0$ we get

$$F(s) \leq \frac{\lambda_1(\Omega) - \eta}{2}s^2 + \frac{s^p}{\tau^p}F(s). \tag{21}$$

For any $u \in W_0^{1,2}(\Omega)$ with $\rho_\varepsilon/2 \leq \|u\|_{W_0^{1,2}(\Omega)} \leq 2\rho_\varepsilon$, by Lemma 2.1, we obtain $\|u\|_{L^\infty(\Omega)} \leq C_1\|u\|_{W_0^{1,2}(\Omega)} \leq 2\rho_\varepsilon C_1$ and $\|u\|_{L^p(\Omega)} \leq C_2\|u\|_{W_0^{1,2}(\Omega)}$ for constants C_1 and C_2. Then inserting (21) into (8), together with (H_2) we have

$$\begin{aligned}
J_f(u) &\geq \frac{1}{2}\|u\|^2_{W_0^{1,2}(\Omega)} - \frac{\lambda_1(\Omega) - \eta}{2}\int_\Omega u^2 d\mu \\
&\quad - \frac{1}{\tau^p}\int_\Omega u^pF(u)d\mu - \varepsilon\|h\|_{\mathcal{H}(\Omega)}\|u\|_{W_0^{1,2}(\Omega)} \\
&\geq \frac{1}{2}\|u\|^2_{W_0^{1,2}(\Omega)} - \frac{\lambda_1(\Omega) - \eta}{2\lambda_1(\Omega)}\|u\|^2_{W_0^{1,2}(\Omega)} \\
&\quad - \frac{1}{\tau^p}\max_{s\in[0,2\rho_\varepsilon C_1]} F(s)\|u\|^p_{L^p(\Omega)} - \varepsilon\|h\|_{\mathcal{H}(\Omega)}\|u\|_{W_0^{1,2}(\Omega)} \\
&\geq \frac{\eta}{2\lambda_1(\Omega)}\|u\|^2_{W_0^{1,2}(\Omega)} - \frac{C_2^pM_{2\rho_\varepsilon C_1}}{\tau^p}\|u\|^p_{W_0^{1,2}(\Omega)} - \varepsilon\|h\|_{\mathcal{H}(\Omega)}\|u\|_{W_0^{1,2}(\Omega)} \\
&= \|u\|_{W_0^{1,2}(\Omega)}\Big(\frac{\eta}{2\lambda_1(\Omega)}\|u\|_{W_0^{1,2}(\Omega)} - C\|u\|^{p-1}_{W_0^{1,2}(\Omega)} - \varepsilon\|h\|_{\mathcal{H}(\Omega)}\Big),
\end{aligned}$$

where $C = (C_2^pM_{2\rho_\varepsilon C_1})/\tau^p$. Let $\rho_\varepsilon = \sqrt{\varepsilon}$. It follows that

$$\lim_{\varepsilon\to 0^+} \frac{\frac{\eta}{4\lambda_1(\Omega)}\varepsilon^{\frac{1}{2}} - C2^{p-1}\varepsilon^{\frac{p-1}{2}} - \varepsilon\|h\|_{\mathcal{H}(\Omega)}}{\frac{\eta}{4\lambda_1(\Omega)}\varepsilon^{\frac{1}{2}}} = 1.$$

Thus there exists some $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$,

$$\frac{\eta}{4\lambda_1(\Omega)}\varepsilon^{\frac{1}{2}} - C2^{p-2}\varepsilon^{\frac{p-1}{2}} - \varepsilon\|h\|_{\mathcal{H}(\Omega)} \geq \frac{\eta}{8\lambda_1(\Omega)}\varepsilon^{\frac{1}{2}}.$$

Let $\delta_\varepsilon = \eta\varepsilon/(16\lambda_1(\Omega))$. Then there holds $J_\varepsilon(u) \geq \delta_\varepsilon$ if $\varepsilon \in (0, \varepsilon_1)$. □

Lemma 4.2. *Let f satisfy conditions (H_1)-(H_3). Then J_f satisfies the $(PS)_c$ condition.*

Proof. For any $c \in \mathbb{R}$, take $\{u_k\} \subset W_0^{1,2}(\Omega)$. We obtain

$$\frac{1}{2}\int_{\Omega\cup\partial\Omega} |\nabla u_k|^2 d\mu - \int_\Omega F(u_k)d\mu - \varepsilon\int_\Omega hu_k d\mu = c + o_k(1), \tag{22}$$

$$\Big|\int_{\Omega\cup\partial\Omega} |\nabla u_k|^2 d\mu - \int_\Omega f(u_k)u_k d\mu - \varepsilon\int_\Omega hu_k d\mu\Big| = \|u_k\|o_k(1). \tag{23}$$

Inserting (23) into (22), in view of (H_3), one has

$$\begin{aligned}\|u_k\|^2_{W_0^{1,2}(\Omega)} &\leq 2c + \frac{2}{q}\int_\Omega f(u_k)u_k d\mu + 2\varepsilon\int_\Omega hu_k d\mu + o_k(1)\\ &\leq 2c + \frac{2}{q}\|u_k\|^2_{W_0^{1,2}(\Omega)} + 2\varepsilon(1 - \frac{1}{q})\|h\|_{\mathcal{H}(\Omega)}\|u_k\|_{W_0^{1,2}(\Omega)}\\ &\quad + \|u_k\|_{W_0^{1,2}(\Omega)}o_k(1) + o_k(1)\\ &\leq 2c + \frac{2}{q}\|u_k\|^2_{W_0^{1,2}(\Omega)} + \frac{4\varepsilon^2(q-1)^2}{q(q-2)}\|h\|_{\mathcal{H}(\Omega)}\\ &\quad + \frac{q-2}{4q}\|u_k\|^2_{W_0^{1,2}(\Omega)} + \frac{q-2}{4q}\|u_k\|^2_{W_0^{1,2}(\Omega)} + o_k(1),\end{aligned}$$

which implies that

$$\|u_k\|^2_{W_0^{1,2}(\Omega)} \leq \frac{4qc}{(q-2)} + \frac{8\varepsilon^2(q-1)^2}{(q-2)^2}\|h\|^2_{\mathcal{H}(\Omega)} + o_k(1).$$

Hence, there exists some $u \in W_0^{1,2}(\Omega)$ such that up to a subsequence, $u_k \to u$ in $W_0^{1,2}(\Omega)$. □

Lemma 4.3. *Let f satisfy conditions (H_1)-(H_4) and $0 \leq h(x) \not\equiv 0$ for any $x \in \Omega$. Then there exists some strictly positive function $\tilde{u} \in W_0^{1,2}(\Omega)$ such that $J_f(\tilde{u}) < 0$ with $\|\tilde{u}\|_{W_0^{1,2}(\Omega)} > \rho_\varepsilon$.*

Proof. Equivalently, we can prove that there exists $u \in W_0^{1,2}(\Omega)$ such that $J_f(tu) \to -\infty$ as $t \to +\infty$ for any $\varepsilon > 0$. In view of (H_3), there exist constants $c_1 > 0$, $c_2 > 0$ such that $F(s) \geq c_1 s^q - c_2$ for any $s \in [0, +\infty)$ and $q > 2$. For any fixed $x_0 \in \Omega$, we take $u(x)$ the same as Lemma 3.3. Noting that Ω contains finite vertexes, together with $q > 2$ we have

$$J_f(tu) \le -c_1\mu(x_0)t^q + \frac{1}{2}\Big(\sum_{\substack{x\sim x_0\\ x\in\Omega\cup\partial\Omega}} \mu(x)|\nabla u|^2(x)\Big)t^2 - \varepsilon h(x_0)\mu(x_0)t + c_2\mu(x_0)$$
$$\to -\infty$$

as $t \to +\infty$. This ends the proof of Lemma 4.3. $\square$

Lemma 4.4. *Let f satisfy conditions (H_1)-(H_3) and $0 \le h(x) \not\equiv 0$ for any $x \in \Omega$. Then there exists some $\nu_1 > 0$ such that $J_f(u) < 0$ if $0 < \|u\|_{W_0^{1,2}(\Omega)} < \nu_1$.*

Proof. We consider the equation

$$\begin{cases} -\Delta u = h, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{24}$$

Then we prove (24) has a weak solution. It follows from Young inequality that

$$J_h(u) \ge \frac{1}{4}\|u\|^2_{W_0^{1,2}(\Omega)} - \|h\|^2_{\mathcal{H}(\Omega)}.$$

Hence $J_h(u)$ has a lower bound in $W_0^{1,2}(\Omega)$ and certainly has infimum. It is not difficult for the reader to find u^* by variational methods, which is the minimizer of functional $J_h(u)$ in $W_0^{1,2}(\Omega)$. It yields that $dJ_h(u^*)(u^*) = 0$, namely,

$$\int_\Omega hu^* d\mu = \int_{\Omega\cup\partial\Omega} |\nabla u^*|^2 d\mu > 0. \tag{25}$$

Now we calculate the derivative of $J_f(tu^*)$ and have

$$\frac{d}{dt}J_f(tu^*) = t\|u^*\|^2_{W_0^{1,2}(\Omega)} - \int_\Omega u^* f(tu^*) d\mu - \varepsilon \int_\Omega hu^* d\mu.$$

By (25), together with $f(0) = 0$ we obtain

$$\frac{d}{dt}\Big|_{t=0} J_f(tu^*) < 0.$$

Set $u = u^*/\|u^*\|_{W_0^{1,2}(\Omega)}$. This ends the proof of Lemma 4.4. $\square$

Next, we will prove (5) has two distinct strictly positive solutions. By solution analysis in Section 2, we just need to prove (5) has two distinct weak solutions.

On the one hand, we can verify that $J_f(u)$ satisfies all the hypotheses of Lemma 2.2 by Lemmas 4.1-4.3: $J_f(u) \in C^1(W_0^{1,2}(\Omega), \|\cdot\|)$, $J_f(0) = 0$; $J_f(\tilde{u}) < 0$ with $\|\tilde{u}\|_{W_0^{1,2}(\Omega)} > \rho_\varepsilon$; $J_f(u) \ge \delta_\varepsilon > 0$ with $\|u\|_{W_0^{1,2}(\Omega)} = \rho_\varepsilon$. Furthermore, J_f satisfies the $(PS)_c$ condition. Thus we can use mountain-pass theorem and find the minimizer of $J_f(u)$ such that the appropriate critical value is strictly positive. And the critical point of $J_f(u)$ is a weak solution of the equation (5).

On the other hand, by Lemmas 4.1 and 4.4, we can get $c = \inf_{\|u\|\le 2\rho_\varepsilon} J_f(u) < 0$. Take a function sequence $\{u_k\} \subset W_0^{1,2}(\Omega)$ with $\|u_k\|_{W_0^{1,2}(\Omega)} \le 2\rho_\varepsilon$ such that $J_f(u_k) \to c$ as $k \to +\infty$. Since $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$, by Lemma 2.1, we can find some $u_c \in W_0^{1,2}(\Omega)$ such that up to a subsequence $\{u_k\}$,

$$\|u_c\|_{W_0^{1,2}(\Omega)} = \lim_{k\to+\infty} \|u_k\|_{W_0^{1,2}(\Omega)},$$

$$\int_\Omega hu_c d\mu = \lim_{k\to+\infty} \int_\Omega hu_k d\mu.$$

By (H_2), there exists some constant C such that

$$|F(u_k) - F(u_c)| \le C|u_k - u_c|,$$

which leads to

$$\int_\Omega F(u_c)d\mu = \lim_{k\to+\infty} \int_\Omega F(u_k)d\mu.$$

Then we conclude $J_f(u_c) = \lim_{k\to+\infty} J_f(u_k) = c < 0$ and u_c is the minimizer of $J_f(u)$ on $B_{2\rho_\varepsilon}$. Moreover, it follows from Lemma 4.1 that $\|u_c\|_{W_0^{1,2}(\Omega)} < \rho_\varepsilon/2$. Then we can get the Euler-Lagrange equation as follows

$$-\Delta u_c = f(u_c) + \varepsilon h.$$

Therefore, we obtain another weak solution of (5) and complete the proof of Theorem 1.2.

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