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WEAKLY EQUIVARIANT CLASSIFICATION OF SMALL COVERS OVER A PRODUCT OF SIMPLICIES

Aslı Güçlükan İlhan and Sabri̇ Kaan Gürbüzer

Abstract. Given a dimension function ω, we introduce the notion of an ω-vector weighted digraph and an ω-equivalence between them. Then we establish a bijection between the weakly $(\mathbb{Z}/2)^n$-equivariant homeomorphism classes of small covers over a product of simplices $\Delta^{\omega(1)} \times \cdots \times \Delta^{\omega(m)}$ and the set of ω-equivalence classes of ω-vector weighted digraphs with m-labeled vertices, where n is the sum of the dimensions of the simplicies. Using this bijection, we obtain a formula for the number of weakly $(\mathbb{Z}/2)^n$-equivariant homeomorphism classes of small covers over a product of three simplices.

1. Introduction

Let P be a simple convex polytope of dimension n. A small cover over P is an n-dimensional smooth closed manifold M with a locally standard $\mathbb{Z}_2^n$-action whose orbit space is P. Two small covers over P are said to be Davis-Januskiewicz equivalent if there is a weakly $\mathbb{Z}_2^n$-equivariant homeomorphism between them covering the identity on P. For every small cover M over P, there is an associated function from the set of codimension one faces of P to $\mathbb{Z}_2^n$ called a characteristic function. The general linear group over $\mathbb{Z}_2$ acts freely on the set of characteristic functions over P by composition. It is well-known that there is a one-to-one correspondence between the orbit space of this action and the set of Davis-Januskiewicz equivalence classes of small covers over P [6].

The group of automorphisms of the face poset of P acts on the set of characteristic functions by composition. The orbit space of this right action is in a one-to-one correspondence with the set of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P [10]. In particular, there is a bijection between the double coset of this action and the left action of $\mathrm{GL}(n, \mathbb{Z}_2)$ and the set of

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weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P. Classifications of small covers over a specific polytope have been studied by many authors [1–3, 8].

In [3], Choi shows that there is a bijection between the set of Davis-Januszkiewicz equivalence classes of small covers over an n-cube and the set of acyclic digraphs with n-labeled vertices. The automorphism group of the face poset of an n-cube is the wreath product $\mathbb{Z}_2 \wr S_n$. The corresponding action of the group $\mathbb{Z}_2 \wr S_n$ on acyclic digraphs with n labeled vertices is given by local complementation and reordering of vertices [9]. The notions of a local complementation are generalized to digraphs by FonDerFlaass [7]. In [4], a one-to-one correspondence between the diffeomorphism classes of small covers over an n-cube and the acyclic digraphs with n-labeled vertices up to local complementation and slide operations is established.

In this paper, we give a classification of small covers over a product of simplices up to weakly $\mathbb{Z}_2^n$-equivariant homeomorphism in terms of digraphs. For this, we introduce the notion of an ω-vector weighted digraph for a given dimension function $\omega : \{1, 2, \ldots, m\} \to \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers. An ω-vector weighted digraph is a digraph with labeled vertices $\{v_1, \ldots, v_m\}$, where each edge directed from v_i has an associated non-zero vector in $\mathbb{Z}_2^{\omega(i)}$. It turns out that there is a bijection between the set of Davis-Januszkiewicz equivalence classes of small covers over a product of simplices, namely $\Delta^{\omega(1)} \times \cdots \times \Delta^{\omega(m)}$, and the set of acyclic ω-vector weighted digraphs. Hence the number of Davis-Januszkiewicz equivalence classes of small covers over a product of simplices is given by the formula (3.1) in Proposition 3.2. This formula was first obtained by Choi [3] by defining a surjection to the set of underlying digraphs and counting the sizes of the preimages.

In Section 4, we show that the action of the automorphism group of the face poset of a product of simplices corresponds to three operations on ω-vector weighted digraphs. The first two are reordering vertices that have the same image under the dimension function and permuting the weights of the edges from a fixed vertex. The third one is a generalization of the local complementation at a vertex v_i which we called (σ, k)-local complementation since a permutation $\sigma \in S_{\omega(i)}$ and an integer $1 \leq k \leq \omega(i)$ are also involved. We say that two ω-vector weighted digraphs are ω-equivalent if one can be obtained from the other by applying a sequence of these operations. Hence there is a bijection between the set of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over a product of simplices and the set of ω-equivalence classes of acyclic ω-vector weighted digraphs. Using this bijection, we give a formula for the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over a product of three simplices. These numbers are closely related with the number of permutations whose cycle decompositions have certain types. For this reason, we give some formulas involving the number of permutations of a given type in Section 5.

2. Preliminaries

Let P be a simple convex polytope of dimension n and $\mathcal{F}(P) = \{F_1, \ldots, F_m\}$ be the set of facets of P. A function $\lambda : \mathcal{F}(P) \to \mathbb{Z}_2^n$ satisfying the non-singularity condition that

$$F_{i_1} \cap \cdots \cap F_{i_n} \neq \emptyset \quad \Rightarrow \langle \lambda(F_{i_1}), \ldots, \lambda(F_{i_n}) \rangle = \mathbb{Z}_2^n$$

is called a characteristic function. For any $p \in P$, let $\mathbb{Z}_2^n(p)$ be the subgroup of $\mathbb{Z}_2^n$ generated by $\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})$, where the intersection $\bigcap_{j=1}^k F_{i_j}$ is the minimal face containing p in its relative interior. Then the manifold $M(\lambda) = (P \times \mathbb{Z}_2^n)/\sim$, where

$$(p, g) \sim (q, h) \text{ if } p = q \text{ and } g^{-1}h \in \mathbb{Z}_2^n(p),$$

is a small cover over P.

Theorem 2.1 ([6])**.** *For every small cover M over P, there is a characteristic function λ with a $\mathbb{Z}_2^n$-homeomorphism $M(\lambda) \to M$ covering the identity on P.*

Let $\Lambda(P)$ be the set of all characteristic functions on P. It is well-known that certain group actions on $\Lambda(P)$ can be used to classify small covers over P. The group $GL(n, \mathbb{Z}_2)$ acts freely on $\Lambda(P)$ by $g \cdot \lambda = g \circ \lambda$. For any λ, λ' in $\Lambda(P)$, the small covers $M(\lambda)$ and $M(\lambda')$ are Davis-Januszkiewicz equivalent if and only if there is an element $g \in GL(n, \mathbb{Z}_2)$ such that $g \cdot \lambda = \lambda'$. Therefore the set of Davis-Januszkiewicz equivalence classes of small covers over P corresponds bijectively to the coset $GL(n, \mathbb{Z}_2)\backslash\Lambda(P)$ by the above theorem.

Another action on $\Lambda(P)$ that gives such a classification is the action of the group of automorphisms of the poset set $(\mathcal{F}(P), \subset)$, which is denoted by $\mathrm{Aut}(\mathcal{F}(P))$. The group $\mathrm{Aut}(\mathcal{F}(P))$ acts $\Lambda(P)$ on right by $\lambda \cdot h = \lambda \circ h$. As shown in [10], for any λ, λ' in $\Lambda(P)$, there is a $\mathbb{Z}_2^n$-equivariant homeomorphism between small covers $M(\lambda)$ and $M(\lambda')$ if and only $\lambda \cdot h = \lambda'$ for some $h \in \mathrm{Aut}(\mathcal{F}(P))$. Hence there is a bijection between the orbit space of this action and the set of $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P [10]. By combining this with the above theorem, Lu and Masuda [10] obtain the following result.

Theorem 2.2 ([10])**.** *There is a one-to-one correspondence between the set of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P and the double coset $GL(n, \mathbb{Z}_2)\backslash\Lambda(P)/\mathrm{Aut}(\mathcal{F}(P))$.*

3. ω-vector weighted digraphs

In [5], Choi, Masuda and Suh introduce the notion of a vector matrix to associate a quasitoric manifold over a product of simplices. Given a dimension function $\omega : \{1, 2, \ldots, m\} \to \mathbb{N}$, a vector matrix of size m is a matrix $A = [\mathbf{v_{ij}}]$ whose entries in the i-th row are vectors in $\mathbb{Z}_2^{\omega(m)}$. We denote the k-th entry of $\mathbf{v_{ij}}$ by $(\mathbf{v_{ij}})_k$. Choi uses the vector matrices over $\mathbb{Z}_2$ to classify small covers over a product of simplices up to Davis-Januszkiewicz equivalences. More precisely, given a function $\omega : \{1, 2, \ldots, m\} \to \mathbb{N}$, let $M_\omega(m)$ be the set of all vector matrices $A = [\mathbf{v_{ij}}]$ of size m over $\mathbb{Z}_2$ whose entries in the i-th row are vectors in $\mathbb{Z}_2^{\omega(i)}$ satisfying the following condition: Every principal minor of $A_{k_1 \cdots k_p}$ is 1, where $A_{k_1 \cdots k_m}$ is the $(m \times m)$-matrix whose (i, j)-th entry is $(\mathbf{v_{ij}})_{k_i}$ for any $1 \leq k_i \leq \omega(i)$ and $1 \leq i \leq m$. In [3], Choi shows that there is a bijection between the set $M_\omega(m)$ and the Davis-Januszkiewicz equivalence classes of small covers over a product of simplices $P = \Delta^{\omega(1)} \times \cdots \times \Delta^{\omega(m)}$.

It is well-known that there is a one-to-one correspondence between acyclic digraphs with m labeled vertices and the set of $\mathbb{Z}_2$-matrices of size m all of whose principal minors are 1 [3, Theorem 2.2]. Here we introduce the notion of ω-vector weighted digraphs to generalize this bijection to $M_\omega(m)$.

Definition 3.1. Let $\omega : \{1, 2, \ldots, m\} \to \mathbb{N}$ be a function. A digraph with labeled vertices $\{v_1, \ldots, v_m\}$ is called ω-vector weighted if every edge (v_i, v_j) is assigned with a non-zero vector $\omega(v_i, v_j)$ in $\mathbb{Z}_2^{\omega(i)}$.

For convenience, we say that the weight of (v_i, v_j) is the zero vector if there is no edge from v_i to v_j. For the abuse of notation, we also denote the dimension of the weight vector of any edge directed from a vertex v by $\omega(v)$. Note that an ω-vector weighted digraph is indeed a vector weighted digraph when the weight vectors are equidimensional. An ω-vector weighted digraph is called acyclic if it does not contain any directed cycle.

Let $G = (V, E)$ be an ω-vector weighted digraph with labeled vertices $\{v_1, \ldots, v_m\}$. We define the adjacency matrix of G as an $(m \times m)$ ω-vector matrix whose (i, j)-the entry is $\omega(v_i, v_j)$. We denote it by $A_\omega(G)$. For any $1 \leq k_i \leq \omega(i)$ and $1 \leq i \leq m$, $\big(A_\omega(G)\big)_{k_1 \cdots k_p}$ is an adjacency matrix of some subgraph of the underlying acyclic digraph and hence we have the following.

Proposition 3.2. *There is a one-to-one correspondence between the set of acyclic ω-vector weighted digraphs with labeled vertices $v_1, \ldots, v_m$ and $M_\omega(m)$. In particular,*

$$|M_\omega(m)| = \sum_{G \in \mathcal{G}_m} \prod_{v_i \in V(G)} (2^{\omega(i)} - 1)^{\mathrm{outdeg}(v_i)}, \tag{3.1}$$

where $\mathrm{outdeg}(v_i)$ *is the number of edges directed from* v_i.

This formula was first obtained by Choi [3] by defining a surjection to the set of underlying digraphs whose preimages are the set of possible weights.

The local complementation of a digraph $G = (V(G), E(G))$ at vertex v is the digraph $G * v$, where $V(G * v) = V(G)$ and $E(G * v)$ is the symmetric difference of the sets $E(G)$ and $\{(u, w) : (u, w) \in N_G^-(v) \times N_G^+(v)\}$. Here, $N_G^+(v)$ denotes the set of all out-neighbors of v and $N_G^-(v)$ is the set of all in-neighbors of v. This notion can be easily generalized to ω-vector weighted digraphs by letting $E(G * v) = E(G) \cup N_G^-(v) \times N_G^+(v)$, where the weight of the edge (u, w) is $\omega(u, w) + \omega(u, v)$ if $(u, w) \in N_G^-(v) \times N_G^+(v)$ and is $\omega(u, w)$, otherwise. Note that if the sum $\omega(u,w)+\omega(u,v)$ is the zero vector, this means that there is no edge from u to w in $G*v$. If we assume that weights of the edges of the digraphs (without weight) are $1 \in \mathbb{Z}_2$, this definition agrees with the definition of the local complementation of a digraph. A permutation $\sigma \in S_{\omega(v)}$ acts on ω-vector weighted digraphs that contain v as a vertex by permuting the coordinates of the weights of the edges from v. Now we introduce two more generalizations of a local complementation which also permutes the coordinates of the associated weights.

Definition 3.3. Let v be a vertex of a ω-vector weighted digraph G and σ be a permutation in $S_{\omega(v)}$. The σ-local complementation of G at vertex v is obtained by permuting the weights of edges from v in $G * v$ by σ. We denote the obtained ω-vector weighted digraph by $G *_\sigma v$.

Definition 3.4. Let v be a vertex of a ω-vector weighted digraph G and $\sigma \in S_{\omega(v)}$. For any $1 \le k \le \omega(v)$, the (σ,k)-local complementation of G at vertex v is the ω-vector weighted digraph $G \underset{(\sigma,k)}{*} v$, where $V(G \underset{(\sigma,k)}{*} v) = V(G)$ and the edge set of $G \underset{(\sigma,k)}{*} v$ is the union of the sets $E(G)$ and $\{(u,w) \in N_G^-(v) \times N_G^+(v) : (\omega(v,w))_k = 1\}$. The weight of the edge (u,w) is given by

i) $\omega(u,w)+\omega(u,v)$ if $(u,w) \in N_G^-(v) \times N_G^+(v)$ and $(\omega(v,w))_k = 1$,
ii) $\sigma \cdot \omega(v,w)$ if $u = v$ and $(\omega(v,w))_k = 0$,
iii) $\sigma \cdot \omega(v,w) + e_{\sigma^{-1}(k)}$ if $u = v$ and $(\omega(v,w))_k = 1$, where e_i is the vector all of whose coordinates are 1 except the i-th one,
iv) $\omega(u,w)$, otherwise.

Example 3.5. Let $\omega : \{1,2,3,4\} \to \mathbb{N}$ be the dimension function defined by $\omega(1) = 2$, $\omega(2) = \omega(3) = \omega(4) = 3$. Let G be the ω-vector weighted digraph given in Figure 1.

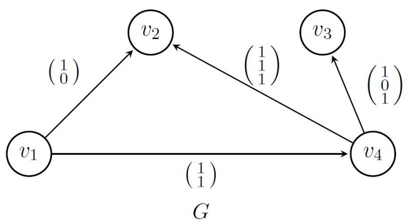

FIGURE 1. The ω-weighted acyclic digraph.

For $\sigma = (123) \in S_3$, the σ-local complementation of G and the $(\sigma,2)$-local complementation of G at vertex v_4 are given in Figure 2.

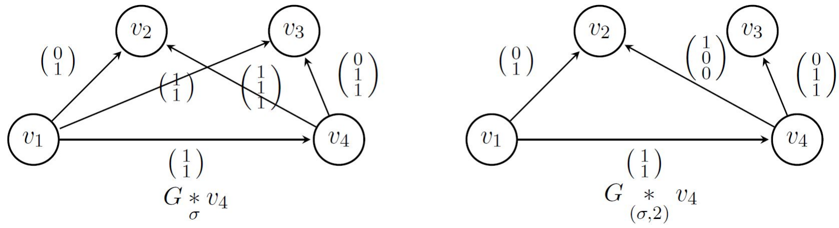

FIGURE 2. A σ-local complementation and a $(\sigma, 2)$-local complementation.

4. Small covers over a product of simplices

A. Güçlükan İlhan [9] shows that there is a bijection between the weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over an n-cube and non-isomorphic acyclic digraphs with n-labeled vertices up to local complementation. In this section, we generalize this result to a product of simplices using Theorem 2.2. The automorphism group of a product of simplices depends not only on the dimension of the simplices but also the number of equidimensional simplices appearing in the product (see [1]). For this reason, we let

$$P = \prod_{i=1}^{l} P_i, \text{ where } P_i = \underbrace{\Delta^{n_i} \times \cdots \times \Delta^{n_i}}_{m_i},$$

with $1 \leq n_1 < n_2 < \cdots < n_l$ and $\sum_{i=1}^{l} n_i m_i = n$. Here the set of facets of P_i is

$$\{f^i_{j,k} = \Delta^{n_i} \times \cdots \times \Delta^{n_i} \times \underbrace{\tilde{f}^i_k}_{\text{j-th}} \times \Delta^{n_i} \times \cdots \times \Delta^{n_i} : 1 \leq k \leq n_i + 1,\ 1 \leq j \leq m_i\},$$

where $\{\tilde{f}^i_0, \ldots, \tilde{f}^i_{n_i}\}$ is the set of facets of the simplex Δ^{n_i}. Therefore the set of faces of P is given by

$$\mathcal{F}(P) = \{F^i_{j,k} : 1 \leq k \leq n_i + 1,\ 1 \leq j \leq m_i,\ 1 \leq i \leq l\},$$

where $F^i_{j,k} = P_1 \times \cdots \times P_{i-1} \times f^i_{j,k} \times P_{i+1} \times \cdots \times P_l$. Note that, there are $(n + m)$-facets, where $m = \sum_{i=1}^{l} m_i$. The automorphism group of P is $\prod_{i=1}^{l} (S_{n_i+1} \wr S_{m_i})$, where $S_{n_i+1} \wr S_{m_i}$ is the wreath product of $S_{n_{i+1}}$ with S_{m_i}. Here, $\sigma \in S_{n_{i+1}}$ sends $F^i_{j,k}$ to $F^i_{j,\sigma(k)}$ and fixes other facets. The permutation $\mu \in S_{m_i}$ sends $F^i_{j,k}$ to $F^i_{\mu(j),k}$ and fixes the others (see Lemma 3.2 in [1]).

Let $M^i_{j,k} = \left(\sum_{\alpha=1}^{i-1} m_\alpha n_\alpha\right) + (j-1)n_i + k$ and $N^i_j = \left(\sum_{\alpha=1}^{i-1} m_\alpha\right) + j$, where $1 \leq i \leq l$, $1 \leq j \leq m_i$ and $1 \leq k \leq n_i$. To every characteristic function λ over P, one can associate an $(n \times (n+m))$ matrix Λ whose $M^i_{j,k}$-th column is $\lambda(F^i_{j,k})$ and $(n + N^i_j)$-th column is $\lambda(F^i_{j,n_i+1})$. By reordering the facets and choosing a basis, we can choose a representative of the orbit of Λ of the form $(I_n|\Lambda_*)$, where Λ_* is an $(n \times m)$-matrix. Following Choi [3], we call Λ_* the reduced submatrix of λ. As shown in [3], Λ_* can be seen as an element of $M_\omega(m)$, where $\omega(t) = n_i$ if $t = N^i_j$ for some $1 \le j \le n_i$. We call the function ω defined in this way the dimension function of P. Since there is a bijection between the Davis-Januszkiewicz equivalence classes of small covers over P and the reduced submatrices, we have the following result by Proposition 3.2.

Corollary 4.1. *The Davis-Januszkiewicz equivalence classes of small covers over P are in a one-to-one correspondence with the acyclic ω-vector weighted digraphs with labeled m vertices.*

An arbitrary element $g \in \mathrm{Aut}(\mathcal{F}(P))$ can be written as a product of elements of the form

$$(1_1, \ldots, 1_{i-1}, (\mathrm{id}^i_1, \ldots, \mathrm{id}^i_{j-1}, \sigma^i_j, \mathrm{id}^i_{j+1}, \ldots, \mathrm{id}^i_{m_i}; \mathrm{id}^i), 1_{i+1}, \ldots, 1_l)$$

and

$$(1_1, \ldots, 1_{i-1}, (\mathrm{id}^i_1, \ldots, \mathrm{id}^i_{m_i}; \mu^i), 1_{i+1}, \ldots, 1_l),$$

where $\sigma^i_j \in S_{n_i+1}$ and $\mu^i \in S_{m_i}$, that we also denote by σ^i_j and μ^i. Here 1_i, id^i_j, and id^i denote the identity elements in $S_{n_i+1} \wr S_{m_i}$, S_{n_i+1}, and S_{m_i}, respectively. Let $G = (V, E)$ be an acyclic ω-vector weighted digraph with labeled vertices $v_1, \ldots, v_m$. Clearly, the corresponding action of μ^i sends G to an acyclic ω-vector weighted digraph obtained by reordering the vertices $\{v_p : N^i_1 \le p \le N^i_{m_i}\}$ by $(\mu^i)^{-1}$, where the weight of the edge (v_p, v_q) in the resulting graph is $\omega(v_{\mu(p)}, v_{\mu(q)})$.

When all the simplices are 1-dimensional, σ^i_j is an element of the cyclic group of order 2 and when it is non-trivial, the corresponding action on the acyclic digraphs is the local complementation at vertex $v_{N^i_j}$ [9]. Let $^- : S_{n_i+1} \to S_{n_i}$ be defined by $\overline{\sigma}^i_j(t) = \sigma^i_j(t)$ if $\sigma^i_j(t) \ne n+1$, and $\overline{\sigma}^i_j(t) = \sigma^i_j(n+1)$, otherwise.

Lemma 4.2. *If σ^i_j fixes $n_i + 1$, then σ^i_j acts by permuting the weights of the edges from $v_{N^i_j}$. If $\sigma^i_j(n_i+1) \ne n_i + 1$, then σ^i_j act as $(\overline{\sigma}^i_j, \sigma^i_j(n_i+1))$-local complementation at the vertex $v_{N^i_j}$.*

Proof. For simplicity, we denote σ^i_j by σ. Let $[\mathbf{0}]$ and $[\mathbf{1}]$ denote the vectors whose coordinates are all 0 and 1, respectively and $\mathbf{I} = [\delta_{ij}]$ be an ω-weighted vector matrix, where $\delta_{ij} = [\mathbf{1}]$ if $i = j$ and $[\mathbf{0}]$, otherwise. For a given an acyclic ω-vector weighted digraph G, the reduced submatrix of the associated characteristic function is given by $\Lambda_* = A_\omega(G) + \mathbf{I}$. Assume that $\sigma \cdot [I_{n\times n} | \Lambda_*] = [P_\sigma | Q_\sigma]$. Then $A_\omega(G \cdot \sigma)$ is equal to $P_\sigma^{-1} Q_\sigma + \mathbf{I}$ since $([P_\sigma | Q_\sigma]) = ([I_n | P_\sigma^{-1} \cdot Q_\sigma])$. Here P_σ^{-1} is the $\mathbb{Z}_2$-inverse of P_σ.

If $\sigma(n_i + 1) = \sigma(n_i + 1)$, then $Q_\sigma = \Lambda_*$ and P_σ is the block diagonal matrix in which all the blocks are identity matrices of the corresponding dimensions except the N^i_j-th block that is the permutation matrix of $\overline{\sigma}$. In this case, the inverse of P_σ is the block diagonal matrix of the same form whose N^i_j-th block is the permutation matrix of $\overline{\sigma}^{-1}$. Therefore $P_\sigma^{-1} Q_\sigma$ is the vector matrix obtained by permuting the $M^i_{j,1}, \dots, M^i_{j,n_i}$-th rows of Λ_* by σ. Hence it acts on the corresponding digraph by permuting the weights of the edges from $v_{N^i_j}$ by σ.

Now suppose that $\sigma(n_i+1) \neq n_i+1$. Let $A_\omega(G) = [\mathbf{v}_{\alpha\beta}]$. Then, we have

$$(P_\sigma)_{pq} = \begin{cases} 1 & \text{if } p = q = M^a_{b,c} \text{ with } (a,b) \neq (i,j), \\ 1 & \text{if } p = M^i_{j,\sigma(k)} \text{ and } q = M^i_{j,k},\ \sigma(k) \neq n_i+1, \\ (\mathbf{v}_{\mathbf{N^a_b},\mathbf{N^i_j}})_c & \text{if } p = M^a_{b,c},\ q = M^i_{j,k}, \text{ and } \sigma(k) = n_i+1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(Q_\sigma)_{rs} = \begin{cases} 1 & \text{if } s = N^i_j,\ r = M^i_{j,\sigma(n_i+1)}, \\ 0 & \text{if } s = N^i_j,\ r \neq M^i_{j,\sigma(n_i+1)}, \\ (\mathbf{v}_{\mathbf{N^a_b},\mathbf{s}})_c & \text{if } r = M^a_{b,c},\ (a,b) \neq (i,j), \end{cases}$$

for $1 \leq p,q,r \leq n$ and $1 \leq s \leq m$. Since $(\mathbf{v_{pp}})_k = 1$ for all p, the $\mathbb{Z}_2$-inverse of P_σ is given by

$$(P^{-1}_\sigma)_{pq} = \begin{cases} 1 & \text{if } p = q = M^a_{b,c} \text{ with } (a,b) \neq (i,j), \\ 1 & \text{if } p = M^i_{j,\sigma^{-1}(k)} \text{ and } q = M^i_{j,k},\ k \neq \sigma(n_i+1), \\ (\mathbf{v}_{\mathbf{N^a_b},\mathbf{N^i_j}})_c & \text{if } p = M^a_{b,c},\ q = M^i_{j,k}, \text{ and } \sigma(n_i+1) = k, \\ 0 & \text{otherwise.} \end{cases}$$

Since $A_\omega \in M_\omega(m)$, $P^{-1}_\sigma Q_\alpha$ is the matrix $[\mathbf{v}'_{\mathbf{pq}}]$, where

(4.1)
$$(\mathbf{v}'_{\alpha\beta})_k = \begin{cases} 1 & \text{if } \alpha = \beta, \\ (\mathbf{v}_{\mathbf{N^i_j},\beta})_{\sigma(n_i+1)} & \text{if } \alpha = N^i_j,\ k = \sigma^{-1}(n_i+1), \\ (\mathbf{v}_{\mathbf{N^i_j},\beta})_{\sigma(n_i+1)} + (\mathbf{v}_{\mathbf{N^i_j},\beta})_{\sigma(k)} & \text{if } \alpha = N^i_j,\ k \neq \sigma^{-1}(n_i+1), \\ (\mathbf{v}_{\alpha,\mathbf{N^i_j}})_k & \text{if } \alpha \neq N^i_j,\ \beta = N^i_j, \\ (\mathbf{v}_{\alpha,\beta})_k + (\alpha, \mathbf{v}_{\mathbf{N^i_j}})_k (\mathbf{v}_{\mathbf{N^i_j},\beta})_{\sigma(n_i+1)} & \text{otherwise.} \end{cases}$$

Therefore, by subtracting $\mathbf{I}$, we obtain the adjacency matrix of the $(\overline{\sigma}^i_j, \sigma^i_j(n_i+1))$-local complementation of G at the vertex $v_{N^i_j}$. □

Remark 4.3. Note that if we define $(\mathbf{v}_{\alpha\beta})_{n_i+1}$ to be zero, then the formula (4.1) also gives the adjacency matrix of $G \cdot \sigma$ when $\sigma(n_i+1) = n_i+1$. With this assumption, one can also express the entries of the adjacency matrix of $G \cdot \sigma$ explicitly when $\sigma = \prod_{i,j} \sigma^i_j$. Indeed, the k-th coordinate of the (α,β)-th entry of $A_\omega(G \cdot \sigma)$ is given by

$$\begin{cases} 0, \text{ if } \alpha = \beta; \\ (\mathbf{v}_{\alpha\beta})^\sigma + \sum_{(a_1,\dots,a_p)\in S_{\alpha,\beta}} (\mathbf{v}_{\alpha,\mathbf{a_1}})^\sigma \cdot (\mathbf{v}_{\mathbf{a_1},\mathbf{a_2}})^\sigma \cdots (\mathbf{v}_{\mathbf{a_{p-1}},\mathbf{a_p}})^\sigma \cdot (\mathbf{v}_{\mathbf{a_p},\beta})^\sigma, \\ \text{ if } \alpha = N_b^a \text{ and } \sigma_b^a(k) = n'_a + 1; \\ (\mathbf{v}_{\alpha\beta})^\sigma + (\mathbf{v}_{\alpha\beta})^\sigma_k \\ + \sum_{(a_1,\dots,a_p)\in S_{\alpha,\beta}} \left((\mathbf{v}_{\alpha,\mathbf{a_1}})^\sigma + (\mathbf{v}_{\alpha,\mathbf{a_1}})^\sigma_k\right) \cdot (\mathbf{v}_{\mathbf{a_1},\mathbf{a_2}})^\sigma \cdots (\mathbf{v}_{\mathbf{a_{p-1}},\mathbf{a_p}})^\sigma \cdot (\mathbf{v}_{\mathbf{a_p},\beta})^\sigma, \\ \text{otherwise.} \end{cases}$$

Here $S_{\alpha,\beta} = \{(a_1,\dots,a_p) : a_k \neq a_l, a_k \in \{1,\dots,N_m^l\} \setminus \{\alpha,\beta\}\}$ and the entry $(\mathbf{v_{i,j}})_{\sigma_b^a(k)}$ is denoted by $(\mathbf{v_{i,j}})^\sigma_k$ and the entry $(\mathbf{v_{i,j}})_{\sigma_b^a(n_a+1)}$ is denoted by $(\mathbf{v_{i,j}})^\sigma$, where $i = N_b^a$.

Definition 4.4. We say two ω-vector weighted digraphs are ω-equivalent if one is obtained (up to graph isomorphisms) from the other one by applying a sequence of the following operations:

(1) Reordering vertices whose images under the dimension function are the same,
(2) Permutation of the weights of the edges from vertex v by an element of $S_{\omega(v)}$,
(3) (σ, k)-local complementation.

Then the following theorem directly follows from the above lemma.

Theorem 4.5. *There is a bijection between the weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over $\Delta^{n_1} \times \cdots \times \Delta^{n_m}$ and the set of ω-equivalence classes of ω-vector weighted digraphs with m-labeled vertices, where ω is defined by $\omega(i) = n_i$ and $n = n_1 + \cdots + n_m$.*

In the following example, we count this number for the product of two simplices and in the last section we consider the case where digraphs have 3-vertices.

Example 4.6. Let $P = \Delta^{n_1} \times \Delta^{n_2}$ and $\omega : \{1,2\} \to \mathbb{N}$ be the dimension function of P. Then an acyclic ω-vector weighted digraph is of the one of the types given in Figure 3,

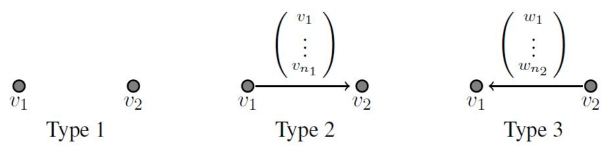

FIGURE 3. Types of the acyclic ω-vector weighted digraphs on two vertices.

where $(v_1, \ldots, v_{n_1}) \in \mathbb{Z}_2^{n_1} \setminus \{\mathbf{0}\}$ and $(w_1, \ldots, w_{n_2}) \in \mathbb{Z}_2^{n_2} \setminus \{\mathbf{0}\}$. Note that if two acyclic ω-vector weighted digraphs on two vertices are ω-equivalent, then they have the same type. Moreover the operation (3) can be reduced the one that replaces the zeros and ones in the weights of vertices from the fixed vertex except the one in the fixed coordinate. Let us first consider the case $n_1 \neq n_2$. In this case, two ω-weighted digraphs G_1 and G_2 of Type 2 are ω-equivalent if and only if $u_1 = u_2$ or $u_1 + u_2 = n_1 - 1$, where u_i is the number of zero coordinates of the weight vector of the edge (v_1, v_2) in G_i for $i = 1, 2$. Therefore the number of equivalence classes of Type 2 is $\left\lfloor \frac{n_1+1}{2} \right\rfloor$. This is also true for the acyclic ω-vector weighted digraphs of Type 3. So there are $1 + \left\lfloor \frac{n_1+1}{2} \right\rfloor + \left\lfloor \frac{n_2+1}{2} \right\rfloor$ ω-equivalence classes of acyclic ω-vector weighted digraphs with labeled vertices v_1, v_2. Hence there are $1 + \left\lfloor \frac{n_1+1}{2} \right\rfloor + \left\lfloor \frac{n_2+1}{2} \right\rfloor$ weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over $P = \Delta^{n_1} \times \Delta^{n_2}$ when $n_1 \neq n_2$.

When $n_1 = n_2$, the reordering of the vertices v_1 and v_2 is also allowed. Therefore the number of weakly $\mathbb{Z}_2^{2n_1}$-equivariant homeomorphism classes of small covers over P is $1 + \left\lfloor \frac{n_1+1}{2} \right\rfloor$ in this case.

5. Some results on the number of permutations of certain types

In this section, we give some results on the number of permutations of certain types that are used in the next section to find a formula for the number of acyclic ω-vector weighted digraphs on labeled 3 vertices up to ω-equivalence. It is well-known that the number of permutations of n elements with m cycle is given by the unsigned Stirling number of the first kind denoted by $c(n, m)$. The Stirling number of the first kind is originally defined as the coefficient of the expansion of the rising factorial $x^{\bar{n}}$ into powers of x, that is,

$$x^{\bar{n}} = \sum_{m=0}^{n} c(n, m) x^m. \tag{5.1}$$

They also satisfy the following recurrence relations

$$\begin{aligned} c(n, m) &= \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!} c(n-k, m-1), \\ c(n, m) &= c(n-1, m-1) + (n-1)c(n-1, m). \end{aligned}$$

Let us denote by $c_d(n, m)$ the permutations of n elements with m cycles all of which has length divisible by d. Clearly, when n is not divisible by d, $c_d(n, m)$ is zero. Now let $n = dt$. If the cycle containing dt has length dk, then there are $\binom{dt-1}{dk-1}(dk-1)! = \frac{(dt-1)!}{(dt-dk)!}$ ways to choose this cycle, and $c_d(dt - dk, m - 1)$ ways to choose permutations consists of the $dt - dk$ elements outside that cycle.

Therefore, we have the following relation

$$c_d(dt,m) = \sum_{k=1}^{t} \frac{(dt-1)!}{(dt-dk)!} c_d(dt-dk, m-1).$$

Another way to calculate the number $c_d(dt, m)$ is to use the following recurrence relation

$$c_d(dt+d,m) = (dt+1)^{\overline{d-1}} c_d(dt, m-1) + (dt)^{\overline{d}} c_d(dt,m). \tag{5.2}$$

Indeed we can divide the permutations of $dt+d$ elements whose cycles are all divisible by d into two types: the one in which the cycle containing $dt+d$ has length d and the others. Note that there are $(dt+1)^{\overline{d-1}} c_d(dt, m-1)$ permutations of the first type since there are $(dt+d-1)(dt+d-2)\cdots(dt+1) = (dt+1)^{\overline{d-1}}$ ways to choose the other elements of the cycle containing $dt+d$. On the other hand, when the cycle containing $dt+d$ has length greater than d by deleting the element $dt+d$ and the first $d-1$ elements coming next to it, we obtain a permutation of dt elements that consists of exactly m cycles of length divisible by d and vice a versa. Here we can choose the elements that go next to $dt+d$ in the cycle containing $dt+d$ in $(dt+d-1)(dt+d-2)\cdots(dt+1)$ ways. This leaves dt remaining elements. For any permutation of the remaining elements of the same type, we can choose one of these elements, say x and place the d elements in order to the left of x. Therefore the number of permutations of the second type is $(dt+d-1)(dt+d-2)\cdots(dt)c_d(dt,m)$.

Lemma 5.1. *For every $t \in \mathbb{N}$, we have the following relation*

$$(x)^{\overline{t}} = \frac{t!}{(dt)!} \sum_{m=0}^{t} c_d(dt,m)(xd)^m. \tag{5.3}$$

Proof. We prove by induction on t. The case $t=1$ is trivial. Let A_t denote the right hand side of the equation (5.3). Since $c_d(dt,-1) = c_d(dt, t+1) = 0$, by (5.2) we have

$$\begin{aligned} A_{t+1} &= \frac{t!}{d(dt)!} \sum_{m=0}^{t+1} c_d(dt, m-1)(xd)^m + \frac{t(t!)}{(dt)!} \sum_{m=0}^{t+1} c_d(dt,m)(xd)^m \\ &= \frac{x \cdot t!}{(dt)!} \sum_{m=0}^{t} c_d(dt,m)(xd)^m + \frac{t(t!)}{(dt)!} \sum_{m=0}^{t} c_d(dt,m)(xd)^m \\ &= (x+t) \frac{t!}{(dt)!} \sum_{m=0}^{t} c_d(dt,m)(xd)^m. \end{aligned}$$

Since $(x+t)x^{\overline{t}} = x^{\overline{t+1}}$, the result follows by induction. □

Let $c(n,m,e)$ denote the number of permutation of n-elements with m-cycles exactly e of them have even lengths. Considering the cases where the cycle containing n is even or odd, one can easily obtain the following formula

$$\begin{aligned} c(n,m,e) = & \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-1)!}{(n-2t-1)!} c(n-2t-1, m-1, e) \\ & + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-1)!}{(n-2t)!} c(n-2t, m, e-1). \end{aligned}$$

Another recurrence relation including these numbers is

$$\begin{aligned} c(n,m,e) = & \, c(n-1, m-1, e) + (n-1)c(n-2, m-1, e-1) \\ & + (n-1)(n-2)c(n-2, m, e). \end{aligned} \tag{5.4}$$

The above relation can be proved as above by considering the cases where the length of the cycle containing n is 1, 2 or ≥ 2. The following result easily follows by induction substituting $e = 0$ in the above formula.

Lemma 5.2. *For every $n \in \mathbb{N}$, $\sum_{m=1}^{n} 2^m c(n, m, 0) = 2(n!)$.*

We also need the following results to count the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over a product of three simplices.

Lemma 5.3. *For every $n \in \mathbb{N}$,*

$$\sum_{m=1}^{n} 2^m \sum_{e=1}^{m} c(n, m, e) = (n-1)(n!).$$

Proof. The result follows from the relation $\sum_{e=1}^{m} c(n, m, e) = c(n, m) - c(n, m, 0)$ and the corresponding relations for $c(n, m)$ and $c(n, m, 0)$. □

Proposition 5.4. *For every $n \in \mathbb{N}$, we have the following formula*

$$\sum_{m=1}^{n} 2^m \sum_{e=1}^{m} 2^e c(n, m, e) = \begin{cases} (2k)!(k^2 + 2k - 1) & \text{when } n = 2k, \\ (2k+1)!k(k+3) & \text{when } n = 2k+1. \end{cases}$$

Proof. We prove by induction. The case $n = 1$ is trivial. Suppose that the above equations holds for all integers less than n. Using the equation (5.4) and Lemma 5.2, we have

$$\begin{aligned} & \sum_{m=1}^{n} \sum_{e=1}^{m} 2^{m+e} c(n, m, e) \\ = & \sum_{m=1}^{n-1} \sum_{e=1}^{m} 2^{m+e} c(n-1, m, e) + 4(n-1) \sum_{m=1}^{n-2} 2^m c(n-2, m, 0) \\ & + \Big(4(n-1) + (n-1)(n-2)\Big) \Big(\sum_{m=1}^{n-2} \sum_{e=1}^{m} 2^{m+e} c(n-2, m, e) \Big) \end{aligned}$$

$$= 2\sum_{m=1}^{n-1} 2^m \sum_{e=1}^{m} 2^e c(n-1,m,e) + 8(n-1)!$$
$$+ (n-1)(n+2)\Big(\sum_{m=1}^{n-2} 2^m \sum_{e=1}^{m} 2^e c(n-2,m,e)\Big).$$

When $n = 2k$, one has

$$\begin{aligned}
&\sum_{m=1}^{n} 2^m \sum_{e=1}^{m} 2^e c(n,m,e) \\
&= 2(2k-1)!(k-1)(k+2) + 8(2k-1)! \\
&\quad + (2k-1)(2k+2)(2k-2)!((k-1)^2 + 2(k-1) - 1) \\
&= (2k)!(k^2 + 2k - 1)
\end{aligned}$$

by induction. The case $n = 2k+1$ can be shown similarly. □

As an immediate consequences of above results, we have the following.

Corollary 5.5. *For every $n \in \mathbb{N}$, we have*

$$\sum_{m=1}^{n} 2^m \sum_{e=1}^{m} (2^e - 1)c(n,m,e) = (n!)\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor.$$

6. Products of three simplices

In this section, we give a formula for the number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over $P = \Delta^{n_1} \times \Delta^{n_2} \times \Delta^{n_3}$, where $n_1 \le n_2 \le n_3$ and $n_1 + n_2 + n_3 = n$. By Theorem 4.5, the number of such classes is equivalent to the number of ω-equivalence classes of acyclic ω-vector weighted digraphs on 3-labeled vertices $\{v_1, v_2, v_3\}$, where $\omega(i) = n_i$, $1 \le i \le 3$. Since this number also depends on the number of vertices whose images under the dimension function are the same, we need to consider the cases where $n_1 < n_2 < n_3$, $n_1 = n_2 < n_3$ and $n_1 = n_2 = n_3$, separately.

We first consider the case where $n_1 < n_2 < n_3$. The other cases follow easily from this one. Note that there are 25 different acyclic digraph with labeled vertices $\{v_1, v_2, v_3\}$ and hence we can classify the acylic ω-vector weighted digraphs as shown in Figure 4. By duality, it suffices to understand the number of the ω-equivalence classes of Type 1, Type 2, Type 8, Type 11, Type 17 and Type 23. As a set, the equivalence class of the ω-vector weighted digraphs of the type 1, 2, 8 or 17 consists of digraphs of the same type, respectively. However, an ω-vector weighted digraph of Type 11 can be ω-equivalent to that of Type 23. There is only one ω-vector weighted digraph of Type 1. As discussed in Example 4.6, there are $\lfloor \frac{n_1+1}{2} \rfloor$ different ω-equivalence classes of Type 2. Clearly, two ω-vector weighted digraphs of Type 17 are ω-equivalent if and only if the numbers of zero coordinates of the weight vectors v and w in each of the graphs are either the same or their sum is $n_2 - 1$ and $n_3 - 1$, respectively. Therefore the number of ω-equivalence classes of Type 17 is $\lfloor \frac{n_2+1}{2} \rfloor \cdot \lfloor \frac{n_3+1}{2} \rfloor$.

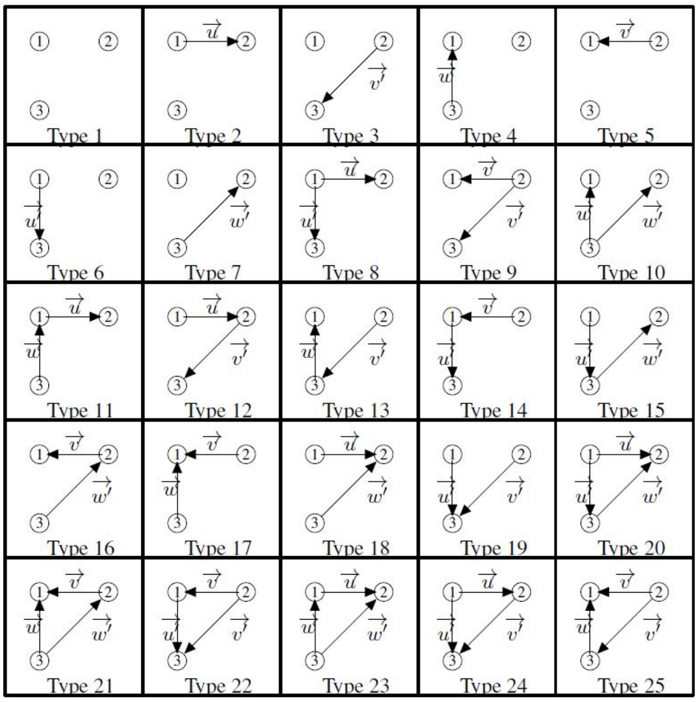

FIGURE 4. Types of ω-vector weighted digraphs with 3 labeled vertices.

Lemma 6.1. *The number of ω-equivalence classes of Type 8 is equal to $f(n_1)$, where*

$$f(n) = \begin{cases} \frac{2k^3+9k^2+k}{6} & \textit{when } n = 2k, \\ \frac{(k+1)(k^2+5k+3)}{3} & \textit{when } n = 2k+1. \end{cases}$$

Proof. We use Burnside's lemma. Let $X = \left(\mathbb{Z}_2^{n_1} \setminus \{\mathbf{0}\}\right) \times \left(\mathbb{Z}_2^{n_1} \setminus \{\mathbf{0}\}\right)$. Then the set of ω-equivalence classes of Type 8 is in a one-to-one correspondence with the orbit space of the action of S_{n_1+1} on X defined by

$$\sigma \cdot (v, w) = \begin{cases} (\overline{\sigma}(v), \overline{\sigma}(w)) & \text{if } v \in \mathcal{V}_\sigma \text{ and } w \in \mathcal{V}_\sigma, \\ (\overline{\sigma}(v) + e, \overline{\sigma}(w)) & \text{if } v \notin \mathcal{V}_\sigma \text{ and } w \in \mathcal{V}_\sigma, \\ (\overline{\sigma}(v), \overline{\sigma}(w) + e) & \text{if } v \in \mathcal{V}_\sigma \text{ and } w \notin \mathcal{V}_\sigma, \\ (\overline{\sigma}(v) + e, \overline{\sigma}(w) + e) & \text{otherwise,} \end{cases}$$

where $e = e_{\sigma^{-1}(n_1+1)}$ and $\mathcal{V}_\sigma = \{v \in \mathbb{Z}_2^{n_1} \setminus \{\mathbf{0}\} : (v)_{\sigma(n_1+1)} = 0 \text{ if } \sigma(n_1+1) \neq n_1 + 1\}$. Let $\lambda_1^{m_1} \cdots \lambda_k^{m_k}$ denote the cycle type of σ with $\lambda_1 < \lambda_2 < \cdots < \lambda_k$

and $m = \sum_i m_i$. If σ fixes $n_1 + 1$, then (v, w) is fixed by σ if and only if the values of the coordinates of v and w corresponding to the same cycle are the same. Therefore the number of fixed points of σ is $(2^{m-1} - 1)^2$ when $\sigma(n_1 + 1) = n_1 + 1$ since the cycle decomposition of $\overline{\sigma}$ is $\lambda_1^{m_1-1} \cdots \lambda_k^{m_k}$ and at least one of the coordinates of v and w are non-zero.

Suppose that $\sigma(n_1 + 1) \neq n_1 + 1$. In this case the number of disjoint cycles of σ and $\overline{\sigma}$ are the same. Note that if $\overline{\sigma}(v) + e = v$ for some non-zero vector v in $\mathbb{Z}_2^{n_1}$, then λ_i is even for $1 \leq i \leq k$. Therefore if there is a cycle of odd length in the cycle decomposition of σ and (v, w) is fixed by σ, then $(v)_{\sigma(n_1+1)} = (w)_{\sigma(n_1+1)} = 0$. In this case, we have $(v, w) = (\overline{\sigma}(v), \overline{\sigma}(w))$. Since all the coordinates of v and w corresponding to cycle containing $\sigma(n_1+1)$ are 0, the number of fixed points of σ is $(2^{m-1}-1)^2$ when λ_i is odd for some i. On the other hand the number of elements $v \in \mathcal{V}_\sigma$ satisfying the condition $\overline{\sigma}(v)+e = v$ is 2^{m-1}. Therefore if all the λ_i's are even, then σ fixes $(2^{m-1} - 1)^2$ elements in $\mathcal{V}_\sigma \times \mathcal{V}_\sigma$, $(2^{m-1})^2$ elements in $\mathcal{V}'_\sigma \times \mathcal{V}'_\sigma$ and $2(2^{m-1})(2^{m-1} - 1)$ elements in $\mathcal{V}_\sigma \times \mathcal{V}'_\sigma \cup \mathcal{V}'_\sigma \times \mathcal{V}_\sigma$. Therefore there are $(2^m - 1)^2$ elements of X fixed by σ when σ consists of cycles of even lengths only.

Therefore the number of ω-equivalence classes of Type 8 is given by the following formula

$$\frac{1}{(n_1+1)!}\left(\sum_{m=1}^{n_1+1}(2^{m-1}-1)^2 c(n_1+1, m) + \sum_{m=1}^{n_1+1}(3\cdot 4^{m-1} - 2^m) c_2(n_1+1, m)\right)$$

by Burnside lemma. Since $(2^{m-1}-1)^2 = \frac{4^m}{4} - 2^m + 1$, the first sum is equal to $\frac{n_1^3+9n_1^2+2n_1}{24}$ by formula (5.1). Since $c_2(n_1+1, m) = 0$ for even n_1, the number of ω-equivalence classes of Type 8 is $\frac{2k^3+9k^2+k}{6}$ when $n_1 = 2k$. When $n_1 = 2k+1$, the second sum is equal to $\frac{1}{(2k+2)!}\left(\frac{3\cdot\overline{2^{k+1}}}{4} - \overline{1^{k+1}}\right) = \frac{3k+2}{4}$ by Lemma 5.1 and hence the number of ω-equivalence classes of Type 8 is $\frac{(k+1)(k^2+5k+3)}{3}$. □

Since an ω-vector weighted digraph of Type 11 can only be ω-equivalent to that of Type 11 or Type 23 and vice a versa, we need to consider their union that is obtained by allowing w' to be zero in Type 23. Using the same idea of the above proof, we obtain the following result.

Lemma 6.2. *The number of w-equivalence classes of Type* 11 *and Type* 23 *is equal to $h(n_1, n_3)$, where the function $h(n, m)$ is defined by*

$$\begin{cases} \frac{nm(m^2+9m+14)}{48} & \text{if } n, m \equiv 0 \pmod 2, \\ \frac{n(m^3+9m^2+23m+15)}{48} & \text{if } n \equiv 0,\ m \equiv 1 \pmod 2, \\ \frac{nm(m^2+9m+14)+3m(m+2)}{48} & \text{if } n \equiv 1,\ m \equiv 0 \pmod 2, \\ \frac{n(m^3+9m^2+23m+15)+3(m^2+2m-3)}{48} & \text{if } n \equiv 1, 3,\ m \equiv 1 \pmod 4, \\ \frac{n(m^3+9m^2+23m+15)+3(m^2+2m+1)}{48} & \text{if } n \equiv 1, 3, m \equiv 3 \pmod 4. \end{cases}$$

Proof. Let $X = \left(\mathbb{Z}_2^{n_1} \setminus \{\mathbf{0}\}\right) \times \left(\mathbb{Z}_2^{n_3} \setminus \{\mathbf{0}\}\right) \times \mathbb{Z}_2^{n_3}$. The group $S_{n_1+1} \times S_{n_3+1}$ acts on the set X, where $(\sigma, \beta) \in S_{n_1+1} \times S_{n_3+1}$ acts by sending (u, w, w') to

$$\begin{cases} (\overline{\sigma}(u), \overline{\beta}(w), \overline{\beta}(w')) & \text{if } u \in \mathcal{V}_\sigma \text{ and } w, w' \in \mathcal{V}_\beta, \\ (\overline{\sigma}(u), \overline{\beta}(w), \overline{\beta}(w') + e_2) & \text{if } u \in \mathcal{V}_\sigma,\ w \in \mathcal{V}_\beta \text{ and } w' \notin \mathcal{V}_\beta, \\ (\overline{\sigma}(u), \overline{\beta}(w) + e_2, \overline{\beta}(w')) & \text{if } u \in \mathcal{V}_\sigma\ w \notin \mathcal{V}_\beta \text{ and } w' \in \mathcal{V}_\beta, \\ (\overline{\sigma}(u), \overline{\beta}(w) + e_2, \overline{\beta}(w') + e_2) & \text{if } u \in \mathcal{V}_\sigma \text{ and } w, w' \notin \mathcal{V}_\beta, \\ (\overline{\sigma}(u) + e_1, \overline{\beta}(w), \overline{\beta}(w + w')) & \text{if } u \notin \mathcal{V}_\sigma \text{ and } w, w' \in \mathcal{V}_\beta, \\ (\overline{\sigma}(u) + e_1, \overline{\beta}(w), \overline{\beta}(w + w') + e_2) & \text{if } u \notin \mathcal{V}_\sigma\ w \in \mathcal{V}_\beta \text{ and } w' \notin \mathcal{V}_\beta, \\ (\overline{\sigma}(u) + e_1, \overline{\beta}(w) + e_2, \overline{\beta}(w + w') + e_2) & \text{if } u \notin \mathcal{V}_\sigma\ \ w \notin \mathcal{V}_\beta \text{ and } w' \in \mathcal{V}_\beta, \\ (\overline{\sigma}(u) + e_1, \overline{\beta}(w) + e_2, \overline{\beta}(w + w')) & \text{otherwise,} \end{cases}$$

where $e_1 = e_{\sigma^{-1}(n_1+1)}$, $e_2 = e_{\beta^{-1}(n_3+1)}$ and $\mathcal{V}_\sigma$ is defined as in the above proof. Then the number of ω-equivalence classes of the union is equal to the size of the orbit space of the $S_{n_1+1} \times S_{n_3+1}$-action on the set X. We find the size of the orbit space of this action by calculating the number of fixed points of (α, β) as above. For this, we consider four cases depending on whether α and β fix $n_1 + 1$ and $n_3 + 1$, respectively. Let the number of disjoint cycles of α and β be m_1 and m_2, respectively and the number of cycles of even lengths in the cycle decompositions of α and β be e_1 and e_2, respectively.

Case 1: Suppose that $\alpha(n_1 + 1) = n_1 + 1$ and $\beta(n_3 + 1) = n_3 + 1$. We need to find the pairs (u, w, w') satisfying $\overline{\sigma}(u) = u, \overline{\beta}(w) = w$ and $\overline{\beta}(w') = w'$. There are $(2^{m_1-1} - 1)(2^{m_2-1} - 1)2^{m_2-1}$ elements fixed by (σ, β). Therefore the number of points fixed by a pair of this type is

$$\begin{aligned} &\sum_{m_1=1}^{n_1+1} \sum_{m_2=1}^{n_3+1} (2^{m_1-1} - 1)(2^{m_2-1} - 1)2^{m_2-1} c(n_1, m_1 - 1)c(n_3, m_2 - 1) \\ = &\ \frac{(n_1!)n_1(n_3 + 1)!n_3(n_3 + 5)}{6} \end{aligned}$$

by the formula (5.1).

Case 2: Now, suppose that $\alpha(n_1 + 1) = n_1 + 1$ and $\beta(n_3 + 1) \neq n_3 + 1$. The fixed points of a pair of this type can be counted as in the above lemma by taking the cases where $w = 0$ into an account. Hence the number of fixed points is equal to

$$\begin{cases} (2^{m_1-1} - 1)(4^{m_2} - 2^{m_2}), & \text{when all the cycles of } \beta \text{ have even lengths,} \\ (2^{m_1-1} - 1)(4^{m_2-1} - 2^{m_2-1}), & \text{otherwise.} \end{cases}$$

Note that the number of permutations that do not fix n_3+1 and have m disjoint cycles is equal to $n_3 c(n_3, m)$. There are $c_2(n_3+1, m)$ permutations that do not fix $n_3 + 1$ and have m disjoint cycles all of which have even lengths. Therefore the sum of the number of fixed points of pairs of this type is

$$\begin{aligned}
&\sum_{m_1=1}^{n_1+1}\sum_{m_2=1}^{n_3+1}(2^{m_1-1}-1)c(n_1,m_1-1)\Big((4^{m_2-1}-2^{m_2-1})(n_3c(n_3,m_2)\\
&\qquad\qquad -c_2(n_3+1,m_2))+(4^{m_2}-2^{m_2})c_2(n_3+1,m_2)\Big)\\
&=(n_1!)n_1\left(n_3\sum_{m_2=1}^{n_3+1}(4^{m_2-1}-2^{m_2-1})c(n_3,m_2)\right.\\
&\qquad\qquad\left.+\sum_{m_2=1}^{n_3+1}\big(3\cdot 4^{m_2-1}-2^{m_2-1}\big)c_2(n_3+1,m_2)\right)\\
&=(n_1!)n_1\left(\frac{(n_3+1)!n_3(n_3+6)(n_3-1)}{24}\right.\\
&\qquad\qquad\left.+\sum_{m_2=1}^{n_3+1}\Big(\frac{3}{4}4^{m_2}-\frac{1}{2}2^{m_2}\Big)c_2(n_3+1,m_2)\right).
\end{aligned}$$

When n_3 is even, $c_2(n_3+1,m_2)=0$. Otherwise, the last sum is equal to

$$(n_3+1)!\cdot\Big(\frac{3}{4}\frac{n_3+3}{2}-\frac{1}{2}\Big)=(n_3+1)!\cdot\frac{3n_3+5}{8}.$$

Therefore the above sum is equal to $(n_1!)n_1\frac{(n_3+1)!n_3(n_3+6)(n_3-1)}{24}$ when n_3 is even and to $(n_1!)n_1\frac{(n_3+1)!(n_3^3+5n_3^2+3n_3+15)}{24}$ when n_3 is odd.

Case 3: Let $\alpha(n_1+1)\neq n_1+1$ and $\beta(n_3+1)=n_3+1$. Then the element (α,β) fixes $(2^{m_1-1}-1)(2^{m_2-1}-1)2^{m_2-1}$ many (u,w,w') for which $\alpha(n_1+1)$-th coordinate of u is zero, that is, $(u)_{\alpha(n_1+1)}=0$. Let us now consider (u,w,w')'s with $(u)_{\alpha(n_1+1)}\neq 0$, that is, $u\notin S_\alpha$. In this case $\alpha(u)+e_1=u$ has a solution if and only if all the cycles of α have even lengths and there are $2^{m_1-1}-1$ many u's satisfying this relation. We also need to find (w,w') satisfying the equations $\overline{\beta}(w)=w$, $\overline{\beta}(w+w')=w'$ and $w\neq 0$. Let $(i_1,i_2,\dots,i_k)$ be a cycle of $\overline{\beta}$. By the first equation, $w_{i_1}=\cdots=w_{i_j}$, say $w_{i_1}=a$. By the second equation, we have

$$w'_{i_j}=a+w_{i_j+1}\text{ for }1\le j\le k-1\text{ and }w'_{i_k}=a+w_{i_1}.$$

Adding up these equations gives $ka\equiv 0 \pmod 2$. If k is even, the matrix $(w|w')$ is a block matrix $\begin{pmatrix}A\\ \vdots\\ A\end{pmatrix}$, where A is the one of the following matrices $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$, or $\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)$. If k is odd, then $a=0$ and hence the values of the coordinates of w' corresponding to this cycle are either all 0 or all 1. Since $w\neq 0$, the number of (w,w')'s satisfying the above equations is $4^{e_2}2^{(m_2-e_2)-1}-2^{e_2}2^{(m_2-e_2)-1}=2^{m_2-1}(2^{e_2}-1)$. Therefore the number of the points fixed by (σ,β) is

$$(2^{m_1-1}-1)(4^{m_2-1}-2^{m_2-1})+2^{m_1-1}2^{m_2-1}(2^{e_2}-1),$$

when all the cycles of α have even lengths and otherwise it is equal to

$$(2^{m_1-1}-1)(4^{m_2-1}-2^{m_2-1}).$$

Hence the number of all the elements of X fixed by an element of this type is given by the following formula:

$$n_1 \sum_{m_1=1}^{n_1+1} \sum_{m_2=1}^{n_3+1} (2^{m_1-1}-1)(4^{m_2-1}-2^{m_2-1})c(n_1,m_1)c(n_3,m_2-1)$$
$$+\sum_{m_1=1}^{n_1+1} \sum_{m_2=1}^{n_3+1} \sum_{e_2=1}^{m_2} 2^{m_1-1}2^{m_2-1}(2^{e_2}-1)c_2(n_1+1,m_1)c(n_3,m_2-1,e_2).$$

The first term of this sum is $(n_1!)(n_3+1)!\frac{n_1(n_1-1)}{2}\frac{n_3(n_3+5)}{6}$. The second term is zero when n_1 is an even number. When n_1 is an odd number, the second term depends on the parity of n_3. Indeed it is equal to $\frac{(n_1+1)!\cdot n_3!\cdot(n_3)^2}{8}$ when n_3 is even and to $\frac{(n_1+1)!\cdot(n_3+1)!\cdot(n_3-1)}{8}$ when n_3 is odd. So the above sum is equal to

$$\begin{cases} \frac{n_1!(n_1)(n_1-1)(n_3+1)!n_3(n_3+5)}{12} & \text{if } n_1 \equiv 0 \pmod 2, \\ \frac{n_1!(n_1)(n_1-1)(n_3+1)!n_3(n_3+5)}{12}+\frac{(n_1+1)!\cdot n_3!\cdot(n_3)^2}{8} & \text{if } n_1 \equiv 1, n_3 \equiv 0 \pmod 2, \\ \frac{n_1!(n_1)(n_1-1)(n_3+1)!n_3(n_3+5)}{12}+\frac{(n_1+1)!\cdot(n_3+1)!\cdot(n_3-1)}{8} & \text{if } n_1 \equiv n_3 \equiv 1 \pmod 2. \end{cases}$$

Case 4: Let $\alpha(n_1+1) \neq n_1+1$ and $\beta(n_3+1) \neq n_3+1$. It follows similarly as in the above lemma that the number of (u,w,w') satisfying $(u)_{\alpha(n_3+1)}=0$ and fixed by an element of this type is equal to

$$\begin{cases} (2^{m_1-1}-1)(4^{m_2}-2^{m_2}), & \text{when all the cycles of } \beta \text{ have even lengths,} \\ (2^{m_1-1}-1)(4^{m_2-1}-2^{m_2-1}), & \text{otherwise.} \end{cases}$$

As above $\overline{\sigma}(u)+e_1=u$ has a solution if and only if all the cycles of σ have even lengths. From now on we assume that σ consists of cycles of even lengths only and we solve the corresponding equations for w and w' depending on whether they are elements of $\mathcal{V}_\beta$ or not.

Let $(i_1,\ldots,i_k)$ be a cycle of β. If both w and w' are in $\mathcal{V}_\beta$, we need to solve the equations

$$\overline{\beta}(w)=w,\ \overline{\beta}(w+w')=w',\ \text{and } (w)_{\beta(n_3+1)}=(w)'_{\beta(n_3+1)}=0.$$

By the first equation, we have $w_{i_1}=\cdots=w_{i_k}=a$ for some $a\in\{0,1\}$. If $\beta(n_3+1)\in\{i_1,\ldots,i_k\}$, then $w_{i_j}=w'_{i_j}=0$ for $1\le j\le k$ by the last equation. Otherwise w' must satisfy the equations

$$w'_{i_j}=a+w'_{i_j+1} \text{ for } 1\le j\le k-1 \text{ and } w'_{i_k}=a+w'_{i_1}.$$

There are 4 solutions when k is even and 2 solutions when k is odd. Since w can not be 0, the number of (u,w,w')'s with $u\notin\mathcal{V}_\sigma$ and $w,w'\in\mathcal{V}_\beta$ fixed by

(α, β) is

$$\begin{cases} 2^{m_1-1}2^{m_2-1}(2^{e_2-1}-1), & \text{when the cycle containing } n_3+1 \text{ has even length,} \\ 2^{m_1-1}2^{m_2-1}(2^{e_2}-1), & \text{otherwise.} \end{cases}$$

If $w \in \mathcal{V}_\beta$ and $w' \notin \mathcal{V}_\beta$, we need to solve the equations

$$\overline{\beta}(w) = w, \ \ \overline{\beta}(w+w') + e_2 = w', \ (w)_{\beta(n_3+1)} = 0 \text{ and } (w)'_{\beta(n_3+1)} = 1.$$

Then $w_{i_1} = \cdots = w_{i_k} = a$ for some $a \in \{0,1\}$. If $\beta(n_3+1) \in \{i_1, \ldots, i_k\}$, say $\beta(n_3+1) = i_1$, then $a = 0$ and w' satisfies the equations

$$w'_{i_j} = 1 + w'_{i_j+1} \text{ for } 1 \le j \le k-1, \text{ and } w'_{i_k} = w'_{i_1} = 1.$$

Hence $k-1 \equiv 0 \pmod 2$, i.e., k must be odd. When k is odd, the above system has a unique solution. Now suppose that $\beta(n_3+1) \notin \{i_1, \ldots, i_k\}$. By the second equation, we have

$$w'_{i_j} = w_{i_j} + a + 1 \text{ for } 1 \le j \le k-1 \text{ and } w'_{i_k} = w_{i_1} + a + 1.$$

Adding up these equations, we obtain $k(a+1) \equiv 0 \pmod 2$. If k is even, the matrix $(w|w')$ must consist of the blocks of one of the forms $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, or $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. If k is odd, then $a = 1$ and hence the values of the coordinates of w' corresponding to this cycle are either all 0 or all 1. Therefore the number of (u, w, w')'s with $u \notin \mathcal{V}_\sigma$, $w \in \mathcal{V}_\beta$, and $w' \notin \mathcal{V}_\beta$ fixed by (α, β) is

$$\begin{cases} 0, & \text{when the cycle containing } n_3+1 \text{ has odd length,} \\ 2^{m_1-1}(4^{m_2-1} - 2^{m_2-1}), & \text{when all the cycles of } \beta \text{ have even lengths,} \\ 2^{m_1-1}2^{m_2+e_2-2}, & \text{otherwise.} \end{cases}$$

Now suppose that $w \notin S_\beta$ and $w' \in S_\beta$. Then (u, w, w') is a fixed points of (α, β) if

$$\overline{\beta}(w) + e_2 = w, \ \ \overline{\beta}(w+w') + e_2 = w', \ (w)_{\beta(n_3+1)} = 1 \text{ and } (w)'_{\beta(n_3+1)} = 0.$$

If $\beta(n_3+1) \in \{i_1, \ldots, i_k\}$, say $\beta(n_3+1) = i_1$, then w satisfies the equations $w_{i_j} = 1 + w_{i_j}$ for $1 \le j \le k-1$ and $w_{i_k} = w_{i_1} = 1$. Hence $k - 1 \equiv 0 \pmod 2$, i.e., k is odd. Let $k = 2k' + 1$. Then w_{i_j} is equal to 1 if j is odd and 0, otherwise. Therefore w' satisfies the equations $w'_{i_{2j+1}} = w_{i_{2j+2}} + w'_{i_{2j+2}} + 1$, $w'_{i_{2j+2}} = w_{i_{2j+3}} + w'_{i_{2j+3}}$ for $1 \le j \le k'-1$ and $w'_{i_1} = 0, w_{i_{2k'+1}} = 1$. This forces k' to be odd. Hence the cycle containing n_3+1 must be divisible by 4. In this case, we have a unique solution. If $\beta(n_3+1) \notin \{i_1, \ldots, i_k\}$, we need to solve the simultaneous equations $w_{i_j} = w_{i_j+1} + 1$, $w'_{i_j} = w'_{i_j+1} + w_{i_j+1} + 1$, $1 \le j \le k-1$ and $w_{i_k} = w_{i_1} + 1$, $w'_{i_k} = w'_{i_1} + 1$.

Algebraically manipulating as above, one can show that this system has a solution if and only if k is divisible by 4. In this case the matrix $(w|w')$ is a block matrix $\begin{pmatrix} A \\ \vdots \\ A \end{pmatrix}$, where A is one of the following matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$

or $\begin{pmatrix} 0&1\\ 1&1\\ 0&0\\ 1&0 \end{pmatrix}$. Therefore the number of (u, w, w')'s with $u \notin \mathcal{V}_\sigma$, $w \in \mathcal{V}_\beta$, and $w' \notin \mathcal{V}_\beta$ that are fixed by (α, β) is

$$\begin{cases} 2^{m_1-1}4^{m_2-1}, & \text{when all the cycles of } \beta \text{ have lengths divisible by 4,} \\ 0, & \text{otherwise.} \end{cases}$$

The last case we need to consider is the one where neither w nor w' are in $\mathcal{V}_\beta$. In this case, to be a fixed point, w and w' must satisfy the equations

$$\overline{\beta}(w) + e_2 = w, \quad \overline{\beta}(w + w') = w', \text{ and } (w)_{\beta(n_3+1)} = (w)'_{\beta(n_3+1)} = 1.$$

If $\beta(n_3 + 1) \in \{i_1, \ldots, i_k\}$, say $\beta(n_3 + 1) = i_1$, then w satisfies the equations $w_{i_j} = 1 + w_{i_j}$ for $1 \le j \le k - 1$ and $w_{i_k} = w_{i_1} = 1$. This system has a solution only if k is odd, say $k = 2k' + 1$. Since the solution is $w_{i_j} = 1$ if j is odd, and $w_{i_{j+1}} = 0$, otherwise, w' must satisfy the equations $w'_{i_j} = w'_{i_{j+1}} + 1$ if j is odd, and $w'(i_j) = w'_{i_{j+1}}$, otherwise. This system has a solution if and only if k' is odd, i.e., the cycle containing n_3+1 is divisible by 4. Otherwise, w and w' must satisfy the equations $w_{i_j} = w_{i_j+1}$, and $w'_{i_j} = w'_{i_{j+1}} + w'_{i_{j+1}}$, simultaneously. As before such a system has a solution if and only if k is divisible by 4. In this case the matrix $(w|w')$ is a block matrix $\begin{pmatrix} A \\ \vdots \\ A \end{pmatrix}$, where A is the one of the following matrices $\begin{pmatrix} 1&0\\ 0&0\\ 1&1\\ 0&1 \end{pmatrix}$, $\begin{pmatrix} 1&1\\ 0&1\\ 1&0\\ 0&0 \end{pmatrix}$, $\begin{pmatrix} 0&0\\ 1&1\\ 0&1\\ 1&0 \end{pmatrix}$ or $\begin{pmatrix} 0&1\\ 1&0\\ 0&0\\ 1&1 \end{pmatrix}$. Therefore the number of (u, w, w')'s with $u \notin \mathcal{V}_\sigma$, $w \in \mathcal{V}_\beta$, and $w' \notin \mathcal{V}_\beta$ fixed by an element of this type is

$$\begin{cases} 2^{m_1-1}4^{m_2-1}, & \text{when all the cycles of } \beta \text{ have lengths divisible by 4,} \\ 0, & \text{otherwise.} \end{cases}$$

To sum up, the number of elements fixed by (α, β) when $\alpha(n_1 + 1) \neq n_1 + 1$ and $\beta(n_3 + 1) \neq n_3 + 1$ is equal to

$$\begin{cases} (2^{m_1-1}-1)(4^{m_2-1}-2^{m_2-1}), & \text{if } \alpha \notin \mathcal{E}^2_{n_1+1},\ \beta \notin \mathcal{E}^1_{n_3+1}, \\ (2^{m_1-1}-1)(4^{m_2}-2^{m_2}), & \text{if } \alpha \notin \mathcal{E}^2_{n_1+1},\ \beta \in \mathcal{E}^1_{n_3+1}, \\ (2^{m_1-1}-1)(4^{m_2-1}-2^{m_2-1})+2^{m_1-1}2^{m_2-1}(2^{e_2}-1), & \text{if } \alpha \in \mathcal{E}^2_{n_1+1},\ \beta \notin \mathcal{E}^2_{n_3+1}, \\ (2^{m_1-1}-1)(4^{m_2}-2^{m_2})+2^{m_1}(2 \cdot 4^{m_2-1}-2^{m_2-1}), & \text{if } \alpha \in \mathcal{E}^2_{n_1+1},\ \beta \in \mathcal{E}^4_{n_3+1}, \\ (2^{m_1-1}-1)(4^{m_2}-2^{m_2})+2^{m_1}(4^{m_2-1}-2^{m_2-1}), & \text{otherwise,} \end{cases}$$

where $\mathcal{E}^d_n$ is the set of all permutations of n elements which consists of cycles whose lengths are divisible by d. Therefore the number of elements of X fixed by such (α, β)'s is equal to $I = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$, where

$$I_1 = \sum_{m_1=1}^{n_1+1} \sum_{m_2=1}^{n_3+1} \Bigg(\big(2^{m_1-1} - 1\big)\big(4^{m_2-1} - 2^{m_2-1}\big)\big(n_1 c(n_1, m_1) - c_2(n_1 + 1, m_1)\big)$$

$$\cdot \big(n_3 c(n_3, m_2) - c_2(n_3+1, m_2)\big)\Bigg)$$

$$I_2 = \sum_{m_1=1}^{n_1+1} \sum_{m_2=1}^{n_3+1} \Bigg(\big(2^{m_1-1}-1\big)\big(4^{m_2}-2^{m_2}\big)\big(n_1 c(n_1, m_1) - c_2(n_1+1, m_1)\big) \cdot c_2(n_3+1, m_2)\Bigg)$$

$$I_3 = \sum_{m_1=1}^{n_1+1} \sum_{m_2=1}^{n_3+1} \Bigg(\big(2^{m_1-1}-1\big)\big(4^{m_2-1}-2^{m_2-1}\big)\big(n_3 c(n_3, m_2) - c_2(n_3+1, m_2)\big) \cdot c_2(n_1+1, m_1)\Bigg)$$

$$I_4 = \sum_{m_1=1}^{n_1+1} \sum_{m_2=1}^{n_3+1} \Bigg(2^{m_1-1}2^{m_2-1} c_2(n_1+1, m_1)\Big(-\big(2^{m_2}-1\big)c_2(n_3+1, m_2) + \sum_{e_2=1}^{m_2} (2^{e_2}-1)(c(n_3+1, m_2, e_2) - c(n_3, m_2-1, e_2)\big)\Big)\Bigg)$$

$$I_5 = \sum_{m_1=1}^{n_1+1} \sum_{m_2=1}^{n_3+1} \Bigg(\Big(\big(2^{m_1-1}-1\big)\big(4^{m_2}-2^{m_2}\big) + 2^{m_1}\big(4^{m_2-1}-2^{m_2-1}\big)\Big) \cdot c_2(n_1+1, m_1) c_2(n_3+1, m_2)\Bigg)$$

and

$$I_6 = \sum_{m_1=1}^{n_1+1} \sum_{m_2=1}^{n_3+1} 2^{m_1} 4^{m_2-1} c_2(n_1+1, m_1) c_4(n_3+1, m_2).$$

Let $f : \mathbb{Z}_+^3 \to \mathbb{R}$ be the function defined by $f(x, n, d) = \frac{1}{\left(\frac{n+1}{d}\right)!}\left(\frac{1}{x}\right)^{\overline{\frac{n+1}{d}}}$. Then the above sums are given by the following formulas

$$I_1 = \begin{cases} \frac{n_1!(n_3+1)!(n_1^2-n_1)(n_3^3+5n_3^2-6n_3)}{48} & \text{if } n_1 \equiv n_3 \equiv 0 \pmod 2, \\ \frac{n_1!(n_3+1)!(n_1^2-n_1)(n_3^3+5n_3^2-9n_3+3)}{48} & \text{if } n_1 \equiv 0, n_3 \equiv 1 \pmod 2, \\ \frac{n_1!(n_3+1)!\left(n_1^2-2n_1-1+2(n_1+1)f(2,n_1,2)\right)(n_3^3+5n_3^2-6n_3)}{48} & \\ \text{if } n_1 \equiv 1, n_3 \equiv 0 \pmod 2, & \\ \frac{n_1!(n_3+1)!\left(n_1^2-2n_1-1+2(n_1+1)f(2,n_1,2)\right)(n_3^3+5n_3^2-9n_3+3)}{48} & \\ \text{if } n_1 \equiv n_3 \equiv 1 \pmod 2. & \end{cases}$$

$$I_2 = \begin{cases} 0 & \text{if } n_3 \equiv 0 \pmod 2. \\ \frac{n_1!(n_3+1)!(n_1^2-n_1)(n_3+1)}{4} & \text{if } n_1 \equiv 0, n_3 \equiv 1 \pmod 2, \\ \frac{n_1!(n_3+1)!\left(n_1^2-2n_1-1+2(n_1+1)f(2,n_1,2)\right)(n_3+1)}{4} & \text{if } n_1 \equiv n_3 \equiv 1 \pmod 2. \end{cases}$$

$$I_3 = \begin{cases} 0 & \text{if } n_1 \equiv 0 \pmod 2 \\ \frac{(n_1+1)!(n_3+1)!(1-2f(2,n_1,2))(n_3^3+5n_3^2-6n_3)}{48} & \text{if } n_1 \equiv 1, n_3 \equiv 0 \pmod 2, \\ \frac{(n_1+1)!(n_3+1)!(1-2f(2,n_1,2))(n_3^3+5n_3^2-9n_3+3)}{48} & \text{if } n_1 \equiv n_3 \equiv 1 \pmod 2. \end{cases}$$

$$I_4 = \begin{cases} 0 & \text{if } n_1 \equiv 0 \pmod 2. \\ \frac{(n_1+1)!n_3!(n_3^3+n_3^2+2n_3)}{16} & \text{if } n_1 \equiv 1, n_3 \equiv 0 \pmod 2, \\ \frac{(n_1+1)!(n_3+1)!(n_3^2-2n_3+1)}{16} & \text{if } n_1 \equiv n_3 \equiv 1 \pmod 2. \end{cases}$$

$$I_5 = \begin{cases} (n_1+1)!(n_3+1)!\left(\frac{3n_3+1-4(n_3+1)f(2,n_1,2)}{8}\right) & \text{if } n_1 \equiv n_3 \equiv 1 \pmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

$$I_6 = \begin{cases} \frac{(n_1+1)!(n_3+1)!}{4} & \text{if } n_1 \equiv 1 \pmod 2, n_3 \equiv 3 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Hence when n_1 and n_3 are even,

$$I = \frac{n_1!(n_3+1)!(n_1^2-n_1)(n_3^3+5n_3^2-6n_3)}{48}$$

when n_1 is even and n_3 is odd,

$$I = \frac{n_1!(n_3+1)!(n_1^2-n_1)(n_3^3+5n_3^2+3n_3+15)}{48}$$

when n_1 is odd and n_3 is even,

$$I = \frac{(n_1+1)!n_3!(n_3^3+n_3^2+2n_3)}{16} + \frac{n_1!(n_3+1)!(n_1^2-n_1)(n_3^3+5n_3^2-6n_3)}{48}$$

when n_1 is odd and $n_3 \equiv 1 \pmod 4$,

$$I = \frac{(n_1+1)!(n_3+1)!(n_3^3+8n_3^2+3n_3+12)}{48} + \frac{n_1!(n_3+1)!(n_1^2-2n_1-1)(n_3^3+5n_3^2+3n_3+15)}{48}$$

and when n_1 is odd and $n_3 \equiv 3 \pmod 4$,

$$I = \frac{(n_1+1)!(n_3+1)!(n_3^3+8n_3^2+3n_3+24)}{48} + \frac{n_1!(n_3+1)!(n_1^2-2n_1-1)(n_3^3+5n_3^2+3n_3+15)}{48}.$$

Therefore, the number of orbits of the action of $S_{n_1+1} \times S_{n_3+1}$ on X is given by the formula

$$= \begin{cases} \frac{n_1 n_3(n_3^2+9n_3+14)}{48} & \text{if } n_1 \equiv n_3 \equiv 0 \pmod 2, \\ \frac{n_1(n_3^3+9n_3^2+23n_3+15)}{48} & \text{if } n_1 \equiv 0,\ n_3 \equiv 1 \pmod 2\ , \\ \frac{n_1 n_3(n_3^2+9n_3+14)+3n_3(n_3+2)}{48} & \text{if } n_1 \equiv 1,\ n_3 \equiv 0 \pmod 2, \\ \frac{n_1(n_3^3+9n_3^2+23n_3+15)+3(n_3^2+2n_3-3)}{48} & \text{if } n_1 \equiv 1 \pmod 2,\ n_3 \equiv 1 \pmod 4, \\ \frac{n_1(n_3^3+9n_3^2+23n_3+15)+3(n_3^2+2n_3+1)}{48} & \text{if } n_1 \equiv 1 \pmod 2,\ n_3 \equiv 3 \pmod 4, \end{cases}$$

by Burnside's lemma. □

As an immediate result of the above calculations, we have the following theorem.

Theorem 6.3. *Let $P = \Delta^{n_1} \times \Delta^{n_2} \times \Delta^{n_3}$ with $n_1 \le n_2 \le n_3$, $n = n_1+n_2+n_3$ and f and h be functions given in Lemma 6.1 and Lemma 6.2, respectively. The number of weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers over P is equal to*

$$1 + \sum_{i=1}^{3}\Big(\Big\lfloor \frac{n_i+1}{2} \Big\rfloor + f(n_i)\Big) + \sum_{1\le i<j\le 3} \Big\lfloor \frac{n_i+1}{2} \Big\rfloor \Big\lfloor \frac{n_j+1}{2} \Big\rfloor + \sum_{1\le i\ne j\le 3} h(n_i,n_j)$$

when $n_1 < n_2 < n_3$,

$$1 + \Big\lfloor \frac{n_1+1}{2} \Big\rfloor + \Big\lfloor \frac{n_3+1}{2} \Big\rfloor + f(n_1) + f(n_3) + \Big\lfloor \frac{n_1+1}{2} \Big\rfloor \Big\lfloor \frac{n_3+1}{2} \Big\rfloor + \Big\lfloor \frac{n_1+1}{2} \Big\rfloor^2$$
$$+ h(n_1,n_1) + h(n_1,n_3) + h(n_3,n_1)$$

when $n_1 = n_2 < n_3$, and

$$1 + \Big\lfloor \frac{n_1+1}{2} \Big\rfloor + f(n_1) + \Big\lfloor \frac{n_1+1}{2} \Big\rfloor^2 + h(n_1,n_1)$$

when $n_1 = n_2 = n_3$.

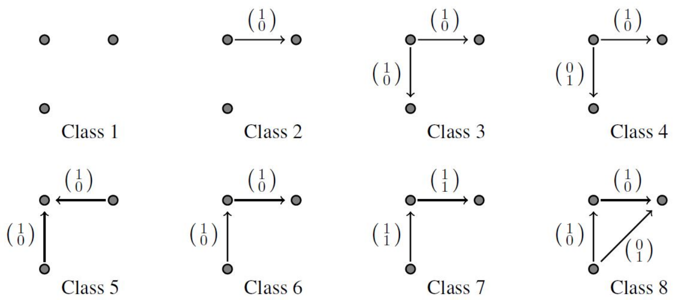

FIGURE 5. The weakly $\mathbb{Z}_2^6$-equivariant homeomorphism classes of small covers over $\Delta^2 \times \Delta^2 \times \Delta^2$.

Example 6.4. By the above theorem, there are 8 different weakly $\mathbb{Z}_2^6$-equivariant homeomorphism classes of small covers over $P = \Delta^2 \times \Delta^2 \times \Delta^2$. The corresponding ω-equivalence classes of acyclic ω-weighted digraphs are listed in Figure 5.

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Aslı Güçlükan İlhan
Department of Mathematics
Dokuz Eylul University
İzmir, Turkey
Email address: asli.ilhan@deu.edu.tr

Sabri Kaan Gürbüzer
Department of Mathematics
Dokuz Eylul University
İzmir, Turkey
Email address: kaan.gurbuzer@deu.edu.tr