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LAZHAR TYPE INEQUALITIES FOR p-CONVEX FUNCTIONS

Tekin Toplu, İmdat İşcan*, and Selahattin Maden

Abstract. The aim of this study is to establish some new Jensen and Lazhar type inequalities for p-convex function that is a generalization of convex and harmonic convex functions. The results obtained here are reduced to the results obtained earlier in the literature for convex and harmonic convex functions in special cases.

1. Introduction

Definition 1.1. *The function* $f : I \subset \mathbb{R} \to \mathbb{R}$, *is said to be convex, if the following inequality holds*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

for all $x, y \in I$ *and* $t \in [0,1]$. *We say* f *is concave if* $(-f)$ *is convex.*

Our next theorem is Discrete Jensen's inequality for convex functions [6].

Theorem 1.2. *Let* f *be a convex function defined on a interval* I. *If* $x_1, x_2 \cdots, x_n \in I$ *and* $\lambda_1, \lambda_2 \cdots \lambda_n \ge 0$ *with* $\sum_{i=1}^{n} \lambda_i = 1$,*then*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \le \sum_{i=1}^{n} \lambda_i f(x_i) \tag{1}$$

In [7], Popoviciu gave the following theorem.

Theorem 1.3. *Let* f *be a real-valued continuous function on an interval* I. *Then* f *is convex if and only if*

$$\begin{aligned} &\frac{f(x_1)+f(x_2)+f(x_3)}{3} + f\left(\frac{x_1+x_2+x_3}{3}\right) \\ &\ge \frac{2}{3}\left[f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_1}{2}\right)\right] \end{aligned}$$

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*Corresponding author

In [2], Lazhar Bougoffa gave a generalization of a variant the Popoviciu's inequality as follows.

Theorem 1.4. *If f is a convex function and $x_1, x_2, \cdots, x_n$ lie in its domain, then*

$$\begin{aligned} & \sum_{i=1}^{n} f(x_i) - f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \\ \geq \quad & \frac{n-1}{n}\left[f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right) + f\left(\frac{x_n + x_1}{2}\right)\right] \end{aligned}$$

In [2] Bougoffa also gave a variant of generalized Popoviciu inequality as follows.

Theorem 1.5. *If f is a convex function and $x_1, x_2, \cdots, x_n$ lie in its domain, then*

$$(2)(n-1)\left[f(b_1) + f(b_2) \cdots + f(b_n)\right] \leq n\left[f(a_1) + \cdots + f(a_n) - f(a)\right],$$

where $a = \frac{a_1 + \cdots + a_n}{n}$ and $b_i = \frac{na - a_i}{n-1}$, $i = 1, \cdots, n$.

In [4], İşcan gave definition of harmonically convexity as follows.

Definition 1.6. *Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t) f(x)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Dragomir [3] proved the following Jensen type inequality (discrete version) for harmonically convex functions:

Theorem 1.7. *If $f : [a, b] \subset (0, +\infty) \to \mathbb{R}$ is harmonically convex function, then*

$$f\left(\frac{1}{\sum_{i=1}^{n} \frac{t_i}{x_i}}\right) \leq \sum_{i=1}^{n} t_i f(x_i) \tag{3}$$

for all $x_1, \cdots, x_n \in [a, b]$,$t_1 \cdots, t_n \geq 0$ with $t_1 + \cdots + t_n = 1$

In [1], Azócar et al gave Lazhar type inequalities for harmonic convex function.

Theorem 1.8. *$f : [a, b] \subseteq (0, +\infty) \to \mathbb{R}$ is a harmonically convex function and $x_1, x_2 \cdots x_n \in [a, b]$,then*

$$\begin{aligned} & \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i) - f\left(\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}\right)\right] \\ \geq \quad & f\left(\frac{2x_1x_2}{x_1 + x_2}\right) + \cdots + f\left(\frac{2x_{n-1}x_n}{x_{n-1} + x_n}\right) + f\left(\frac{2x_nx_1}{x_n + x_1}\right) \end{aligned}$$

In [1], Azócar et al proved inequality for harmonically convex functions as following theorem.

Theorem 1.9. *If $f : [a, b] \subseteq (0, \infty) \to \mathbb{R}$ is a harmonically convex function and $a_1, \cdots, a_n \in [a, b]$,then*

$$\sum_{i=1}^{n} f(b_i) \leq \frac{n}{n-1}\left[\sum_{i=1}^{n} f(a_i) - f\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{a_i}}\right)\right],$$

where $b_i = \frac{n-1}{n\alpha^{-1} - a_i^{-1}}, i = 1, , 2, ..., n$ and $\alpha = \frac{n}{a_1^{-1} + a_2^{-1} + ... + a_n^{-1}}$.

In [8], Zhang and Wan gave definition of p-convex function as follows.

Definition 1.10. *Let I be a p-convex set. A function $f : I \to \mathbb{R}$ is said to be a p-convex function or belongs to class $PC(I)$, if*

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq tf(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Remark 1.11. [8] *An interval I is said to be a p-convex set if $[tx^p + (1-t)y^p]^{\frac{1}{p}} \in I$ for all for all $x, y \in I$ and $t \in [0, 1]$,where $p = 2k + 1$ or $p = n/m$, $n = 2r + 1$, $m = 2s + 1$ and $k, r, s \in \mathbb{N}$.*

Remark 1.12. [5] *If $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$, then $[tx^p + (1-t)y^p]^{\frac{1}{p}} \in I$ for all for all $x, y \in I$ and $t \in [0, 1]$.*

According to Remark 1.12, we can give different version of the definition of p-convex function as below.

Definition 1.13. [5] *Let $I \subset (0, \infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be p-convex function, if*

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq tf(x) + (1-t) f(y) \tag{4}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality is reserved, then f is said to be p-concave.

According to definition above, it can easily be seen that p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0, \infty)$ for $p = 1$ and $p = -1$, respectively. In [5] İşcan gave the relation between convex function and p-convex function as follows

Proposition 1.14. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ and $h(x) = x^p$ for $p \in \mathbb{R}\backslash\{0\}$. If we consider $g : J = h(I) \to \mathbb{R}$ defined as $g(t) = \left(f \circ h^{-1}\right)$ then, if f is p-convex on I, if and only if g is convex on J.*

The aim of this study is to establish some new Jensen and Lazhar type inequalities for p-convex functions that are generalizations of results obtained inTheorem 1.2, Theorem 1.4 and Theorem 1.5 for convex functions and obtained in Theorem 1.7, Theorem 1.8 and Theorem 1.9 for harmonic convex functions.

2. Main Results

In this section, we present some new lazhar type inequalities for p-convex functions. For these, firstly we prove the following Jensen inequality for p-convex functions.

Theorem 2.1. *Let $p \in \mathbb{R}\backslash\{0\}$, $n \in \mathbb{N}$ and $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function. If $x_1, x_2 \cdots, x_n \in I$, then*

$$f\left(\left[\sum_{i=1}^{n} t_i x_i^p\right]^{\frac{1}{p}}\right) \leq \sum_{i=1}^{n} t_i f(x_i) \tag{5}$$

where $t_1, t_2, t_3 \ldots, t_n \geq 0$, $t_1 + t_2 + t_3 + \cdots + t_n = 1$.

Proof. It can easily be seen by induction. For;
i) $n = 1$, it is obvious that $t = 1$ also and inequality holds;

$$f\left([tx^p]^{\frac{1}{p}}\right) \leq tf(x) \tag{6}$$

ii) For n we assume inequality holds

$$f\left(\left[\sum_{i=1}^{n} t_i x_i^p\right]^{\frac{1}{p}}\right) \leq \sum_{i=1}^{n} t_i f(x_i) \tag{7}$$

iii) Now we show for $i = n + 1$.
Let $\sum_{i=1}^{n+1} t_i = 1$ for $t_i \geq 0$. Now we take $\alpha = \sum_i^n t_i$ Firstly we will prove for $\alpha \neq 0$

$$\begin{aligned}
f\left(\left[\sum_{i=1}^{n+1} t_i x_i^p\right]^{\frac{1}{p}}\right) &= f\left(\left[\alpha\left(\left[\sum_{i=1}^{n} \frac{t_i}{\alpha} x_i^p\right]^{\frac{1}{p}}\right)^p + t_{n+1}x_{n+1}^p\right]^{\frac{1}{p}}\right) \\
&\leq \alpha f\left(\left[\sum_{i=1}^{n} \frac{t_i}{\alpha} x_i^p\right]^{\frac{1}{p}}\right) + t_{n+1} f(x_{n+1}) \\
&\leq \alpha \sum_{i=1}^{n} \frac{t_i}{\alpha} f(x_i) + t_{n+1} f(x_{n+1}) = \sum_{i=1}^{n+1} f(x_i)
\end{aligned}$$

Secondly for $\alpha = 0$ the proof is obvious.

$$f(x_{n+1}) = \sum_{i=1}^{n+1} t_i f(x_i) \tag{8}$$

Thus we have desired inequality. This completes the proof. □

Remark 2.2. *We can give alternative proof for (5) inequality by using Proposition 1.14. Let $f : I \subset (0,\infty) \to \mathbb{R}$ and $x_1, x_2, \cdots, x_n \in I$. Then*

$h(x_1), h(x_2)\cdots, h(x_n) \in J$ where $h(x_i) = x_i^p$ for $i \geq 1, 2, \cdots, n$. Since $\left(f \circ h^{-1}\right)$ is convex function on J, with respect to Theorem 1.2, then we have

$$\left(f \circ h^{-1}\right)\left(\sum_{i=1}^{n} \lambda_i h(x_i)\right) \leq \sum_{i=1}^{n} \lambda_i \left(f \circ h^{-1}\right)(h(x_i)) \tag{9}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. Since $h(x) = x^p$, from the inequality (9), we obtain,

$$f\left(\left(\sum_{i=1}^{n} \lambda_i x_i^p\right)^{\frac{1}{p}}\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

Remark 2.3. *In (5) inequality, if we choose specially $p = 1$, it is obvious that inequality reduces to (1) inequality.*

Remark 2.4. *In (5) inequality, if we choose specially $p = -1$, in this case inequality reduces to (3) inequality.*

Now we will give Lazhard type inequalities for p-convex functions.

Theorem 2.5. *Let $p \in \mathbb{R}\backslash\{0\}$, $n \in \mathbb{N}$ and $f : I \subset (0, \infty) \to \mathbb{R}$ be a p-convex function. If $x_1, x_2, \cdots, x_n \in I$ then*

$$\begin{aligned} &\sum_{i=1}^{n} f(x_i) - f\left(\left[\sum_{i=1}^{n} \frac{x_i^p}{n}\right]^{\frac{1}{p}}\right) \\ \geq\ & \frac{n-1}{n}\left[f\left(\left[\frac{x_1^p + x_2^p}{2}\right]^{\frac{1}{p}}\right) + f\left(\left[\frac{x_2^p + x_3^p}{2}\right]^{\frac{1}{p}}\right) + \cdots + f\left(\left[\frac{x_n^p + x_1^p}{2}\right]^{\frac{1}{p}}\right)\right]. \end{aligned} \tag{10}$$

Proof. Since f is p-convex, we have

$$\begin{aligned} &f\left(\left[\frac{{x_1}^p + {x_2}^p}{2}\right]^{\frac{1}{p}}\right) + f\left(\left[\frac{{x_2}^p + {x_3}^p}{2}\right]^{\frac{1}{p}}\right) + \cdots + f\left(\left[\frac{{x_n}^p + {x_1}^p}{2}\right]^{\frac{1}{p}}\right) \\ &\qquad \leq f(x_1) + f(x_2) + \ldots + f(x_n) \end{aligned}$$

hence, by Jensen inequality for p-convex functions we get

$$\begin{aligned} f(x_1) + f(x_2) + .. + f(x_n) = \sum_{i=1}^{n} f(x_i) &= \frac{n}{n-1}\sum_{i=1}^{n} f(x_i) - \frac{1}{n-1}\sum_{i=1}^{n} f(x_i) \\ &= \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} \frac{1}{n} f(x_i)\right] \\ &\leq \frac{n}{n-1}\left[\sum_{i=1}^{n} f(x_i) - f\left(\left[\sum_{i=1}^{n} \frac{x_i^p}{n}\right]^{\frac{1}{p}}\right)\right]. \end{aligned}$$

Thus we have desired inequality. This completes the proof.

Remark 2.6. *An alternative proof for Theorem 10 can be given by using Proposition 1.14. Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *and* $x_1, x_2, \cdots, x_n \in I$. *Then* $h(x_1), h(x_2)\cdots, h(x_n) \in J$ *where* $h(x_i) = x_i^p$ *for* $i \geq 1, 2, \cdots, n$. *Since* $\left(f \circ h^{-1}\right)$ *is convex function on* J,*with respect to Theorem 1.4, then we have,*

$$\sum_{i=1}^{n} \left(f \circ h^{-1}\right)\left(h\left(x_i\right)\right) - \left(f \circ h^{-1}\right)\left(\frac{1}{n}\sum_{i=1}^{n} h\left(x_i\right)\right)$$
$$\geq \frac{n-1}{n}\left[f \circ h^{-1}\left(\frac{h\left(x_1\right)+h\left(x_2\right)}{2}\right) + \cdots + f \circ h^{-1}\left(\frac{h\left(x_n\right)+h\left(x_1\right)}{2}\right)\right]$$

Since $h(x) = x^p$

$$\sum_{i=1}^{n} f\left(x_i\right) - f\left(\left[\sum_{i=1}^{n}\frac{x_i^p}{n}\right]^{\frac{1}{p}}\right)$$
$$\geq \frac{n-1}{n}\left[f\left(\left[\frac{x_1^p+x_2^p}{2}\right]^{\frac{1}{p}}\right) + f\left(\left[\frac{x_2^p+x_3^p}{2}\right]^{\frac{1}{p}}\right) + \cdots + f\left(\left[\frac{x_n^p+x_1^p}{2}\right]^{\frac{1}{p}}\right)\right]$$

□

Remark 2.7. *In (10) inequality, if we choose specially* $p = 1$, *inequality reduces to (1.4) inequality.*

Remark 2.8. *In (10) inequality, if we choose specially* $p = -1$, *inequality reduces to (1.8) inequality.*

Theorem 2.9. *Let* $p \in \mathbb{R}\backslash\{0\}$, $n \in \mathbb{N}\backslash\{1\}$ *and* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a* p*-convex function. If* $a_1, a_2, \cdots, a_n \in I$, *then*

$$(11) \qquad (n-1)\left[f\left(b_1\right)+f\left(b_2\right)+...+f\left(b_n\right)\right]$$
$$\leq n\left[f\left(a_1\right)+f\left(a_2\right)+...+f\left(a_n\right) - f\left(\left[\sum_{i=1}^{n}\frac{a_i^p}{n}\right]^{\frac{1}{p}}\right)\right],$$

where $a = \left[\frac{a_1{}^p+a_2{}^p+...+a_n{}^p}{n}\right]^{\frac{1}{p}}$ *and* $b_i = \left[\frac{na^p - a_i{}^p}{n-1}\right]^{\frac{1}{p}}$.

Proof. By using Jensen inequality for p-convex functions and p-convexity of f,

$$f\left(b_1\right)+f\left(b_2\right)+\cdots+f\left(b_n\right)$$
$$\leq f\left(a_1\right)+f\left(a_2\right)+\cdots+f\left(a_n\right) = \sum_{i=1}^{n} f\left(a_n\right)$$

and

$$\begin{aligned}\sum_{i=1}^{n} f(a_n) &= \frac{n}{n-1}\left[f(a_1)+f(a_2)+...+f(a_n)\right] \\ &\quad -\frac{1}{n-1}\left[f(a_1)+f(a_2)+...+f(a_n)\right] \\ &= \frac{n}{n-1}\left[f(a_1)+f(a_2)+...+f(a_n)\right] \\ &\quad -\frac{n}{n-1}\left[\frac{1}{n}f(a_1)+\frac{1}{n}f(a_2)+...+\frac{1}{n}f(a_n)\right] \\ &\leq \frac{n}{n-1}\left[f(a_1)+f(a_2)+...+f(a_n)-f\left(\left[\sum_{i=1}^{n}\frac{a_i^p}{n}\right]^{\frac{1}{p}}\right)\right]\end{aligned}$$

Thus we have desired inequality. This completes the proof. □

Remark 2.10. *We can also give an alternative proof for Theorem 11, by using Proposition 1.14. Let $f : I \subset (0,\infty) \to \mathbb{R}$ and $x_1, x_2, \cdots, x_n \in I$. Then $h(x_1), h(x_2) \cdots, h(x_n) \in J$ where $h(x_i) = x_i^p$ for $i \geq 1, 2, \cdots, n$. Since $\left(f \circ h^{-1}\right)$ is convex function on J, with respect to Theorem 1.5, then we have,*

$$(n-1)\left[f\circ h^{-1}(b_1) + f\circ h^{-1}\left(b_2+\cdots+f\circ h^{-1}(b_n)\right)\right]$$
$$\leq n\left[f\circ h^{-1}(h(a_1)) + f\circ h^{-1}(h(a_2)) + \cdots + f\circ h^{-1}(h(a_n)) - f\circ h^{-1}h(a)\right]$$

where $b_i^p = \frac{nh(a)-h(a_i)}{n-1}$ and $h(a) = \frac{h(a_1)+h(a_2)+\cdots+h(a_n)}{n}$, since $f\circ h^{-1}$ is convex on J, then

$$\begin{aligned}&(n-1)\left[f(b_1)+f(b_2)+...+f(b_n)\right] \\ \leq\; & n\left[f(a_1)+f(a_2)+...+f(a_n)-f\left(\left[\sum_{i=1}^{n}\frac{a_i^p}{n}\right]^{\frac{1}{p}}\right)\right]\end{aligned}$$

Remark 2.11. *In (11) inequality, if we choose specially $p = 1$, inequality reduces to (1.5) inequality.*

Remark 2.12. *In (11) inequality, if we choose specially $p = -1$, inequality reduces to (1.9) inequality.*

3. Some applications for special means

Let $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$. Let us recall the following special means of numbers $\mathbf{a} = (a_1, a_2, ..., a_n)$, $a_i \geq 0, i = 1, 2, ..., n$:

1. The weighted arithmetic mean

$$A(\mathbf{a};\alpha) := \sum_{i=1}^{n} \alpha_i a_i.$$

2. The weighted harmonic mean

$$H(\mathbf{a};\alpha) := \left(\sum_{i=1}^{n}\alpha_i a_i^{-1}\right)^{-1}, \quad a_i > 0.$$

3. The weighted mean power of order p

$$M_p(\mathbf{a};\alpha) := \left(\sum_{i=1}^{n}\alpha_i a_i^{p}\right)^{1/p}, \ a_i > 0, \ p \in \mathbb{R}\setminus\{0\}.$$

It should be emphasized that

$$H(\mathbf{a};\alpha) = M_{-1}(\mathbf{a};\alpha) \le M_1(\mathbf{a};\alpha) = A(\mathbf{a};\alpha)$$

and $M_p(\mathbf{a};\alpha) \le M_q(\mathbf{a};\alpha)$ for $p \le q$.

Proposition 3.1. *Let* $0 < p \le q,\ n \in \mathbb{N}$ *with* $x_1, x_2 \cdots, x_n \in (0,\infty)$ *and* $t_1, t_2, t_3 \ldots, t_n \ge 0$, $\sum_{i=1}^{n} t_i = 1$. *Then, we have the following inequality:*

$$M_p(\mathbf{x};\mathbf{t}) = \left[\sum_{i=1}^{n} t_i x_i^p\right]^{\frac{1}{p}} \le \left[\sum_{i=1}^{n} t_i x_i^q\right]^{\frac{1}{q}} = M_q(\mathbf{x};\mathbf{t}),$$

where $\mathbf{t} = (t_1, t_2, t_3 \ldots, t_n),\ \mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Proof. The assertion follows from inequality (5), for $f:(0,\infty)\to\mathbb{R},\ f(x) = x^q$. □

If we take $p = 1$ in the above proposition, we get the following result.

Corollary 3.2. *Let* $1 \le q,\ n \in \mathbb{N}$ *with* $x_1, x_2 \cdots, x_n \in (0,\infty)$ *and* $t_1, t_2, t_3 \ldots, t_n \ge 0$, $\sum_{i=1}^{n} t_i = 1$. *Then, we have the following inequality:*

$$A(\mathbf{x};\mathbf{t}) \le M_q(\mathbf{x};\mathbf{t}),$$

where $\mathbf{t} = (t_1, t_2, t_3 \ldots, t_n),\ \mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Proposition 3.3. *Let* $p < 0,\ n \in \mathbb{N}$ *with* $x_1, x_2 \cdots, x_n \in (0,\infty)$ *and* $t_1, t_2, t_3 \ldots, t_n \ge 0$, $\sum_{i=1}^{n} t_i = 1$. *Then, we have the following inequality:*

$$H(\mathbf{x};\mathbf{t}) \le M_p(\mathbf{x};\mathbf{t}),$$

where $\mathbf{t} = (t_1, t_2, t_3 \ldots, t_n),\ \mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Proof. The assertion follows from inequality (5), for $f:(0,\infty)\to\mathbb{R},\ f(x) = -x^{-1}$. □

Proposition 3.4. *Let* $0 < p \le q,\ n \in \mathbb{N}$ *with* $x_1, x_2 \cdots, x_n \in (0,\infty)$. *Then, we have the following inequality:*

$$\begin{aligned} & nM_q^q(\mathbf{x};\mathbf{t}_n) - M_p^q(\mathbf{x};\mathbf{t}_n) \\ \ge\ & \frac{n-1}{n}\left[M_p^q\left((x_1,x_2);\frac{1}{2}\right) + M_p^q\left((x_2,x_3);\frac{1}{2}\right) + \cdots + M_p^q\left((x_n,x_1);\frac{1}{2}\right)\right], \end{aligned}$$

where $\mathbf{t}_n = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}),\ \mathbf{x} = (x_1, x_2, \ldots, x_n)$.

Proof. The assertion follows from inequality (10), for $f:(0,\infty)\to\mathbb{R},\ f(x)=x^q$. □

If we take $p=1$ in the above proposition, we get the following result.

Corollary 3.5. *Let $1\le q,\ n\in\mathbb{N}$ with $x_1,x_2\cdots,x_n\in(0,\infty)$. Then, we have the following inequality:*

$$\begin{aligned}&nM_q^q(\mathbf{x};\mathbf{t}_n)-A^q(\mathbf{x};\mathbf{t}_n)\\ \ge\ &\frac{n-1}{n}\left[A^q\left((x_1,x_2);\frac{1}{2}\right)+A^q\left((x_2,x_3);\frac{1}{2}\right)+\cdots+A^q\left((x_n,x_1);\frac{1}{2}\right)\right],\end{aligned}$$

where $\mathbf{t}_n=(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}),\ \mathbf{x}=(x_1,x_2,\dots,x_n)$.

Proposition 3.6. *Let $p<0,\ n\in\mathbb{N}$ with $x_1,x_2\cdots,x_n\in(0,\infty)$. Then, we have the following inequality:*

$$\begin{aligned}&nH^{-1}(\mathbf{x};\mathbf{t}_n)-M_p^{-1}(\mathbf{x};\mathbf{t}_n)\\ \le\ &\frac{n-1}{n}\left[M_p^{-1}\left((x_1,x_2);\frac{1}{2}\right)+M_p^{-1}\left((x_2,x_3);\frac{1}{2}\right)+\cdots+M_p^{-1}\left((x_n,x_1);\frac{1}{2}\right)\right],\end{aligned}$$

where $\mathbf{t}_n=(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}),\ \mathbf{x}=(x_1,x_2,\dots,x_n)$.

Proof. The assertion follows from inequality (10), for $f:(0,\infty)\to\mathbb{R},\ f(x)=-x^{-1}$. □

Proposition 3.7. *Let $0<p\le q,\ n\in\mathbb{N}\setminus\{1\}$ with $a_1,a_2,\cdots,a_n\in(0,\infty)$ then*

$$(n-1)M_q(\mathbf{b};\mathbf{t}_n)\le\ nM_q^q(\mathbf{a};\mathbf{t}_n)-M_p^q(\mathbf{a};\mathbf{t}_n),$$

where $\mathbf{a}=(a_1,a_2,\dots,a_n),\mathbf{t}_n=(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}),a=M_p(\mathbf{a};\mathbf{t}_n),\ b_i=\left[\frac{na^p-a_i{}^p}{n-1}\right]^{\frac{1}{p}}$ *and* $\mathbf{b}=(b_1,b_2,\dots,b_n)$.

Proof. The assertion follows from inequality (11), for $f:(0,\infty)\to\mathbb{R},\ f(x)=x^q$. □

If we take $p=1$ in the above proposition, we get the following result.

Corollary 3.8. *Let $1\le q,\ n\in\mathbb{N}\setminus\{1\}$ with $a_1,a_2,\cdots,a_n\in(0,\infty)$ then*

$$(n-1)M_q(\mathbf{b};\mathbf{t}_n)\le\ nM_q^q(\mathbf{a};\mathbf{t}_n)-A^q(\mathbf{a};\mathbf{t}_n),$$

where $\mathbf{a}=(a_1,a_2,\dots,a_n),\mathbf{t}_n=(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}),a=A(\mathbf{a};\mathbf{t}_n),\ b_i=\left[\frac{na^p-a_i{}^p}{n-1}\right]^{\frac{1}{p}}$ *and* $\mathbf{b}=(b_1,b_2,\dots,b_n)$.

Proposition 3.9. *Let $p<0,\ n\in\mathbb{N}\setminus\{1\}$ with $a_1,a_2,\cdots,a_n\in(0,\infty)$ then*

$$(n-1)H(\mathbf{b};\mathbf{t}_n)\ge\ M_p^{-1}(\mathbf{a};\mathbf{t}_n)-nH^{-1}(\mathbf{a};\mathbf{t}_n),$$

where $\mathbf{a}=(a_1,a_2,\dots,a_n),\mathbf{t}_n=(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}),a=M_p(\mathbf{a};\mathbf{t}_n),\ b_i=\left[\frac{na^p-a_i{}^p}{n-1}\right]^{\frac{1}{p}}$ *and* $\mathbf{b}=(b_1,b_2,\dots,b_n)$.

Proof. The assertion follows from inequality (11), for $f : (0, \infty) \to \mathbb{R}$, $f(x) = -x^{-1}$. $\square$

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Tekin Toplu
Institue of Science, Giresun University,
Giresun, Türkiye.
E-mail: tekintoplu@gmail.com

İmdat İşcan
Department of Mathematics, Giresun University,
Giresun, Türkiye.
E-mail: imdat.iscan@giresun.edu.tr

Selahattin Maden
Department of Mathematics, Ordu University,
Ordu, Türkiye.
E-mail: smaden@odu.edu.tr