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NEW CONGRUENCES FOR ℓ-REGULAR OVERPARTITIONS

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Dedicated to Prof. B. Ramakrishnan on his 60th Birthday

Abstract. For a positive integer ℓ, $\overline{A}_\ell(n)$ denotes the number of overpartitions of n into parts not divisible by ℓ. In this article, we find certain Ramanujan-type congruences for $\overline{A}_{r\ell}(n)$, when $r \in \{8, 9\}$ and we deduce infinite families of congruences for them. Furthermore, we also obtain Ramanujan-type congruences for $\overline{A}_{13}(n)$ by using an algorithm developed by Radu and Sellers [15].

1. Introduction

A partition of n is a non-increasing sequence of positive integers whose sum is n and the positive integers are called parts of the partitions. The number partitions of n is denoted by $p(n)$. The function $p(n)$ was first studied by Euler [10], where he showed that the number of partitions of n into odd parts is same as the number of partition of n into distinct parts. In 2004, S. Corteel and J. Lovejoy [8] introduced overpartitions. An overpartition of n is a partition of n, in which the first occurrence of a number may be overlined. For example, the overpartitions of 4 are 4, $\overline{4}$, $3+1$, $\overline{3}+1$, $3+\overline{1}$, $\overline{3}+\overline{1}$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $1+1+1+1$, $\overline{1}+1+1+1$. In 2003, Lovejoy [12] introduced the function $\overline{A}_\ell(n)$ which counts the number of overpartitions of n into parts which are not divisible by ℓ. For example $A_3(4) = 10$. Its relevant partitions are 4, $\overline{4}$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $1+1+1+1$, $\overline{1}+1+1+1$.

For any complex numbers a and q, the q shifted factorial is defined as

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^{n+1}), \quad |q|<1.$$

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The generating function for $\overline{A}_\ell(n)$ is

$$\sum_{n=0}^{\infty}\overline{A}_\ell(n)q^n=\frac{(-q;q)_\infty(q^\ell;q^\ell)_\infty}{(q;q)_\infty(-q^\ell;q^\ell)_\infty}=\frac{\varphi(-q^\ell)}{\varphi(-q)}, \tag{1.1}$$

where the function $\varphi(q)$ is as defined in (2.1).

In 2016, Shen [18] obtained $2,3$ and 4 dissections of the generating function $\overline{A}_\ell(n)$ when $\ell=3,4$ and deduced some congruences modulo $3,6$ and 24. In 2018, Ray and Barman [4] found infinite families of congruences $\overline{A}_{2\ell}(n)$ modulo 4 and $\overline{A}_{4\ell}(n)$ modulo $4,8$ and 16.

Andrews [3] introduced the singular overpartitions function $\overline{C}_{k,i}(n)$, which counts the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. Note that for $n\geq 0$,

$$\overline{A}_3(n)=\overline{C}_{3,1}(n).$$

Many researchers investigated the arithmetic properties of $\overline{C}_{3,1}(n)$. Andrews [3] proved the Ramanujan-type congruences

$$\overline{C}_{3,1}(9n+3)\equiv\overline{C}_{3,1}(9n+6)\equiv 0 \pmod{3} \quad \text{for } n\geq 0.$$

Chen et al. [7] investigated the parity of $\overline{C}_{3,1}(n)$ and proved that $\overline{C}_{3,1}(n)$ is always even. Naika and Gireesh [13] found infinite families of congruences for $\overline{C}_{3,1}(n)$ modulo $12,18,48$ and 72. They conjectured that $\overline{C}_{3,1}(12n+11)\equiv 0 \pmod{144}$ for all $n\geq 0$, which was proved by Barman and Ray [4]. Ahmed and Baruah [1] proved many congruences for $\overline{C}_{3,1}(n)$ modulo $4,18$ and 36. Recently, Ray and Chakraborty [16] proved that if $q_i^{2a i}\geq \ell$, then $\overline{A}_\ell(n)$ is almost always divisible by q_i^j where j is a fixed integer and $\ell=q_1^{a_1}q_2^{a_2}\cdots q_m^{a_m}$ with $q_i\geq 3$ are primes. They also obtained infinite families of congruences for $\overline{A}_5(n)$ and certain Ramanujan type congruences for $\overline{A}_7(n)$. In this article, we find new Ramanujan type simple congruences for $\overline{A}_{8\ell}(n)$ and $\overline{A}_{9\ell}(n)$.

Theorem 1. *For any positive integer ℓ, we have the following congruences*:

$$\overline{A}_{8\ell}(8n+1)\equiv 0 \pmod{2}, \tag{1.2}$$

$$\overline{A}_{8\ell}(8n+3)\equiv 0 \pmod{8}, \tag{1.3}$$

$$\overline{A}_{8\ell}(8n+5)\equiv 0 \pmod{8}, \tag{1.4}$$

$$\overline{A}_{8\ell}(8n+7)\equiv 0 \pmod{64}, \tag{1.5}$$

$$\overline{A}_{8\ell}(8n+6)\equiv 0 \pmod{4}, \tag{1.6}$$

$$\overline{A}_{8\ell}(4n+3)\equiv 0 \pmod{4}. \tag{1.7}$$

Next, we prove three infinite families of congruences for $\overline{A}_{8\ell}(n)$ modulo 16.

Theorem 2. *Let $p\geq 5$ be a prime number and β be a non-negative integer.*

(a) *If* $\left(\frac{-2}{p}\right) = -1$, *then for any positive integer* ℓ, *we have*

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8p^{2\beta}n + p^{2\beta})q^n \equiv 2(q;q)_\infty(q^2;q^2)_\infty \pmod{4}. \tag{1.8}$$

(b) *If* $\left(\frac{-8}{p}\right) = -1$, *then for any positive integer* ℓ, *we have*

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8p^{2\beta}n + 3p^{2\beta})q^n \equiv 8(q;q)_\infty(q^8;q^8)_\infty \pmod{16}. \tag{1.9}$$

(c) *If* $p \equiv 3 \pmod{4}$, *then for any positive integer* ℓ, *we have*

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8p^{2\beta}n + 5p^{2\beta})q^n \equiv 8f(-q^4)^3\psi(q) \pmod{16}. \tag{1.10}$$

Corollary 1.1. *Let* $p \geq 5$ *be a prime number and* β *be a non-negative integer.*

(a) *If* $\left(\frac{-2}{p}\right) = -1$, *then for any positive integer* ℓ, *we have*

$$\overline{A}_{8\ell}(8p^{2\beta}n + (8j+p)p^{2\beta-1}) \equiv 0 \pmod{4},$$

where $j = 1, 2, \ldots, p-1$.

(b) *If* $\left(\frac{-8}{p}\right) = -1$, *then for any positive integer* ℓ, *we have*

$$\overline{A}_{8\ell}(8p^{2\beta}n + (8j+3p)p^{2\beta-1}) \equiv 0 \pmod{16},$$

where $j = 1, 2, \ldots, p-1$.

(c) *If* $p \equiv 3 \pmod{4}$, *then for any positive integer* ℓ, *we get*

$$\overline{A}_{8\ell}(8p^{2\beta}n + (8j+5p)p^{2\beta-1}) \equiv 0 \pmod{16},$$

where $j = 1, 2, \ldots, p-1$.

We also prove three simple congruences for $\overline{A}_{13}(n)$ by using the methods of Radu and Sellers [15].

Theorem 3. *For all* $n \geq 0$, *we have*

$$\overline{A}_{13}(54n + 18) \equiv 0 \pmod{13}, \tag{1.11}$$

$$\overline{A}_{13}(54n + 36) \equiv 0 \pmod{13}, \tag{1.12}$$

$$\overline{A}_{13}(256n + 128) \equiv 0 \pmod{13}. \tag{1.13}$$

In 2018, Barman and Ray [4] proved the following congruences modulo 8 and 16 for $\overline{A}_9(n)$. We show that these results also hold true for $\overline{A}_{9\ell}(n)$.

Theorem 4. *For any positive integer* ℓ, *we have the following congruences modulo* 8 *and* 16 *for* $\overline{A}_{9\ell}(n)$.

$$\overline{A}_{9\ell}(9n + 3) \equiv 0 \pmod{8}, \tag{1.14}$$

$$\overline{A}_{9\ell}(9n + 6) \equiv 0 \pmod{8}, \tag{1.15}$$

$$\overline{A}_{9\ell}(18n+15) \equiv 0 \pmod{16}, \tag{1.16}$$

$$\overline{A}_{9\ell}(36n+21) \equiv 0 \pmod{16}, \tag{1.17}$$

$$\overline{A}_{9\ell}(36n+30) \equiv 0 \pmod{16}. \tag{1.18}$$

In Section 2, we include the preliminaries required in the paper. In Section 3, we give proofs of Theorems 1-4 and Corollary 1.1.

2. Preliminaries

For $|ab| < 1$, we denote Ramanujan's general theta function as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

In Ramanujan's notation, the Jacobi triple product identity [6, Entry 19, Page 36] is given by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

We list three special cases of $f(a,b)$ as follows.

$$\begin{aligned}
\varphi(q) &:= f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty^2 \\
&\qquad\qquad = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2 (q^4;q^4)_\infty^2}, \\
\psi(q) &:= f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}, \\
f(-q) &:= f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.
\end{aligned} \tag{2.1}$$

Further we have

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty},$$

$$\psi(-q) = \frac{(q;q)_\infty (q^4;q^4)_\infty}{(q^2;q^2)_\infty}.$$

Lemma 2.1 (Hirschhorn and Sellers [11])**.**

$$\frac{1}{\varphi(-q)} = \frac{1}{\varphi(-q^2)^2}\left(\varphi(q^4) + 2q\psi(q^8)\right) \tag{2.2}$$

$$= \frac{\varphi(-q^9)}{\varphi(-q^3)^4}\left(\varphi(-q^9)^2 + 2q\varphi(-q^9)\Omega(-q^3) + 4q^2\Omega(-q^3)^2\right) \tag{2.3}$$

$$= \frac{1}{\varphi(-q^4)^4}\left(\varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3\right), \tag{2.4}$$

where Ω *denotes an octagonal number* (*a number of the form* $3n^2+2n$) *and*

$$\Omega(q) := \sum_{n=-\infty}^{\infty} q^{3n^2+2n} = \frac{(q^2;q^2)_\infty^2 (q^3;q^3)_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty (q^4;q^4)_\infty (q^6;q^6)_\infty}$$

and

$$\Omega(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+2n} = \frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(q^2;q^2)_\infty (q^3;q^3)_\infty}. \tag{2.5}$$

Lemma 2.2 (Baruah and Ojah [5])**.**

$$\begin{aligned} \frac{1}{(q;q)_\infty (q^3;q^3)_\infty} &= \frac{(q^8;q^8)_\infty^2 (q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^2 (q^4;q^4)_\infty (q^6;q^6)_\infty^4 (q^{24};q^{24})_\infty^2} \\ &\quad + q \frac{(q^4;q^4)_\infty^5 (q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^4 (q^6;q^6)_\infty^2 (q^8;q^8)_\infty^2 (q^{12};q^{12})_\infty}. \end{aligned} \tag{2.6}$$

The following result follows from the dissection formulas of Ramanujan collected by Berndt [6, Entry 25, Page 40].

Lemma 2.3.

$$\frac{1}{(q;q)_\infty^4} = \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14} (q^8;q^8)_\infty^4} + 4q \frac{(q^4;q^4)_\infty^2 (q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}.$$

The following p-dissection of $f(-q)$ is due to Cui and Gu [9, Theorem 2.2].

Lemma 2.4. *Let* $p \geq 5$ *be a prime number. Then*

$$\begin{aligned} (q;q)_\infty = f(-q) &= \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) \\ &\quad + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}), \end{aligned}$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6} & \text{if} \quad p \equiv 1 \pmod 6; \\ \frac{-p-1}{6} & \text{if} \quad p \equiv -1 \pmod 6. \end{cases}$$

Furthermore, for $\frac{-(p-1)}{2} \leq k \leq \frac{(p-1)}{2}$ *and* $k \neq \frac{\pm p-1}{6}$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

The following lemma gives a p-dissection of $(q^4;q^4)_\infty^3$. This result directly follows from [2, Lemma 2.3] by replacing q with q^4.

Lemma 2.5. *Let* $p \geq 3$ *be a prime. Then we have*

$$(q^4;q^4)_\infty^3 = f(-q^4)^3$$

$$= \sum_{\substack{k=0\\ k\neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{4\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{4pn\frac{pn+2k+1}{2}}$$

$$+ p(-1)^{\frac{p-1}{2}} q^{4\frac{p^2-1}{8}} f(-q^{4p^2})^3.$$

Furthermore, if $0 \leq k \leq p-1$ *and* $k \neq \frac{p-1}{2}$, *then* $2(k^2+k) \not\equiv \frac{p^2-1}{2} \pmod p$.

The following p-dissection of $\psi(q)$ is given by Cui and Gu [9, Theorem 2.1].

Lemma 2.6. *Let* p *be an odd prime. Then*

$$\psi(q) = \sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(\frac{p^2+(2k+1)p}{2}, \frac{p^2-(2k+1)p}{2}\right) + q^{\frac{p^2-1}{8}} \psi(q^{p^2}).$$

Moreover, $\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod p$ *for* $0 \leq k \leq \frac{p-3}{2}$.

The following p-dissection of $\varphi(q)$ is mentioned in Berndt [6, Page 49].

Lemma 2.7. *For any prime* p,

$$\varphi(q) = \varphi(q^{p^2}) + \sum_{r=1}^{p-1} q^{r^2} f\left(q^{p(p-2r)}, q^{p(p+2r)}\right).$$

In order to prove Theorem 3, we recall an algorithm developed by Radu and Sellers [15]. Let M be a positive integer and let $R(M)$ denote the set of integers sequences $r = (r_\delta)_{\delta|M}$ indexed by the positive divisors of M. For $r \in R(M)$ and the positive divisors $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$ of M, we set $r = (r_{\delta_1}, r_{\delta_2}, \ldots, r_{\delta_k})$. We define $c_r(n)$ by

$$\sum_{n=0}^{\infty} c_r(n) q^n := \prod_{\delta|M} (q^\delta; q^\delta)_\infty^{r_\delta} = \prod_{\delta|M} \prod_{n=1}^{\infty} (1-q^{n\delta})^{r_\delta}.$$

Radu and Sellers [15] approach to prove congruences for $c_r(n)$ modulo a positive integer reduced the number of cases that we need to check as compared with the classical method which uses Sturm's bound alone.

Let $m \geq 0$ and s be integers. We denote by $[s]_m$ the residue class of s in $\mathbb{Z}_m$ and we denote by $\mathbb{S}_m$ the set of squares in $\mathbb{Z}_m^*$. For $t \in \{0, 1, \ldots, m-1\}$ and $r \in R(M)$, the subset $P_{m,r}(t) \subseteq \{0, 1, \ldots, m-1\}$ is defined as

$$P_{m,r}(t) := \left\{ t' : \exists [s]_{24m} \text{ such that } t' = ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \pmod m \right\}.$$

Definition 2.8. For positive integers m, M and N, let $r = (r_\delta) \in R(M)$ and $t \in \{0, 1, \ldots, m-1\}$. Let $k = k(m) := \gcd(m^2-1, 24)$ and write

$$\prod_{\delta|M} \delta^{|r_\delta|} = 2^s \cdot j,$$

where s and j are non-negative integers with j odd. The set Δ^* is the collection of all tuples $(m, M, N, (r_\delta), t)$ satisfying the following conditions.

(a) Every prime divisor of m is also a divisor of N.
(b) If $\delta \mid M$, then $\delta \mid mN$ for every $\delta \geqslant 1$ such that $r_\delta \neq 0$.
(c) $kN \sum_{\delta|M} r_\delta mN/\delta \equiv 0 \pmod{24}$.
(d) $kN \sum_{\delta|M} r_\delta \equiv 0 \pmod 8$.
(e) $\frac{24m}{\gcd(-24kt-k\sum_{\delta|M}\delta r_\delta, 24m)}$ divides N.
(f) If $2 \mid m$, then either ($4 \mid kN$ and $8 \mid sN$) or ($2 \mid s$ and $8 \mid (1-j)N$).

We denote by $\Gamma := SL_2(\mathbb{Z})$, the full modular group of 2-by-2 matrices of determinant 1. For a positive integer N, we define

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \equiv 0 \pmod N \right\}.$$

The congruence subgroup Γ_∞ of level N is defined as

$$\Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

For positive integers m, M and N, $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$, $r \in R(M)$ and $a \in R(N)$, we define

$$p_{m,r}(\gamma) := \min_{\lambda \in \{0,1,\ldots,m-1\}} \frac{1}{24} \sum_{\delta|M} r_\delta \frac{\gcd^2(\delta a + \delta k \lambda c, mc)}{\delta m}$$

and

$$p_a^*(\gamma) := \frac{1}{24} \sum_{\delta|N} a_\delta \frac{\gcd^2(\delta, c)}{\delta}.$$

The following lemma is given by Radu [14, Lemma 4.5].

Lemma 2.9. *Let u be a positive integer, $(m, M, N, (r_\delta), t) \in \Delta^*$ and $a = (a_\delta) \in R(N)$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma$ denote a complete set of representatives of the double cosets of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p_a^*(\gamma) \geq 0$ for all $1 \leq i \leq n$. Let $t_{\min} = \min_{t' \in P_{m,r}(t)} t'$ and*

$$\nu := \frac{1}{24} \left\{ \left(\sum_{\delta|M} r_\delta + \sum_{\delta|N} a_\delta \right) [\Gamma : \Gamma_0(N)] - \sum_{\delta|N} \delta a_\delta \right\} - \frac{1}{24m} \sum_{\delta|M} \delta r_\delta - \frac{t_{\min}}{m}.$$

If the congruence $c_r(mn + t') \equiv 0 \pmod u$ holds for all $t' \in P_{m,r}(t)$ and $0 \leq n \leq \lfloor \nu \rfloor$, then $c_r(mn + t') \equiv 0 \pmod u$ holds for all $t' \in P_{m,r}(t)$ and $n \geq 0$.

The next lemma is given by Wang [19, Lemma 4.3]. This result gives the complete set of representatives of the double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$ when N or $\frac{N}{2}$ is a square-free integer.

Lemma 2.10. *If N or $\frac{N}{2}$ is a square-free integer, then*

$$\cup_{\delta|N}\Gamma_0(N)\begin{bmatrix}1 & 0\\ \delta & 1\end{bmatrix}\Gamma_\infty = \Gamma.$$

In order to prove Theorem 2, we need the following lemma.

Lemma 2.11. *Let ℓ be a positive integer. Then we have the following.*

(a)

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}(8n+1)q^n \equiv 2(q;q)_\infty(q^2;q^2)_\infty \pmod 4.$$

(b)

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}(8n+3)q^n \equiv 8(q;q)_\infty(q^8;q^8)_\infty \pmod{16}.$$

(c)

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}(8n+5)q^n \equiv 8f(-q^4)^3\psi(q) \pmod{16}.$$

A proof of above lemma is given in Section 3.2.

3. Proofs

3.1. Proof for Theorem 1

From (1.1), we have

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}(n) = \frac{\varphi(-q^{8\ell})}{\varphi(-q)}.$$

Using (2.2), we get

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}(n)q^n = \frac{\varphi(-q^{8\ell})}{\varphi(-q^2)^2}\left(\varphi(q^4)+2q\psi(q^8)\right).$$

Extracting the terms containing q^{2n} and q^{2n+1} from both sides, we deduce that

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}(2n)q^n = \frac{\varphi(-q^{4\ell})\varphi(q^2)}{\varphi(-q)^2}, \tag{3.1}$$

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}(2n+1)q^n = 2\frac{\varphi(-q^{4\ell})\psi(q^4)}{\varphi(-q)^2}. \tag{3.2}$$

Using (2.4) in (3.2), we obtain

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}(2n+1)q^n$$

$$= 2\frac{\varphi(-q^{4\ell})\psi(q^4)}{\varphi(-q^4)^8}\left(\varphi(q^4)^3 + 2q\varphi(q^4)^2\psi(q^8) + 4q^2\varphi(q^4)\psi(q^8)^2 + 8q^3\psi(q^8)^3\right)^2$$

$$= 2\frac{\varphi(-q^{4\ell})\psi(q^4)}{\varphi(-q^4)^8}\Bigg(\varphi(q^4)^6 + 4q\varphi(q^4)^5\psi(q^8) + 12q^2\varphi(q^4)^4\psi(q^8)^2$$
$$+ 32q^3\varphi(q^4)^3\psi(q^8)^3 + 48q^4\varphi(q^4)^2\psi(q^8)^4 + 64q^5\varphi(q^4)\psi(q^8)^5 + 64q^6\psi(q^8)^6\Bigg).$$

Extracting the terms containing q^{4n+i} for $i = 0, 1, 2, 3$, respectively, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8n+1)q^n = 2\frac{\varphi(-q^{\ell})\psi(q)}{\varphi(-q)^8}\left(\varphi(q)^6 + 48q\varphi(q)^2\psi(q^2)^4\right), \tag{3.3}$$

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8n+3)q^n = 8\frac{\varphi(-q^{\ell})\psi(q)}{\varphi(-q)^8}\left(\varphi(q)^5\psi(q^2) + 16q\varphi(q)\psi(q^2)^5\right), \tag{3.4}$$

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8n+5)q^n = 8\frac{\varphi(-q^{\ell})\psi(q)}{\varphi(-q)^8}\left(3\varphi(q)^4\psi(q^2)^2 + 16q\psi(q^2)^6\right), \tag{3.5}$$

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8n+7)q^n = 64\frac{\varphi(-q^{\ell})\psi(q)}{\varphi(-q)^8}\varphi(q)^3\psi(q^2)^3. \tag{3.6}$$

Now, the congruences (1.2), (1.3), (1.4) and (1.5) directly follows from (3.3), (3.4), (3.5) and (3.6), respectively.

Using (2.2) in (3.1), we get

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(2n)q^n = \frac{\varphi(-q^{4\ell})\varphi(q^2)}{\varphi(-q^2)^4}\left(\varphi(q^4) + 2q\psi(q^8)\right)^2$$
$$= \frac{\varphi(-q^{4\ell})\varphi(q^2)}{\varphi(-q^2)^4}\left(\varphi(q^4)^2 + 4q^2\psi(q^8)^2 + 2q\varphi(q^4)\psi(q^8)\right).$$

Extracting the terms containing q^{2n+1} from both sides, we get

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(4n+2)q^n = 2\frac{\varphi(-q^{2\ell})\varphi(q)\varphi(q^2)\psi(q^4)}{\varphi(-q)^4}. \tag{3.7}$$

From [6, Entry 25, Page 40], we have

$$\varphi(q)\varphi(-q) = \varphi(-q^2)^2, \tag{3.8}$$

$$\varphi(q)\psi(q^2) = \psi(q)^2. \tag{3.9}$$

Using (3.8) and (3.9) in (3.7), we get

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(4n+2)q^n = 2\frac{\varphi(-q^{2\ell})\varphi(q)\psi(q^2)^2}{\varphi(-q)^4} = 2\frac{\varphi(-q^{2\ell})\varphi(-q^2)^2\psi(q^2)^2}{\varphi(-q)^5}.$$

Further using (2.2), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(4n+2)q^n = 2\frac{\varphi(-q^{2\ell})\varphi(-q^2)^2\psi(q^2)^2}{\varphi(-q^2)^{10}} \left(\varphi(q^4) + 2q\psi(q^8)\right)^5.$$

Extracting the terms containing q^{2n+1} from both sides and then replacing q^2 by q, we get the congruence (1.6).

Using (2.2) in (3.2), we get

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(2n+1)q^n = 2\frac{\varphi(-q^{4\ell})\psi(q^4)}{\varphi(-q^2)^4} \left(\varphi(q^4) + 2q\psi(q^8)\right)^2.$$

Now extracting the terms containing q^{2n+1} from both sides and replacing q^2 by q and then using (3.9), we get

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(4n+3)q^n = 4\frac{\varphi(-q^{2\ell})\psi(q^2)\varphi(q^2)\psi(q^4)}{\varphi(-q)^4} = 4\frac{\varphi(-q^{2\ell})\psi(q^2)^3}{\varphi(-q)^4}.$$

This implies the congruence (1.7).

3.2. Proof of Lemma 2.11

(a) From (3.3), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8n+1)q^n \equiv 2\frac{\varphi(-q^{\ell})\psi(q)\varphi(q)^6}{\varphi(-q)^8} \pmod 4. \tag{3.10}$$

Note that from the binomial theorem, for any positive integer r, we have

$$(q^r;q^r)_\infty^2 \equiv (q^{2r};q^{2r})_\infty^2 \pmod 2.$$

Thus we have

$$\varphi(q) \equiv \frac{\left(q^2;q^2\right)_\infty^5}{(q;q)_\infty^2\left(q^4;q^4\right)_\infty^2} = \frac{\left(q^2;q^2\right)_\infty\left(q^2;q^2\right)_\infty^4}{(q;q)_\infty^2\left(q^4;q^4\right)_\infty^2} \equiv 1 \pmod 2. \tag{3.11}$$

Similarly

$$\varphi(-q) \equiv 1 \pmod 2 \quad \text{and} \quad \varphi(-q^{\ell}) \equiv 1 \pmod 2. \tag{3.12}$$

Also we have

$$\psi(q) \equiv \frac{\left(q^2;q^2\right)_\infty^2}{(q;q)_\infty} \equiv (q;q)_\infty\left(q^2;q^2\right)_\infty \pmod 2. \tag{3.13}$$

Using (3.11), (3.12) and (3.13) in (3.10), we conclude that

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8n+1)q^n \equiv 2(q;q)_\infty(q^2;q^2)_\infty \pmod 4.$$

(b) From (3.4), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8n+3)q^n \equiv 8\frac{\varphi(-q^{\ell})\psi(q)\varphi(q)^5\psi(q^2)}{\varphi(-q)^8} \pmod{16}. \tag{3.14}$$

Since $\left(q^4;q^4\right)_\infty^2 \equiv \left(q^8;q^8\right)_\infty \pmod 2$, from (3.12), we get

$$\psi(q)\psi(q^2) \equiv (q;q)_\infty \left(q^2;q^2\right)_\infty \frac{\left(q^4;q^4\right)_\infty^2}{(q^2;q^2)_\infty} \equiv (q;q)_\infty \left(q^8;q^8\right)_\infty \pmod 2. \tag{3.15}$$

Now using (3.11), (3.12) and (3.15) in (3.14), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8n+3)q^n \equiv 8(q;q)_\infty(q^8;q^8)_\infty \pmod{16}.$$

(c) From (3.5), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8n+5)q^n \equiv 8\frac{\varphi(-q^\ell)\psi(q)\varphi(q)^4\psi(q^2)^2}{\varphi(-q)^8} \pmod{16}. \tag{3.16}$$

Note that

$$\psi(q^2)^2 = \frac{\left(q^4;q^4\right)_\infty^4}{(q^2;q^2)_\infty^2} = \left(q^4;q^4\right)_\infty^3 \frac{\left(q^4;q^4\right)_\infty}{(q^2;q^2)_\infty^2} \equiv \left(q^4;q^4\right)_\infty^3 \pmod 2$$

and thus

$$\psi(q^2)^2 \equiv f(-q^4)^3 \pmod 2. \tag{3.17}$$

So using (3.11), (3.12) and (3.17) in (3.16), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8n+5)q^n \equiv 8f(-q^4)^3\psi(q) \pmod{16}.$$

3.3. Proof for Theorem 2

(a) From Lemma 2.11(a), we have

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}(8n+1)q^n \equiv 2(q;q)_\infty(q^2;q^2)_\infty \pmod 4.$$

Thus (1.8) holds true when $\beta = 0$. We use induction on β to complete the proof. We suppose that (1.8) holds true for some $\beta \geq 0$. Note that (1.8) can be written as follows.

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}\left(8\left(p^{2\beta}n + 3\frac{p^{2\beta}-1}{24}\right)+1\right)q^n \equiv 2(q;q)_\infty(q^2;q^2)_\infty \pmod 4.$$

Using Lemma 2.4, we get, modulo 4,

(3.18)

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}\left(8\left(p^{2\beta}n + 3\frac{p^{2\beta}-1}{24}\right)+1\right)q^n$$

$$\equiv 2\left[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}}\right)+(-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2})\right]$$

$$\times\left[\sum_{\substack{m=-\frac{p-1}{2}\\ m\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^m q^{2\frac{3m^2+m}{2}} f\left(-q^{2\frac{3p^2+(6m+1)p}{2}},-q^{2\frac{3p^2-(6m+1)p}{2}}\right)+(-1)^{\frac{\pm p-1}{6}} q^{2\frac{p^2-1}{24}} f(-q^{2p^2})\right].$$

For a prime $p \geq 5$ and $-\frac{p-1}{2} \leq k, m \leq \frac{p-1}{2}$, we note that

$$\frac{3k^2+k}{2}+2\frac{3m^2+m}{2} \equiv 3\frac{p^2-1}{24} \pmod p \tag{3.19}$$

is equivalent to

$$(6k+1)^2+2(6m+1)^2 \equiv 0 \pmod p.$$

Also $\left(\frac{-2}{p}\right)=-1$ implies that $k=m=\frac{\pm p-1}{6}$ is the only solution of (3.19). So extracting the terms containing $q^{pn+3\frac{p^2-1}{24}}$ from both sides of (3.18) and then replacing q^p by q, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{A}_{8\ell}\left(8\left(p^{2\beta+1}n+3\frac{p^{2\beta+2}-1}{24}\right)+1\right)q^n\\ &\equiv 2(q^p;q^p)_\infty(q^{2p};q^{2p})_\infty \pmod 4.\end{aligned} \tag{3.20}$$

Now extracting the terms containing q^{pn} from both sides and then replacing q^p by q, we get

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}\left(8p^{2(\beta+1)}n+p^{2(\beta+1)}\right)q^n \equiv 2(q;q)_\infty(q^2;q^2)_\infty \pmod 4.$$

Hence (1.8) holds for $\beta+1$. This completes the proof of Theorem 2(a).

(b) From Lemma 2.11(b), we have

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}(8n+3)q^n \equiv 8(q;q)_\infty(q^8;q^8)_\infty \pmod{16}.$$

Thus (1.9) holds true when $\beta=0$. We again use induction on β to complete the proof. We suppose that (1.9) holds true for some $\beta \geq 0$. We can write (1.9) as follows.

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}\left(8\left(p^{2\beta}n+9\frac{p^{2\beta}-1}{24}\right)+3\right)q^n \equiv 8(q;q)_\infty(q^8;q^8)_\infty \pmod{16}.$$

Using Lemma 2.4, we get, modulo 16

(3.21)

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{A}_{8\ell}\left(8\left(p^{2\beta}n+9\frac{p^{2\beta}-1}{24}\right)+3\right)q^n\\ &\equiv 8\left[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^k q^{\frac{3k^2+k}{2}}f\left(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}}\right)+(-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{24}}f(-q^{p^2})\right]\\ &\quad\times\left[\sum_{\substack{m=-\frac{p-1}{2}\\ m\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^m q^{8\frac{3m^2+m}{2}}f\left(-q^{8\frac{3p^2+(6m+1)p}{2}},-q^{8\frac{3p^2-(6m+1)p}{2}}\right)+(-1)^{\frac{\pm p-1}{6}}q^{8\frac{p^2-1}{24}}f(-q^{8p^2})\right].\end{aligned}$$

For a prime $p\geq 5$ and $-\frac{p-1}{2}\leq k,m\leq\frac{p-1}{2}$, we note that

$$\frac{3k^2+k}{2}+8\frac{3m^2+m}{2}\equiv 9\frac{p^2-1}{24}\pmod{p} \tag{3.22}$$

is equivalent to

$$(6k+1)^2+8(6m+1)^2\equiv 0\pmod{p}.$$

Since $\left(\frac{-8}{p}\right)=-1$, it follows that $k=m=\frac{\pm p-1}{6}$ is the only solution of (3.22). Now, extracting the terms containing $q^{pn+9\frac{p^2-1}{24}}$ from both sides of (3.21) and then replacing q^p by q, we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{A}_{8\ell}\left(8\left(p^{2\beta+1}n+9\frac{p^{2\beta+2}-1}{24}\right)+3\right)q^n\\ &\equiv 8(q^p;q^p)_\infty(q^{8p};q^{8p})_\infty\pmod{16}.\end{aligned} \tag{3.23}$$

Now extracting the terms containing q^{pn} from both sides and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}\left(8p^{2(\beta+1)}n+3p^{2(\beta+1)}\right)q^n\equiv 8(q;q)_\infty(q^8;q^8)_\infty\pmod{16}.$$

Thus (1.9) holds true for $\beta+1$ and the assertion follows by induction method.

(c) From Lemma 2.11(c), we have

$$\sum_{n=0}^{\infty}\overline{A}_{8\ell}(8n+5)q^n\equiv 8f(-q^4)^3\psi(q)\pmod{16}.$$

Thus (1.10) is true when $\beta=0$. We once again use induction method for the proof. We suppose that (1.10) holds true for some $\beta\geq 0$. We can write (1.10) as follows.

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}\left(8\left(p^{2\beta}n+5\frac{p^{2\beta}-1}{8}\right)+5\right)q^n \equiv 8f(-q^4)^3\psi(q) \pmod{16}.$$

Using Lemma 2.5 and Lemma 2.6, we obtain

(3.24)

$$\begin{aligned}&\sum_{n=0}^{\infty} \overline{A}_{8\ell}\left(8\left(p^{2\beta}n+5\frac{p^{2\beta}-1}{8}\right)+5\right)q^n\\ &\equiv 8\left[\sum_{\substack{k=0\\k\neq\frac{p-1}{2}}}^{p-1}(-1)^k q^{4\frac{k(k+1)}{2}}\sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{4pn\frac{pn+2k+1}{2}}+p(-1)^{\frac{p-1}{2}}q^{4\frac{p^2-1}{8}}f(-q^{4p^2})^3\right]\\ &\times\left[\sum_{m=0}^{\frac{p-3}{2}}q^{\frac{m^2+m}{2}}f\left(\frac{p^2+(2m+1)p}{2},\frac{p^2-(2m+1)p}{2}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right] \pmod{16}.\end{aligned}$$

For a prime $p\geq 5$, $0\leq k\leq p-1$, $k\neq\frac{p-1}{2}$ and $0\leq m\leq\frac{p-1}{2}$, we note that

$$2(k^2+k)+\frac{m^2+m}{2}\equiv 5\frac{p^2-1}{8} \pmod{p}$$

is equivalent to

$$2^2(2k+1)^2+(2m+1)^2\equiv 0 \pmod{p}.$$

Thus these congruences have the only solution $k=m=\frac{p-1}{2}$ when $p\equiv 3 \pmod{4}$. Now, extracting the terms containing $q^{pn+5\frac{p^2-1}{8}}$, from both side of (3.24) and then replacing q^p by q, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} \overline{A}_{8\ell}\left(8\left(p^{2\beta+1}n+5\frac{p^{2\beta+2}-1}{8}\right)+5\right)q^n\\ &\equiv 8f(-q^{4p})^3\psi(q^p) \pmod{16}.\end{aligned}\tag{3.25}$$

Further extracting the terms containing q^{pn} from both sides of (3.25) and then replacing q^p by q, we get

$$\sum_{n=0}^{\infty} \overline{A}_{8\ell}\left(8p^{2(\beta+1)}n+5p^{2(\beta+1)}\right)q^n \equiv 8f(-q^4)^3\psi(q) \pmod{16}.$$

Hence (1.10) holds true for $\beta+1$ and this completes the proof.

3.4. Proof of Corollary 1.1

(a) From (3.20), we get

$$\overline{A}_{8\ell}(8p^{2\beta}n+(8j+p)p^{2\beta-1})\equiv 0 \pmod{4},$$

where $j=1,2,\ldots,p-1$.

(b) From (3.23), we obtain

$$\overline{A}_{8\ell}(8p^{2\beta}n+(8j+3p)p^{2\beta-1})\equiv 0 \pmod{16},$$

where $j=1,2,\dots,p-1$.

(c) From (3.25), we deduce

$$\overline{A}_{8\ell}(8p^{2\beta}n+(8j+5p)p^{2\beta-1})\equiv 0 \pmod{16},$$

where $j=1,2,\dots,p-1$. This completes the proof.

3.5. Proof of Theorem 3

We have

$$\sum_{n=0}^{\infty}\overline{A}_{13}(n)q^n=\frac{(q;q)_\infty^{24}}{(q^2;q^2)_\infty^{12}} \pmod{13}.$$

Let us consider $(m,M,N,r)=(54,2,12,(r_1=24,r_2=-12))$ and $t\in\{18,36\}$. For each $t\in\{18,36\}$, we verify that $(m,M,N,r,t)\in\Delta^*$ and $P_{m,r}(t)=\{t\}$. For each $\delta\mid 12$, we set $\gamma_\delta=\left[\begin{smallmatrix}1&0\\ \delta&1\end{smallmatrix}\right]$. Since $\frac{N}{2}=6$ is a square-free integer, Lemma 2.10 implies that $\{\gamma_\delta:\delta\mid 12\}$ forms a complete set of double coset representatives of $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. For $a=(0,0,0,0,0,0)\in R(12)$, using SAGE [17] we verified that $p_{m,r}(\gamma_\delta)+p_a^*(\gamma_\delta)\geq 0$ for each $\delta\mid 12$. For each $t\in\{18,36\}$, we compute that the upper bound in Lemma 2.9 is $\lfloor\nu\rfloor=11$ and using Mathematica, we verify that $\bar{A}_{13}(54n+t)\equiv 0 \pmod{13}$ for $n\leq 11$. Now (1.11) and (1.12) follows from Lemma 2.9.

To prove (1.13), we consider $(m,M,N,r,t)=(256,2,4,(r_1=24,r_2=-12),128)$. We verify that $(m,M,N,r,t)\in\Delta^*$ and $P_{m,r}(t)=\{128\}$. We proceed in the same manner as above. In this case, the upper bound in Lemma 2.9 is $\lfloor\nu\rfloor=2$ and using Mathematica, we verify that $\bar{A}_{13}(54n+t)\equiv 0 \pmod{13}$ for $n\leq 2$. Hence (1.13) follows from Lemma 2.9.

3.6. Proof of Theorem 4

From (1.1), we get

$$\sum_{n=0}^{\infty}\overline{A}_{9\ell}(n)q^n=\frac{\varphi(-q^{9\ell})}{\varphi(-q)}.$$

Using (2.3), we obtain

$$\sum_{n=0}^{\infty}\overline{A}_{9\ell}(n)q^n=\frac{\varphi(-q^{9\ell})\varphi(-q^9)}{\varphi(-q^3)^4}\left(\varphi(-q^9)^2+2q\varphi(-q^9)\Omega(-q^3)+4q^2\Omega(-q^3)^2\right).$$

Extracting the coefficients of q^{3n} from both sides and then replacing q^3 by q, we get

$$\sum_{n=0}^{\infty}\overline{A}_{9\ell}(3n)q^n=\frac{\varphi(-q^{3\ell})\varphi(-q^3)^3}{\varphi(-q)^4}.$$

Using (2.3), we get

(3.26)

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{A}_{9\ell}(3n)q^n\\
&= \frac{\varphi(-q^{3\ell})\varphi(-q^9)^4}{\varphi(-q^3)^{13}}\left(\varphi(-q^9)^2+2q\varphi(-q^9)\Omega(-q^3)+4q^2\Omega(-q^3)^2\right)^4\\
&= \frac{\varphi(-q^{3\ell})\varphi(-q^9)^4}{\varphi(-q^3)^{13}}\Bigg(\varphi(-q^9)^8+8q\varphi(-q^9)^7\Omega(-q^3)+40q^2\varphi(-q^9)^6\Omega(-q^3)^2\\
&\quad +128q^3\varphi(-q^9)^5\Omega(-q^3)^3+304q^4\varphi(-q^9)^4\Omega(-q^3)^4+512q^5\varphi(-q^9)^3\Omega(-q^3)^5\\
&\quad +640q^6\varphi(-q^9)^2\Omega(-q^3)^6+512q^7\varphi(-q^9)\Omega(-q^3)^7+256q^8\Omega(-q^3)^8\Bigg).
\end{aligned}$$

Extracting the terms containing q^{3n+1} from both sides of (3.26) and then replacing q^3 by q, we get

(3.27)

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{A}_{9\ell}(9n+3)q^n\\
&= 8\frac{\varphi(-q^{\ell})\varphi(-q^3)^4}{\varphi(-q)^{13}}\Bigg(\varphi(-q^3)^7\Omega(-q)+38q\varphi(-q^3)^4\Omega(-q)^4+64q^2\varphi(-q^3)\Omega(-q)^7\Bigg).
\end{aligned}$$

Extracting the terms containing q^{3n+2} from both sides of (3.26) and then replacing q^3 by q, we obtain

(3.28)

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{A}_{9\ell}(9n+6)q^n\\
&= 8\frac{\varphi(-q^{\ell})\varphi(-q^3)^4}{\varphi(-q)^{13}}\Bigg(5\varphi(-q^3)^6\Omega(-q)^2+64q\varphi(-q^3)^3\Omega(-q)^5+32q^2\Omega(-q)^8\Bigg).
\end{aligned}$$

Now (1.14) and (1.15) follows immediately from (3.27) and (3.28), respectively.

From (2.5), (3.12) and (3.27), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\overline{A}_{9\ell}(9n+3)q^n &\equiv 8\frac{\varphi(-q^{\ell})\varphi(-q^3)^{11}\Omega(-q)}{\varphi(-q)^{13}}\\
&\equiv 8\frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}\\
&\equiv 8\frac{(q^6;q^6)_\infty^2}{(q;q)_\infty(q^3;q^3)_\infty} \pmod{16}
\end{aligned} \tag{3.29}$$

since $(q;q)_\infty^2 \equiv \left(q^2;q^2\right)_\infty \pmod 2$. Next using (2.6) in (3.29), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{A}_{9\ell}(9n+3)q^n\\ &\equiv \frac{(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^6;q^6)_\infty^2(q^{24};q^{24})_\infty^2}\\ &\quad + q\frac{(q^4;q^4)_\infty^5(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty} \pmod{16}.\end{aligned}$$

By proceeding in the same manner as in Theorem 1.7 in [4], we obtain (1.17) and (1.18).

Next from (2.5), (2.6), (3.12) and (3.28), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{A}_{9\ell}(9n+6)q^n\\ &\equiv 8\frac{\varphi(-q^\ell)\varphi(-q^3)^{10}\Omega(-q)^2}{\varphi(-q)^{13}}\\ &\equiv 8\Bigg(\frac{(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^6;q^6)_\infty^2(q^{24};q^{24})_\infty^2}\\ &\qquad + q\frac{(q^4;q^4)_\infty^5(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^2(q^{12};q^{12})_\infty}\Bigg)^2 \pmod{16}.\end{aligned}$$

Now extracting the terms containing q^{2n+1} from both sides, we obtain (1.16). This completes the proof.

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