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HYPO-CONVERGENCE OF SEQUENCES OF FUZZY SETS AND MAXIMIZATION

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Abstract. In optimization theory, hypo-convergence is considered as an effective tool by providing the convergence of supremum values under some conditions. This feature makes it different from other types of convergence. Therefore, we have defined the hypo-convergence of a sequence of fuzzy sets due to the increasing interest in fuzzy set theory in recent years. After giving a theoretical framework, we deal with the optimization process by using a sequential characterization of hypo-convergence of sequence of fuzzy sets. Since the maximization process in optimization theory is beyond the presence of hypo-convergence, we give some conditions to satisfy the convergence of supremum values. Furthermore, we show how sequence of fuzzy sets and fuzzy numbers differ in the convergence of the supremum values.

1. Introduction

The concept of fuzzy sets has been recognized as an appropriate tool in dealing with uncertain and vague information due to its ability for manipulating the knowledge of ambiguity mathematically. It was introduced by Zadeh [25] for the first time. Applications of this theory can be found in decision theory, artificial intelligence, expert systems, computer science, logic, robotics, operations research and others. Zimmermann [29] explains fuzzy set theory in detail in his book for a newcomer to the field or for somebody who wants to apply fuzzy set theory to his problems. The concept of fuzzy number which is a separate class of fuzzy sets and fuzzy arithmetic was introduced by Zadeh [26, 27, 28] in the year 1975. Matloka [11] defined the ordinary convergence of a sequence of fuzzy numbers and studied on some basic theorems. Nanda [13] showed the completeness of the set of all convergent sequences of fuzzy numbers in a metric space. Limit superior and limit inferior of a bounded sequence of fuzzy numbers were defined by Aytar et al. [2]. Afterwards, some properties of limit superior and limit inferior of a bounded sequence of fuzzy numbers have been obtained by Hong et al. [5], Talo and Çakan [19], Talo [20].

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A fuzzy set can be identified with its hypograph which is also called the endograph in many references. Especially, endograph metric was introduced by Kloeden [8]. The Kuratowski convergence of hypographs leads the way through Γ-convergence of fuzzy sets which was studied by Rojas-Medar and Román-Flores [17]. Moreover, Γ-convergence and endograph metric have attracted a lot of attention from mathematicians working in this field [3, 4, 6, 10, 14]. It should be noted that Γ-convergence in two dimensions is an epi or hypo-convergence (see [1]). Thus, it is related to convergence problems for minimization or maximization of sequences of functions.

Hypo-convergence focuses on hypographs, whereas epi-convergence deals with epigraphs. In literature, epi-convergence is more familiar than hypo-convergence, and it is first studied by Wijsman [23, 24] as infimal convergence at that time. After Wijsman's contributions, many authors contributed to this topic [12, 18, 22].

The importance of hypo-convergence stems from the fact that it is the appropriate notion of convergence for maximization problems. It finds optimal solutions by preserving the convergence of supremum values under some conditions in optimization process. Particularly, Proposition 3.1 in [7] and Theorem 2 in [15] show the importance of the convergence of infimum values of a sequence of functions. Our work focuses on the convergence of supremum values of sequence of fuzzy sets for maximization problems. Hence, we deal with the hypo-convergence of sequences of fuzzy sets and we will use it in the field of optimization as we have used statistical epi-convergence in our previous study [21]. Furthermore, we will show how a sequence of fuzzy sets and fuzzy numbers differ in the convergence of the supremum values.

2. Preliminaries

Definition 2.1. [25] *A fuzzy set is a pair (U, f) where U is a set and $f : U \to [0, 1]$ a membership function. For each $x \in U$, the value $f(x)$ is called the grade of membership of x in (U, f). A fuzzy set on $\mathbb{R}^m$ will be denoted by $F(\mathbb{R}^m)$.*

Definition 2.2. [26, 27, 28] *A mapping $u : \mathbb{R} \to [0, 1]$ is a fuzzy number if it satisfies the following conditions:*

(i) $\exists x_0 \in \mathbb{R}$ such that $u(x_0) = 1$ (u is normal).

(ii) $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0, 1]$ (u is fuzzy convex).

(iii) u is upper semi-continuous.

(iv) The set $u^0 := \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact.

E^1 denotes the set of all fuzzy numbers on $\mathbb{R}$.

λ-level set u^λ of $u \in F(\mathbb{R})$ is defined by

$$u^\lambda = \begin{cases} \overline{\{x \in \mathbb{R} : u(x) \geq \lambda\}} & , \quad (0 < \lambda \leq 1) \\ \{x \in \mathbb{R} : u(x) > \lambda\} & , \quad (\lambda = 0) \end{cases} .$$

The set $u^\lambda = [\underline{u}^\lambda, \overline{u}^\lambda]$ is closed, bounded and non-empty interval for each $\lambda \in [0, 1]$.

In this paper, we will use Kuratowski convergence [9] in order to define hypo convergence of a fuzzy number. Before moving on to definitions, we will use the following collections of subsets of $\mathbb{N}$.

$$\begin{aligned} \mathcal{N} &= \{N \subseteq \mathbb{N} : \mathbb{N} \backslash N \text{ finite}\}, \\ \mathcal{N}^\# &= \{N \subseteq \mathbb{N} : N \text{ infinite}\}. \end{aligned}$$

In a metric space (X, d), $\mathcal{N}(x)$ denotes the family of all open sets containing the point $x \in X$.

Definition 2.3. [16] *The outer and inner limit of a sequence (A_n) of closed subsets of X is the following sets*

$$\begin{aligned} \limsup_n A_n &= \Big\{x \mid \forall\, V \in \mathcal{N}(x),\ \exists N \in \mathcal{N}^\#,\ \forall n \in N : A_n \cap V \neq \emptyset\Big\} \\ &= \Big\{x \mid \exists N \in \mathcal{N}^\#,\ \forall n \in N,\ \exists x_n \in A_n : \lim_{n \in N} x_n = x\Big\}, \end{aligned}$$

$$\begin{aligned} \liminf_n A_n &= \Big\{x \mid \forall\, V \in \mathcal{N}(x),\ \exists N \in \mathcal{N},\ \forall n \in N : A_n \cap V \neq \emptyset\Big\} \\ &= \Big\{x \mid \exists N \in \mathcal{N},\ \forall n \in N,\ \exists x_n \in A_n : \lim_{n \in N} x_n = x\Big\}. \end{aligned}$$

respectively. If the following equality holds,

$$\lim_n A_n = \liminf_n A_n = \limsup_n A_n$$

then, we say that the limit of (A_n) exists.

Definition 2.4. [16] *Let f be a function defined on X, the hypograph of f is the set hypo$(f) = \{(x, \alpha) \in X \times \mathbb{R} \mid \alpha \leq f(x)\}$.*

Definition 2.5. [16] *Given an arbitrary set X, a totally ordered set Y, and a function $f : X \to Y$, the argmaxf is defined by*

$$\text{argmax} f = \{x \in X : f(x) = \sup f\}.$$

Definition 2.6. [16] *Given an arbitrary set X, a totally ordered set Y and a function $f : X \to Y$, the ε-argmax f is defined by*

$$\varepsilon\text{-}argmax f = \{x \in X : f(x) \geq \sup f - \varepsilon\}.$$

3. Main result

In this section, we define hypo-convergence of an upper semicontinuous sequence of fuzzy sets. Then it is followed by a topological definition and a sequential characterization. These definitions will guide us in obtaining necessary condition for solving maximization problems. At the end, we will show how to achieve the convergence ε_n-$argmax u_n \to argmax u$ for hypo-convergent sequence of fuzzy sets. In addition to this, the property of normality for hypo-convergent fuzzy numbers will satisfy the convergence of supremum values.

Definition 3.1. *Let (u_n) be upper semicontinuous sequence of fuzzy sets in $F(\mathbb{R})$. Hypo-limit inferior $h_{st}\text{-}\liminf_n u_n$ is defined by:*

$$hypo(h\text{-}\liminf_n u_n) = \liminf_n hypo(u_n).$$

Similarly hypo-limit superior $h_{st}\text{-}\limsup_n u_n$ is defined by:

$$hypo(h\text{-}\limsup_n u_n) = \limsup_n hypo(u_n).$$

When these two functions equal to each other, we have $h\text{-}\lim_n u_n = h\text{-}\liminf_n u_n = h\text{-}\limsup_n u_n$. Hence the sequence (u_n) is said to be hypo-convergent to $u \in F(\mathbb{R})$. It is symbolized by

$$u_n \xrightarrow{h} u.$$

According to the Definition 3.1, *the relation between Kuratowski convergence and hypo-convergence of sequence of fuzzy sets appears in the following equality.*

$$u_n \xrightarrow{h} u \Longleftrightarrow hypo(u_n) \to hypo(u).$$

Theorem 3.2. *For every $x \in \mathbb{R}$, if we define upper semicontinuous $u \in F(\mathbb{R})$ by*

$$u(x) = \inf_{V \in \mathcal{N}(x)} \limsup_n \sup_{y \in V} u_n(y),$$

then $\limsup_n hypo(u_n) = hypo(u)$.

Proof. We will establish the hypographical inclusions of the sets

$$\limsup_n hypo(u_n) \subset hypo(u)$$

and

$$hypo(u) \subset \limsup_n hypo(u_n).$$

For the first inclusion, let $(x, \mu) \in \limsup_n hypo(u_n)$. Let $V_0 \in \mathcal{N}(x)$, $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ be arbitrary. Since the range of a fuzzy set is $[0, 1]$, it is clear that $V_0 \times (\mu - \varepsilon, 1 + \varepsilon)$ is a neighbourhood of (x, μ). By using the definition of outer limit of a set, $\exists n' \in \mathbb{N}$ with $n' \geq n_0$ we have

$$V_0 \times (\mu - \varepsilon, 1 + \varepsilon) \bigcap hypo(u_{n'}) \neq \emptyset.$$

As a result,

$$\sup_{n\geq n_0}\sup_{y\in V_0} u_n(y) \geq \sup_{y\in V_0} u_{n'}(y) \geq \mu-\varepsilon$$

Hence we have,

$$\limsup_n \sup_{y\in V_0} u_n(y) \geq \mu-\varepsilon$$

Since V_0 and ε are arbitrary, we can take infimum of both sides.

$$\inf_{V\in\mathcal{N}(x)} \limsup_n \sup_{y\in V} u_n(y) = u(x) \geq \mu$$

It shows that $u(x) \geq \mu$ and $(x,\mu)\in hypo(u)$.
For the second inclusion let $(x,\mu)\in hypo(u)$. Let $V_0\in\mathcal{N}(x)$ and $\varepsilon>0$ be arbitrary. Since the range of a fuzzy set is $[0,1]$, we must show $V_0\times(\mu-\varepsilon,1+\varepsilon)$ meets $hypo(u_n)$ frequently. Let $n_0\in\mathbb{N}$ be fixed. We know that

$$\mu-\varepsilon < u(x) \leq \limsup_n \sup_{y\in V_0} u_n(y) \leq \sup_{n\geq n_0}\sup_{y\in V_0} u_n(y).$$

$\exists n'\geq n_0$ such that

$$\sup_{y\in V_0} u_{n'}(y) > \mu-\varepsilon.$$

Hence, we have

$$\Big(V_0\times(\mu-\varepsilon,1+\varepsilon)\Big)\bigcap hypo(u_{n'}) \neq \emptyset$$

which means $(x,\mu)\in\limsup_n hypo(u_n)$. □

Theorem 3.3. *For every $x\in\mathbb{R}$, if we define upper semicontinuous $v\in F(\mathbb{R})$ by*

$$v(x) = \inf_{V\in\mathcal{N}(x)} \liminf_n \sup_{y\in V} u_n(y),$$

then $\liminf_n hypo(u_n) = hypo(v)$.

Proof. We want to show

$$\liminf_n hypo(u_n) \subset hypo(v)$$

and

$$hypo(v) \subset \liminf_n hypo(u_n).$$

For the first inclusion, let $(x,\mu)\in\liminf_n hypo(u_n)$. Let $V_0\in\mathcal{N}(x)$ and $\varepsilon>0$ be arbitrary. Since the range of a fuzzy set is $[0,1]$, it is clear that $V_0\times(\mu-\varepsilon,1+\varepsilon)$ is a neighbourhood of (x,μ). By using the definition of inner limit of a set, $\exists n_0\in\mathbb{N}$ such that

$$V_0\times(\mu-\varepsilon,1+\varepsilon)\bigcap hypo(u_n)\neq\emptyset$$

for each $n \geq n_0$. Thus, for all $n \geq n_0$ we have

$$\sup_{y \in V_0} u_n(y) > \mu - \varepsilon$$

and so

$$\liminf_n \sup_{y \in V_0} u_n(y) \geq \inf_{n \geq n_0} \sup_{y \in V_0} u_n(y) \geq \mu - \varepsilon.$$

Since $V_0 \in \mathcal{N}(x)$ and $\varepsilon > 0$ are arbitrary,

$$\inf_{V \in \mathcal{N}(x)} \liminf_n \sup_{y \in V} u_n(y) = v(x) > \mu$$

and hence $(x, \mu) \in hypo(v)$.

For the second inclusion, let $(x, \mu) \in hypo(v)$. Let $V_0 \in \mathcal{N}(x)$ and $\varepsilon > 0$ be arbitrary. We need to show that $V_0 \times (\mu - \varepsilon, 1 + \varepsilon)$ meets $hypo(u_n)$ eventually. It is obvious that

$$\liminf_n \sup_{y \in V_0} u_n(y) \geq v(x) > \mu - \varepsilon.$$

Then, $\exists n_0 \in \mathbb{N}$ such that

$$\inf_{n \geq n_0} \sup_{y \in V_0} u_n(y) > \mu - \varepsilon.$$

The inequality $\sup_{y \in V_0} u_n(y) > \mu - \varepsilon$ is valid for all $n \geq n_0$. Hence,

$$V_0 \times (\mu - \varepsilon, 1 + \varepsilon) \bigcap hypo(u_n) \neq \emptyset$$

for all $n \geq n_0$ which is equal to

$$(x, \mu) \in \liminf_n hypo(u_n).$$

□

The following definition of hypo-limit inferior and superior of a sequence of fuzzy sets is a result of Theorem 3.2 and Theorem 3.3.

Definition 3.4. *Let (u_n) be an upper semicontinuous sequence of fuzzy sets in $F(\mathbb{R})$. For every $x \in \mathbb{R}$, hypo-limit inferior is defined by*

$$\left(h\text{-}\liminf_n u_n\right)(x) = \inf_{V \in \mathcal{N}(x)} \liminf_n \sup_{y \in V} u_n(y).$$

Hypo-limit superior is defined by

$$\left(h\text{-}\limsup_n u_n\right)(x) = \inf_{V \in \mathcal{N}(x)} \limsup_n \sup_{y \in V} u_n(y).$$

If $\exists u \in F(\mathbb{R})$ such that

$$h\text{-}\liminf_n u_n = h\text{-}\limsup_n u_n = u,$$

then we write $u = h\text{-}\lim_n u_n$ and we say that (u_n) is h-convergent to u on $\mathbb{R}$.

The following theorem shows the relationship between hypo-convergence and convergence of λ-cuts. It is necessary for the proof of Theorem 3.10 that is related to the convergence of argmax values of a sequence of fuzzy sets.

Theorem 3.5. *Let (u_n) be an upper semicontinuous sequence of fuzzy sets in $F(\mathbb{R})$. If for each $\lambda \in [0,1]$, there exists a sequence $\lambda_n \in [0,1]$ satisfying $\lambda_n \to \lambda$ with $u^\lambda = \lim_n u_n^{\lambda_n}$, then $u_n \overset{h}{\to} u$.*

Proof. Assume that $\lambda_n \to \lambda$. Since $u^\lambda = \lim_n u_n^{\lambda_n}$, we can use the inclusion $u^\lambda \subset \liminf_n u_n^{\lambda_n}$. Let $(x,\lambda) \in hypo(u)$. It means $x \in u^\lambda$ and $x \in \liminf_n u_n^{\lambda_n}$. Then, there exists n_0 and a sequence (x_n) satisfying $x_n \to x$ such that $x_n \in u_n^{\lambda_n}$ for each $n \geq n_0$. It is also equal to $u_n(x_n) \geq \lambda_n$ for each $n \geq n_0$. Then,

$$(x_n, \lambda_n) \in hypo(u_n)$$

for each $n \geq n_0$ and $\lim_n (x_n, \lambda_n) = (x, \lambda)$ which is equal to $(x,\lambda) \in \liminf_n hypo(u_n)$ and hence

$$hypo(u) \subset \liminf_n hypo(u_n).$$

Now we want to show $\limsup_n hypo(u_n) \subset hypo(u)$. Suppose contrary that is $(x,\beta) \in \limsup_n hypo(u_n)$ but $(x,\beta) \notin hypo(u)$. Then we have $u(x) < \beta$. There exists a sequence of integers $n_1 < n_2 < n_3 < \ldots$ and $(x_k, \beta_k) \in hypo(u_{n_k})$ such that

$$(x_k, \beta_k) \to (x, \beta).$$

Choose a scalar λ satisfying $u(x) < \lambda < \beta$ with $\lambda_n \to \lambda$ for which $u^\lambda = \lim_n u_n^{\lambda_n}$. Then for all k sufficiently large, the inequality $\lambda_{n_k} < \beta_k$ holds and we get

$$x_k \in u_{n_k}^{\lambda_{n_k}} \text{ and } x \in \limsup_n u_n^{\lambda_n}.$$

Since we have $u^\lambda = \lim_n u_n^{\lambda_n}$, we can use the inclusion $\limsup_n u_n^{\lambda_n} \subset u^\lambda$ and it gives $x \in u^\lambda$. Hence $\lambda < u(x)$ which contradicts $u(x) < \lambda$. Consequently,

$$\limsup_n hypo(u_n) \subset hypo(u).$$

□

The following theorem gives a sequential characterization of the hypo-convergence of a sequence of fuzzy sets. The importance of this theorem stems from its use in the proof of Theorem 3.7 which is important for optimization problems.

Theorem 3.6. *Let (u_n) be an upper semicontinuous sequence of fuzzy sets in $F(\mathbb{R})$ and let x be any point of $\mathbb{R}$. Then, u_n is hypo-convergent to $u \in F(\mathbb{R})$ if and only if at each point x one has*

(i) for every sequence (x_n) with $x_n \to x$,

$$\limsup_n u_n(x_n) \leq u(x),$$

(ii) there exists a sequence (x_n) with $x_n \to x$,

$$\liminf_n u_n(x_n) \geq u(x).$$

Proof. Let us first assume that $u_n \xrightarrow{h} u$. Let (x_n) be an arbitrary sequence convergent to x. Let $\alpha < \limsup_n u_n(x_n)$ be arbitrary. Then there exists an increasing sequence of integers (n_k) such that for each k, $u_{n_k}(x_{n_k}) > \alpha$. That is

$$(x_{n_k}, \alpha) \in hypo(u_{n_k}).$$

Due to our assumption $u_n \xrightarrow{h} u$, we have

$$\limsup_n hypo(u_n) \subset hypo(u).$$

We obtain $(x, \alpha) \in hypo(u)$ which means $\alpha \leq u(x)$. Hence we have verified (i).

In order to show (ii), let $(x, u(x)) \in hypo(u)$. Since we have $u_n \xrightarrow{h} u$, we can use the inclusion $hypo(u) \subset \liminf_n hypo(u_n)$. Then there exists $N \in \mathcal{N}$, for all $n \in N$, $\exists (x_n, \alpha_n) \in hypo(u_n)$ such that

$$(x_n, \alpha_n) \to (x, u(x)).$$

Moreover, for all $n \in N$ we have $u_n(x_n) \geq \alpha_n$. Then we obtain

$$\liminf_n u_n(x_n) \geq u(x).$$

For the converse, assume (i) and (ii) both hold. First we will show $hypo(u) \subset \liminf_n hypo(u_n)$. Let (x, α) be fixed with $\alpha < u(x)$. According to our assumption, there exists a sequence (x_n) such that $\liminf_n u_n(x_n) \geq u(x)$. Then there exists $N \in \mathcal{N}$ such that for all $n \in N$, $\alpha \leq u_n(x_n)$, that is

$$(x_n, \alpha) \in hypo(u_n).$$

Then we have $(x, \alpha) \in \liminf_n hypo(u_n)$.

Now we want to show $\limsup_n hypo(u_n) \subset hypo(u)$. Let $(x, \alpha) \in \limsup_n hypo(u_n)$. Then there exists $N \in \mathcal{N}^{\#}$, for all $n \in N$, $\exists (x_n, \alpha_n) \in hypo(u_n)$ such that

$$(x_n, \alpha_n) \to (x, \alpha).$$

Then, we obtain

$$\begin{aligned} \alpha = \lim_{n \in N} \alpha_n &\leq \liminf_{n \in N} u_n(x_n) \\ &\leq \limsup_n u_n(x_n) \leq u(x) \end{aligned}$$

It gives $(x, \alpha) \in hypo(u)$ which concludes the proof. □

Theorem 3.7. *Let $u_n, u, n = 1, 2, ...,$ be upper semicontinuous fuzzy sets in $F(\mathbb{R})$. Suppose $u_n \xrightarrow{h} u$. If for every $\varepsilon > 0$*

$$\text{argmax} u = \bigcap_{\varepsilon > 0} \liminf_n \big(\varepsilon\text{-argmax} u_n\big),$$

then

$$\sup_{x \in X} u_n(x) \to \sup_{x \in X} u(x).$$

Proof. For each $t \in argmaxu$ and $\varepsilon > 0$, there corresponds a sequence $x_n \in \varepsilon\text{-}argmaxu_n$ such that $x_n \to t$. Since $u_n \xrightarrow{h} u$ by using Theorem 3.6 (i) we obtain

$$\begin{aligned} \sup_{x\in X} u(x) = u(t) &\geq \limsup_n u_n(x_n) \\ &\geq \limsup_n \sup_{x\in X} u_n(x) - \varepsilon. \end{aligned} \tag{3.1}$$

Now assume that $(x, \sup_{x\in X} u(x)) \in hypo(u)$. Since $u_n \xrightarrow{h} u$ we can use the inclusion $hypo(u) \subseteq \liminf_n hypo(u_n)$. Then, there exists $(x_n, y_n) \in hypo(u_n)$ such that

$$(x_n, y_n) \to (x, \sup_{x\in X} u(x)).$$

Observe that there exists $N \in \mathcal{N}$ such that for all $n \in N$ we have $y_n \leq \sup_{x\in X} u_n(x)$. By taking liminf of both sides we get

$$\sup_{x\in X} u(x) = \liminf_n y_n \leq \liminf_n \sup_{x\in X} u_n(x). \tag{3.2}$$

$\sup_{x\in X} u_n(x) \to \sup_{x\in X} u(x)$ follows from (3.1) and (3.2). □

Example 3.8. *Let (u_n) be a sequence of fuzzy sets in $F(\mathbb{R})$ and it is defined by*

$$u_n(x) = \max\{0, 1 - n\big|x + \frac{1}{n}\big|\}.$$

Obviously, the sequence (u_n) is hypo-convergent to the fuzzy set u given below.

$$u(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The sequence (u_n) satisfies the conditions in Theorem 3.7, that is

$$\bigcap_{\varepsilon>0} \liminf_n \big(\varepsilon\text{-}argmaxu_n\big) = argmaxu = \{0\}.$$

Hence we have $\sup_{x\in X} u_n(x) \to \sup_{x\in X} u(x)$ which means that maximizers of (u_n) must maximize u which is an essential property in optimization problems.

Example 3.9. *Let (u_n) be a sequence of fuzzy sets in $F(\mathbb{R})$ and it is defined by*

$$u_n(x) = \begin{cases} 1 - \frac{1}{n}, & \text{if } x = n, \\ 0, & \text{otherwise.} \end{cases}$$

The sequence (u_n) is hypo-convergent to $u(x) = 0$. On the other hand, (u_n) does not satisfy the conditions in Theorem 3.7 even though it is hypo-convergent. We obtain

$$\text{argmax}u = \mathbb{R}$$

for hypo-limit function u but

$$\bigcap_{\varepsilon>0} \liminf_n \left(\varepsilon\text{-argmax}u_n\right) = \emptyset.$$

Hence $\sup_{x\in X} u_n(x) \to 1$ *which is not equal to* $\sup_{x\in X} u(x) = 0$.

Theorem 3.10. *Let* $u_n, u, n = 1, 2, ...,$ *be upper semicontinuous fuzzy sets in* $F(\mathbb{R})$. *Assume that* $u_n \xrightarrow{h} u$. *If* $\sup_{x\in X} u_n(x) \to \sup_{x\in X} u(x)$, *then* $\exists \varepsilon_n \to 0$ *such that*

$$\varepsilon_n\text{-argmax}u_n \to \text{argmax}u. \tag{3.3}$$

Proof. Suppose $u_n \xrightarrow{h} u$ and $\beta_n = \sup_{x\in X} u_n(x) \to \sup_{x\in X} u(x) = \beta$. Since (u_n) is a sequence of fuzzy sets, (β_n) is finite for all $n \in \mathbb{N}$. By Theorem 3.5, there exists a sequence $\lambda_n \to \beta$ such that

$$u_n^{\lambda_n} \to u^\beta = argmaxu.$$

If we write $\varepsilon_n = \beta_n - \lambda_n$, then $\exists N \in \mathcal{N}$ such that $\forall n \in N$ we have

$$\begin{aligned} \varepsilon_n\text{-}argmaxu_n &= \{x \mid u_n(x) \geq \sup u_n - \varepsilon_n\} \\ &= \{x \mid u_n(x) \geq \lambda_n\} \\ &= u_n^{\lambda_n} \to u^\beta = argmaxu. \end{aligned}$$

□

So far we have dealt with hypo-convergence of sequence of fuzzy sets in $F(\mathbb{R})$. We have shown the convergence of supremum values of (u_n) which is necessary for optimization theory. Convergence of argmax of (u_n) is as important as convergence of supremum values of (u_n). In the last theorem, we have given the necessary conditions for the convergence

$$\varepsilon_n\text{-}argmaxu_n \to argmaxu.$$

If we are working on fuzzy numbers, we do not need any condition to achieve this convergence. Hence, the following corollary will be useful for optimization problems on fuzzy numbers.

Corollary 3.11. *Let* $u_n, u, n = 1, 2, ...,$ *be fuzzy numbers in* E^1. *If* $u_n \xrightarrow{h} u$, *then* $\exists \varepsilon_n \to 0$ *such that*

$$\varepsilon_n\text{-argmax}u_n \to \text{argmax}u. \tag{3.4}$$

Proof. The property of normality for fuzzy numbers implies

$$\sup_{x\in X} u_n(x) \to \sup_{x\in X} u(x)$$

whenever $u_n \xrightarrow{h} u$. Thus, the proof is clear from the Theorem 3.10. □

4. Conclusion, future work

Hypo-convergence is used in optimization theory for productivity. Optimal solutions are found by ensuring the convergence of supremum values. Since the maximization process in optimization theory is beyond the presence of hypo-convergence, we have given some conditions to satisfy the convergence of supremum values. These conditions are very important for obtaining correct results in optimization. Theorem 3.7 gives the necessary conditions for $\sup_{x\in X} u_n(x) \to \sup_{x\in X} u(x)$. In its proof, sequential characterization of hypo-convergence is used given by Theorem 3.6. At this stage Example 3.9 is important. In this example, the convergence $\sup_{x\in X} u_n(x) \to \sup_{x\in X} u(x)$ fails even though the sequence (u_n) is hypo-convergent, since (u_n) does not satisfy the conditions in Theorem 3.7. In Theorem 3.10, we have given the necessary conditions for the convergence ε_n-$argmax u_n \to argmax u$. This theorem explains that the maximizers of a sequence of fuzzy sets (u_n) converge to the maximizer of the fuzzy set u. In the proof of the theorem, we have used the relation between the convergence of lambda cuts and hypo-convergence of a sequence of fuzzy sets mentioned in Theorem 3.5. In summary, all the theorems in the main result section are connected.

In some situations, some of the functions may not conform to a usual pattern and affect the efficiency of optimization process. Moreover, obtaining hypo-limit function may fail due to disruption of these functions. For that reason, it may be necessary to use an alternative method that diminishes the effect of such functions by excluding them from consideration. In our next work, we will develop a method to eliminate these functions.

References

[1] H. Attouch, *Variational Convergence for Functions and Operators*, Pitman Advanced Pub. Program, Boston, 1984.

[2] S. Aytar, S. Pehlivan, and M. Mammadov, *The core of a sequence of fuzzy numbers*, Fuzzy Sets and Systems **159** (2008), no. 24, 3369–3379.

[3] T. Fan, *Endographic approach on supremum and infimum of fuzzy numbers*, Inf. Sci. **159** (2004), 221–231.

[4] G. H. Greco, M. P. Moschen, and E. Quelho Frota Rezende, *On the variational convergence for fuzzy sets in metric spaces*, Ann. Univ. Ferrara, Sez. VII, Sci. Mat. **44** (1998), 27–39.

[5] D. H. Hong, E. L. Moon, and J. D. Kim, *A note on the core of fuzzy numbers*, Applied Mathematics Letters **23** (2010), no. 5, 651–655.

[6] H. Huang, *Characterizations of endograph metric and* Γ*-convergence on fuzzy sets*, Fuzzy Sets Syst. **350** (2018), 55–84.

[7] A. J. King and R. Wets, *Epi-consistency of convex stochastic programs*, Stochastics Stochastics Rep. **34** (1991), 83–92.

[8] P.E. Kloeden, *Compact supported endographs and fuzzy sets*, Fuzzy Sets Syst. **4** (1980), no. 2, 193–201.

[9] C. Kuratowski, *Topologie.* Vol. I, PWN, Warszawa, 1958.

[10] G. D. Maso, *An introduction to* Γ*-convergence*, Vol. 8, Boston, 1993.

[11] M. Matloka, *Sequences of fuzzy numbers*, BUSEFAL **28** (1986), 28–37.
[12] U. Mosco,*Convergence of convex sets and of solutions of variational inequalities*, Adv. Math. **3** (1969), 510–585.
[13] S. Nanda, *On sequences of fuzzy numbers*, Fuzzy Sets and Systems **33** (1989), 123–126.
[14] T. Pedraza, J. Rodriguez-Lápez, and S. Romaguera, *Convergence of fuzzy sets with respect to the supremum metric*, Fuzzy Sets Syst. **245** (2014), 83–100.
[15] T. Pennanen, *Epi-convergent discretizations of multistage stochastic programs,* Mathematics of Operations Research **30** (2005), no. 1, 245–256.
[16] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer, Berlin, 1998.
[17] M. Rojas-Medar and H. Román-Flores, *On the equivalence of convergences of fuzzy sets*, Fuzzy Sets Syst. **80** (1996), 217–224.
[18] G. Salinetti and R. J-B. Wets, *On the relation between two types of convergence for convex functions*, J. Math. Anal. Appl. **60** (1977), 211–226.
[19] Ö. Talo and C. Çakan, *The extension of the Knopp core theorem to the sequences of fuzzy numbers,* Information Sciences **276** (2014), 10–20.
[20] Ö. Talo, *Some properties of limit inferior and limit superior for sequences of fuzzy real numbers*, Information Sciences **279** (2014), 560–568.
[21] Ş. Tortop, *Sequential characterization of statistical epi-convergence*, Soft Computing **24** (2020), 18565–18571.
[22] R. J-B. Wets, *Convergence of convex functions, variational inequalities and convex optimization problems,* New York, 1980.
[23] R. A. Wijsman, *Convergence of sequences of convex sets, cones and functions*, Bull. Amer. Math. Soc. **70** (1964), 186–188.
[24] R. A. Wijsman, *Convergence of sequences of convex sets, cones and functions II*, Trans. Amer. Math. Soc. **123** (1966), 32–45.
[25] L. A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338–353.
[26] L. A. Zadeh, *The concept of linguistic variable and its application to approximate reasoning-I*, Information Sciences **8** (1975), no. 3, 199–249.
[27] L. A. Zadeh, *The concept of linguistic variable and its application to approximate reasoning-I*, Information Sciences **8** (1975), no. 4, 301–357.
[28] L. A. Zadeh, *The concept of linguistic variable and its application to approximate reasoning-I*, Information Sciences **9** (1975), no. 1, 43–80.
[29] H.-J. Zimmermann, *Fuzzy Set Theory and its Applications*, 4th Edition, Norwell, Kluwer, MA, 2001.

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