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PROPER BI-SLANT PSEUDO-RIEMANNIAN SUBMERSIONS WHOSE TOTAL MANIFOLDS ARE PARA-KAEHLER MANIFOLDS

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Abstract. In this paper, bi-slant pseudo-Riemannian submersions from para-Kaehler manifolds onto pseudo-Riemannian manifolds are introduced. We examine some geometric properties of three types of bi-slant submersions. We give non-trivial examples of such submersions. Moreover, we obtain curvature relations between the base space, total space and the fibers.

1. Introduction

A $C^{\infty}-$submersion ψ can be defined according to the following conditions. A pseudo-Riemannian submersion ([7],[18],[23],[24],[36],[3]), an almost Hermitian submersion ([43],[13],[4]), a slant submersion ([9],[12],[26],[33]), a para quaternionic submersion ([19]), a Clairaut submersion ([15]), an anti-invariant submersion ([14],[16],[34],[11]), anti-invariant Riemannian submersion from cosymplectic manifolds ([17]),bi-slant submanifold ([8]), bi-slant submer sion([39]), a quasi-bi-slant submersion ([28],[29],[30],[31]), a pointwise slant submersion([22],[40]), a hemi-slant submersion ([41],[38]), a semi-invariant submersion ([25],[35]), a semi-slant $\xi^{\perp}$- Riemannian submersions ([1],[2],[27]), etc. As we know, Riemannian submersions were severally introduced by B. O'Neill ([24]) and A. Gray ([18]) in 1960s. In particular, by using the concept of almost Hermitian submersions, B. Watson ([43]) gave some differential geometric properties among fibers, base manifolds, and total manifolds. Some interesting results concerning para-Kaehler-like statistical submersions were obtained by G.E. Vîlcu ([42]).

In this paper, we examine some geometric properties of three types of proper bi-slant pseudo-Riemannian submersions. Let's list the section of our work. In Section 2, we gather some concepts, which are needed in the following parts. In

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Section 3, we study some geometric properties of three types of proper bi-slant pseudo-Riemannian submersions from almost para-Hermitian manifolds onto pseudo-Riemannian manifolds. We present examples, study the geometry of leaves of distributions. We also obtain necessary and sufficient conditions for a proper bi- slant pseudo-Riemannian submersions to be totally geodesic map. In the final section, we obtain curvature properties between the base space, total space and the fibers.

2. Preliminaries

By a para-Hermitian manifold we mean a triple $(\mathcal{B}, \mathcal{P}, g_{\mathcal{B}})$,where $\mathcal{B}$ is connected differentiable manifold of $2n$- dimensional , $\mathcal{P}$ is a tensor field of type (1,1) and a pseudo-Riemannian metric $g_{\mathcal{B}}$ on $\mathcal{B}$, satisfying

$$\mathcal{P}^2 E_1 = E_1, \quad g_{\mathcal{B}}(\mathcal{P}E_1, \mathcal{P}E_2) = -g_{\mathcal{B}}(E_1, E_2), \tag{1}$$

where E_1, E_2 are vector fields on $\mathcal{B}$. An almost para-Hermitian manifold $\mathcal{B}$ is said to be a para-Kaehler manifold if

$$\nabla \mathcal{P} = 0, \tag{2}$$

where ∇ denotes the Riemannian connection on $\mathcal{B}$ ([21]).

Let $(\mathcal{B}, g_{\mathcal{B}})$ and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be two pseudo-Riemannian manifolds. A pseudo-Riemannian submersion is a smooth map $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ satisfying the following two axioms
(i) the fibres $\psi^{-1}(q)$, $q \in \tilde{\mathcal{B}}$, are $r-$ dimensional pseudo-Riemannian submanifolds of $\mathcal{B}$, where $r = dim(\mathcal{B}) - dim(\tilde{\mathcal{B}})$.
(ii) ψ_* preserves scalar products of vectors normal to fibres.

The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. A vector field U on $\mathcal{B}$ is called basic if U is horizontal and π- related to a vector field U_* on $\tilde{\mathcal{B}}$, i.e., $\pi_* U_p = U_{*\pi_p}$ for all $p \in \mathcal{B}$. We indicate by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. We know that $(\mathcal{B}, g_{\mathcal{B}})$ is called total manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is called base manifold of the submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$.

Define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ by:

$$\mathcal{T}_U \mathcal{W} = h\nabla_{vU} v\mathcal{W} + v\nabla_{vU} h\mathcal{W} \tag{3}$$

and

$$\mathcal{A}_U \mathcal{W} = v\nabla_{hU} h\mathcal{W} + h\nabla_{hU} v\mathcal{W} \tag{4}$$

for every $U, \mathcal{W} \in \chi(\mathcal{B})$, on $\mathcal{B}$ where ∇ is the Levi-Civita connection of $g_{\mathcal{B}}$.

It is easy to see that a pseudo-Riemannian submersion $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. Also, if $\mathcal{A}$ vanishes then the horizontal distribution is integrable. (see [7],[10]). Using (3) and (4), we get

$$\nabla_U W = \mathcal{T}_U W + \hat{\nabla}_U W; \tag{5}$$

$$\nabla_U \zeta = \mathcal{T}_U \zeta + h\nabla_U \zeta; \tag{6}$$

$$\nabla_\zeta U = \mathcal{A}_\zeta U + v\nabla_\zeta U; \tag{7}$$

$$\nabla_\zeta \eta = \mathcal{A}_\zeta \eta + h\nabla_\zeta \eta, \tag{8}$$

for any $\zeta, \eta \in \Gamma(ker\psi_*)^\perp$, $U, W \in \Gamma(ker\psi_*)$. Also, if ζ is basic then $h\nabla_U \zeta = h\nabla_\zeta U = \mathcal{A}_\zeta U$.

It is easily seen that $\mathcal{T}$ is symmetric on the vertical distribution and $\mathcal{A}$ is alternating on the horizontal distribution such that

$$\mathcal{T}_\mathcal{W} U = \mathcal{T}_U \mathcal{W}, \quad \mathcal{W}, U \in \Gamma(ker\psi_*); \tag{9}$$

$$\mathcal{A}_Y V = -\mathcal{A}_V Y = \frac{1}{2} v[Y, V], \quad Y, V \in \Gamma(ker\psi_*)^\perp. \tag{10}$$

Also, it is easily seen that $\mathcal{T}_\mathcal{E}$ and $\mathcal{A}_\mathcal{E}$ are skew-symmetric operators on $\Gamma(T\mathcal{B})$ for any $\mathcal{E} \in \Gamma(T\mathcal{B})$ such that

$$g_\mathcal{B}(\mathcal{T}_\mathcal{W} U, \mathcal{X}) = -g_\mathcal{B}(\mathcal{T}_\mathcal{W} \mathcal{X}, U), \tag{11}$$

$$g_\mathcal{B}(\mathcal{A}_\mathcal{W} U, \mathcal{X}) = -g_\mathcal{B}(\mathcal{A}_\mathcal{W} \mathcal{X}, U). \tag{12}$$

Remark 2.1. *In present paper, we assume that all horizontal vector fields are basic vector fields.*

Let $\psi : (\mathcal{B}, g_\mathcal{B}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_\mathcal{B}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. A pseudo-Riemannian submersion ψ is called a slant submersion if the angle $\varphi(W)$ between PW and space $(ker\psi_*)_q$ is constant for non-null vector field $W \in (ker\psi_*)$ and $q \in \mathcal{B}$, we can say that φ is a slant angle ([16]).

Let $\psi : (\mathcal{B}, g_\mathcal{B}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is a slant submersion with the slant angle φ. If $\varphi = 0$ we can say that the map ψ an invariant submersion [37]. Then, If $\varphi = \frac{\pi}{2}$ we can say that the map ψ an anti-invariant submersion [34]. In other cases, it is called a proper slant submersions.

Let $(\mathcal{B}, g_\mathcal{B})$ and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be pseudo-Riemannian manifolds and $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ is a differentiable map. Then the second fundamental form of ψ is given by

$$(\nabla\psi_*)(X, V) = \nabla^\psi_X \psi_* V - \psi_*(\nabla_X V) \tag{13}$$

for $X, V \in \Gamma(\mathcal{B})$. Here we indicate conveniently by ∇ the Riemannian connections of the metrics $g_\mathcal{B}$ and $g_{\tilde{\mathcal{B}}}$. Recall that ψ is said to be *harmonic* if $trace(\nabla\psi_*) = 0$ and ψ is called a *totally geodesic* map if $(\nabla\psi_*)(X, V) = 0$ for $X, V \in \Gamma(T\mathcal{B})$ ([20]). Note that ∇^ψ is the pullback connection.

Proposition 2.2. *For every vertical vector fields $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$ and for every horizontal vector fields $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$ the following Riemannian curvature tensor R is given by ([24]).*

$$\begin{aligned} R(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4) &= \tilde{R}(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4) - g(\mathcal{T}_{\mathcal{X}_1}\mathcal{X}_3, \mathcal{T}_{\mathcal{X}_2}\mathcal{X}_4) \\ &+ g(\mathcal{T}_{\mathcal{X}_2}\mathcal{X}_3, \mathcal{T}_{\mathcal{X}_1}\mathcal{X}_4), \end{aligned} \tag{14}$$

$$R(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{Y}_1) = g((\nabla_{\mathcal{X}_2}\mathcal{T})_{\mathcal{X}_1}\mathcal{X}_3, \mathcal{Y}_1) - g((\nabla_{\mathcal{X}_1}\mathcal{T})_{\mathcal{X}_2}\mathcal{X}_3, \mathcal{Y}_1), \tag{15}$$

$$\begin{aligned} R(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{X}_1) &= g((\nabla_{\mathcal{Y}_3}\mathcal{A})_{\mathcal{Y}_1}\mathcal{Y}_2, \mathcal{X}_1) + g(\mathcal{A}_{\mathcal{Y}_1}\mathcal{Y}_2, \mathcal{T}_{\mathcal{X}_1}\mathcal{Y}_3) \\ &- g(\mathcal{A}_{\mathcal{Y}_2}\mathcal{Y}_3, \mathcal{T}_{\mathcal{X}_1}\mathcal{Y}_1) - g(\mathcal{A}_{\mathcal{Y}_3}\mathcal{Y}_1, \mathcal{T}_{\mathcal{X}_1}\mathcal{Y}_2), \end{aligned} \tag{16}$$

$$\begin{aligned} R(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4) &= R^*(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4) - 2g(\mathcal{A}_{\mathcal{Y}_1}\mathcal{Y}_2, \mathcal{A}_{\mathcal{Y}_3}\mathcal{Y}_4) \\ &+ g(\mathcal{A}_{\mathcal{Y}_2}\mathcal{Y}_3, \mathcal{A}_{\mathcal{Y}_1}\mathcal{Y}_4) + g(\mathcal{A}_{\mathcal{Y}_1}\mathcal{Y}_3, \mathcal{A}_{\mathcal{Y}_2}\mathcal{Y}_4), \end{aligned} \tag{17}$$

$$\begin{aligned} R(\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{X}_1, \mathcal{X}_2) &= g((\nabla_{\mathcal{X}_1}\mathcal{A})_{\mathcal{Y}_1}\mathcal{Y}_2, \mathcal{X}_2) - g((\nabla_{\mathcal{X}_2}\mathcal{A})_{\mathcal{Y}_1}\mathcal{Y}_2, \mathcal{X}_1) \\ &+ g(\mathcal{A}_{\mathcal{Y}_1}\mathcal{X}_1, \mathcal{A}_{\mathcal{Y}_2}\mathcal{X}_2) - g(\mathcal{A}_{\mathcal{Y}_1}\mathcal{X}_2, \mathcal{A}_{\mathcal{Y}_2}\mathcal{X}_1) \\ &- g(\mathcal{T}_{\mathcal{X}_1}\mathcal{Y}_1, \mathcal{T}_{\mathcal{X}_2}\mathcal{Y}_2) + g(\mathcal{T}_{\mathcal{X}_2}\mathcal{Y}_1, \mathcal{T}_{\mathcal{X}_1}\mathcal{Y}_2), \end{aligned} \tag{18}$$

$$\begin{aligned} R(\mathcal{Y}_1, \mathcal{X}_1, \mathcal{Y}_2, \mathcal{X}_2) &= g((\nabla_{\mathcal{Y}_1}\mathcal{T})_{\mathcal{X}_1}\mathcal{X}_2, \mathcal{Y}_2) + g((\nabla_{\mathcal{X}_1}\mathcal{A})_{\mathcal{Y}_1}\mathcal{Y}_2, \mathcal{X}_2) \\ &- g(\mathcal{T}_{\mathcal{X}_1}\mathcal{Y}_1, \mathcal{T}_{\mathcal{X}_2}\mathcal{Y}_2) + g(\mathcal{A}_{\mathcal{Y}_1}\mathcal{X}_1, \mathcal{A}_{\mathcal{Y}_2}\mathcal{X}_2), \end{aligned} \tag{19}$$

where R, R^* and $\tilde{R}$ are Riemannian curvature of $\mathcal{B}$, $\tilde{\mathcal{B}}$ and $\psi^{-1}(q)$, respectively.

Moreover, if for every vertical vector fields $\mathcal{X}_1, \mathcal{X}_2$ and for every horizontal vector fields $\mathcal{Y}_1, \mathcal{Y}_2$ are orthonormal basis of vertical 2-plane, then we obtain:

$$K(\mathcal{X}_1, \mathcal{X}_2) = \tilde{K}(\mathcal{X}_1, \mathcal{X}_2) + ||\mathcal{T}_{\mathcal{X}_1}\mathcal{X}_2||^2 - g(\mathcal{T}_{\mathcal{X}_1}\mathcal{X}_1, \mathcal{T}_{\mathcal{X}_2}\mathcal{X}_2), \tag{20}$$

$$K(\mathcal{Y}_1, \mathcal{X}_1) = g((\nabla_{\mathcal{Y}_1}\mathcal{T})_{\mathcal{X}_1}\mathcal{X}_1, \mathcal{Y}_1) + ||\mathcal{A}_{\mathcal{Y}_1}\mathcal{X}_1||^2 - ||\mathcal{T}_{\mathcal{X}_1}\mathcal{Y}_1||^2, \tag{21}$$

$$K(\mathcal{Y}_1, \mathcal{Y}_2) = K^*(\mathcal{Y}_1, \mathcal{Y}_2) - 3||\mathcal{A}_{\mathcal{Y}_1}\mathcal{Y}_2||^2, \tag{22}$$

where K, K^* and $\tilde{K}$ are sectional curvature of $\mathcal{B}$, $\tilde{\mathcal{B}}$ and $\psi^{-1}(q)$, respectively ([7]).

3. Bi-slant submersions

Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$.

For any non-null vector field $W \in (ker\psi_*)$, we get

$$\mathcal{P}W = tW + nW, \tag{23}$$

where tW and nW are vertical and horizontal parts of $\mathcal{P}W$.

Also, for non-null vector field $\zeta \in (ker\psi_*)^{\perp}$, we have

$$\mathcal{P}\zeta = B\zeta + C\zeta, \tag{24}$$

where $B\zeta \in ker\psi_*$ and $C\zeta \in (ker\psi_*)^{\perp}$.

In addition, $(ker\psi_*)^{\perp}$ is decomposed as

$$(ker\psi_*)^{\perp} = nD^{\varphi_1} \oplus nD^{\varphi_2} \oplus \mu \tag{25}$$

where μ is the orthogonal complementary distribution of $nD^{\varphi_1} \oplus nD^{\varphi_2}$. We can say that μ is invariant distribution of $(ker\psi_*)^{\perp}$ with respect to P.

Definition 3.1. ([15]) *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper slant submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. We have*
type ∼1 if for every space-like (time-like) vector field $W \in \Gamma(ker\psi_*)$, tW is time-like (space-like), and $\frac{\|tW\|}{\|\mathcal{P}W\|} > 1$,
type ∼ 2 if for every space-like (time-like) vector field $W \in \Gamma(ker\psi_*)$, tW is time-like (space-like), and $\frac{\|tW\|}{\|\mathcal{P}W\|} < 1$,
type ∼ 3 if for every space-like (time-like) vector field $W \in \Gamma(ker\psi_*)$, tW is space-like (time-like).

Now, we can give our definition.

Definition 3.2. *Let $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ be an almost para-Hermitian manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian manifold. A pseudo-Riemannian submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is known a bi-slant submersion if there are two slant distribution $D^{\varphi_1} \in ker\psi_*$ and $D^{\varphi_2} \in ker\psi_*$ such that*

$$ker\psi_* = D^{\varphi_1} \oplus D^{\varphi_2}, \tag{26}$$

where D^{φ_1} and D^{φ_2} have slant angles φ_1 and φ_2, respectively.

Hence, using (23) and (24) we have:

Lemma 3.3. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a bi-slant submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. Then, we obtain the following equations.*
(**a**) $tD^{\varphi_1} \subset D^{\varphi_1}$, (**b**) $tD^{\varphi_2} \subset D^{\varphi_2}$, (**c**) $B\mu = \{0\}$, (**d**) $C\mu = \mu$.

Then, we can easily see that $P^2 = I$ and from (23) and (24) we get:

Lemma 3.4. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a bi-slant submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. Then, we obtain the following equations.*
(**a**) $t^2X + BnX = X$, (**b**) $C^2U + nBU = U$,
(**c**) $tB + BC = \{0\}$, (**d**) $nt + Cn = \{0\}$
for all vector field $X \in D^{\varphi_1}$ and $U \in D^{\varphi_2}$.

The proof of the following Theorems are similar to the proof of ([5],[6]). Therefore we skip its proof.

Theorem 3.5. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, ψ is proper bi-slant submersion of type ∼1 if and only if for any space-like(time-like) vector field $X, Y \in D^{\varphi_1}$ and $U, W \in D^{\varphi_2}$. Then, we have:*
(**a**) $t^2X = \cosh^2\varphi_1 X$. (**b**) $t^2U = \cosh^2\varphi_2 U$.
(**c**) $g_{\mathcal{B}}(tX, tY) = -\cosh^2\varphi_1 g_{\mathcal{B}}(X, Y)$. (**d**) $g_{\mathcal{B}}(tU, tW) = -\cosh^2\varphi_2 g_{\mathcal{B}}(U, W)$.
(**e**) $g_{\mathcal{B}}(nX, nY) = \sinh^2\varphi_1 g_{\mathcal{B}}(X, Y)$. (**f**) $g_{\mathcal{B}}(nU, nW) = \sinh^2\varphi_2 g_{\mathcal{B}}(U, W)$.

Theorem 3.6. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, ψ is proper bi-slant submersion of type ∼2 if and only if for any space-like(time-like) vector field $X, Y \in D^{\varphi_1}$ and $U, W \in D^{\varphi_2}$. Then, we have:*
(**a**) $t^2X = \cos^2\varphi_1 X$. (**b**) $t^2U = \cos^2\varphi_2 U$.
(**c**) $g_{\mathcal{B}}(tX, tY) = -\cos^2\varphi_1 g_{\mathcal{B}}(X, Y)$. (**d**) $g_{\mathcal{B}}(tU, tW) = -\cos^2\varphi_2 g_{\mathcal{B}}(U, W)$.
(**e**) $g_{\mathcal{B}}(nX, nY) = -\sin^2\varphi_1 g_{\mathcal{B}}(X, Y)$. (**f**) $g_{\mathcal{B}}(nU, nW) = -\sin^2\varphi_2 g_{\mathcal{B}}(U, W)$.

Theorem 3.7. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion from an almost para-Hermitian manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, ψ is proper bi-slant submersion of type ∼3 if and only if for any space-like(time-like) vector field $X, Y \in D^{\varphi_1}$ and $U, W \in D^{\varphi_2}$. Then, we have:*
(**a**) $t^2X = -\sinh^2\varphi_1 X$. (**b**) $t^2U = -\sinh^2\varphi_2 U$.
(**c**) $g_{\mathcal{B}}(tX, tY) = \sinh^2\varphi_1 g_{\mathcal{B}}(X, Y)$. (**d**) $g_{\mathcal{B}}(tU, tW) = \sinh^2\varphi_2 g_{\mathcal{B}}(U, W)$.
(**e**) $g_{\mathcal{B}}(nX, nY) = -\cosh^2\varphi_1 g_{\mathcal{B}}(X, Y)$. (**f**) $g_{\mathcal{B}}(nU, nW) = -\cosh^2\varphi_2 g_{\mathcal{B}}(U, W)$.

Let us consider para-Kaehler structure on R^{2n}_n :

$$P(\frac{\partial}{\partial y_{2i}}) = \frac{\partial}{\partial y_{2i-1}},\ P(\frac{\partial}{\partial y_{2i-1}}) = \frac{\partial}{\partial y_{2i}},\ g = (dy^1)^2 - (dy^2)^2 + (dy^3)^2 - ... - (dy^{2n})^2.$$

Here $i \in \{1, ..., n\}$. Also, $(y_1, y_2, ..., y_{2n})$ denotes the cartesian coordinates over R^{2n}_n.

We can easily present non-trivial examples of proper bi-slant pseudo-Riemannian submersions of type∼1, 2 and 3.

Example 3.8. *Let us determine the map* $\psi : R^8_4 \to R^4_2$

$$\psi(y_1,...,y_8) = (y_2 \sinh\beta_1 + y_3 \cosh\beta_1, y_7, y_4 \cosh\beta_2 + y_5 \sinh\beta_2, y_8).$$

So, ψ *is a proper bi-slant pseudo-Riemannian submersion of type* ~ 1*. By direct calculations, we obtain*

$$D^{\varphi_1} =< Y_1 = -\cosh\beta_1 \frac{\partial}{\partial y_2} + \sinh\beta_1 \frac{\partial}{\partial y_3}, Y_2 = \frac{\partial}{\partial y_1} >,$$
$$D^{\varphi_2} =< Y_3 = -\sinh\beta_2 \frac{\partial}{\partial y_4} + \cosh\beta_2 \frac{\partial}{\partial y_5}, Y_4 = \frac{\partial}{\partial y_6} >$$

with bi-slant angles β_1 *and* β_2*.*

Let R^4_2 *be a pseudo-Euclidean space of signature* $(+,+,-,-)$ *with respect to the canonical basis* $(\frac{\partial}{\partial y_1},...,\frac{\partial}{\partial y_8})$*.*

Example 3.9. *Let us determine the map* $\psi : R^8_4 \to R^4_2$

$$\psi(y_1,...,y_8) = (\frac{y_1 - y_3}{\sqrt{2}}, y_4, \frac{\sqrt{3}y_5 - y_7}{2}, y_8).$$

So, ψ *is a proper bi-slant pseudo-Riemannian submersion of type* ~ 2*. By direct calculations, we obtain*

$$D^{\varphi_1} =< Y_1 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3}), Y_2 = \frac{\partial}{\partial y_2} >,$$
$$D^{\varphi_2} =< Y_3 = \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial y_5} + \frac{\partial}{\partial y_7}), Y_4 = \frac{\partial}{\partial y_6} >$$

with bi-slant angles $\varphi_1 = \frac{\pi}{4}$ *and* $\varphi_2 = \frac{\pi}{3}$*.*

Example 3.10. *Let us determine the map* $\psi : R^8_4 \to R^4_2$

$$\psi(y_1,...,y_8) = (y_2 \cosh\beta_1 + y_3 \sinh\beta_1, y_4, y_5 \cosh\beta_2 + y_8 \sinh\beta_2, y_7).$$

So, ψ *is a proper bi-slant pseudo-Riemannian submersion of type* ~ 3*. By direct calculations, we obtain*

$$D^{\varphi_1} =< Y_1 = \sinh\beta_1 \frac{\partial}{\partial y_2} - \cosh\beta_1 \frac{\partial}{\partial y_3}, Y_2 = \frac{\partial}{\partial y_1} >,$$
$$D^{\varphi_2} =< Y_3 = \sinh\beta_2 \frac{\partial}{\partial y_5} - \cosh\beta_2 \frac{\partial}{\partial y_8}, Y_4 = \frac{\partial}{\partial y_6} >$$

with bi-slant angles β_1 *and* β_2*.*

Let R^4_2 *be a pseudo-Euclidean space of signature* $(-,-,+,+)$ *with respect to the canonical basis* $(\frac{\partial}{\partial y_1},...,\frac{\partial}{\partial y_8})$*.*

Using equations (1), (5)∼(8) and (23)∼(24), we get:

Lemma 3.11. *Let* $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be a pseudo-Riemannian submersion of type ∼1, 2, 3 from a para-Kaehler manifold* $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ *onto a pseudo-Riemannian manifold* $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$*. So, we obtain the following equations.*

$$\hat{\nabla}_U tW + \mathcal{T}_U nW = t\hat{\nabla}_U W + \mathcal{B}\mathcal{T}_U W, \tag{27}$$

(28) $$\mathcal{T}_U tW + \mathcal{H}\nabla_U nW = n\hat{\nabla}_U W + \mathcal{C}\mathcal{T}_U W,$$

(29) $$\mathcal{V}\nabla_{\mathcal{X}}\mathcal{Y} + \mathcal{A}_{\mathcal{X}}\mathcal{C}\mathcal{Y} = t\mathcal{A}_{\mathcal{X}}\mathcal{Y} + \mathcal{B}\mathcal{H}\nabla_{\mathcal{X}}\mathcal{Y},$$

(30) $$\mathcal{A}_{\mathcal{X}}\mathcal{B}\mathcal{Y} + \mathcal{H}\nabla_{\mathcal{X}}\mathcal{C}\mathcal{Y} = n\mathcal{A}_{\mathcal{X}}\mathcal{Y} + \mathcal{C}\mathcal{H}\nabla_{\mathcal{X}}\mathcal{Y},$$

(31) $$\hat{\nabla}_U \mathcal{B}\mathcal{X} + \mathcal{T}_U \mathcal{C}\mathcal{X} = t\mathcal{T}_U \mathcal{X} + \mathcal{B}\mathcal{H}\nabla_U \mathcal{X},$$

(32) $$\mathcal{T}_U \mathcal{B}\mathcal{X} + \mathcal{H}\nabla_U \mathcal{C}\mathcal{X} = n\mathcal{T}_U \mathcal{X} + \mathcal{C}\mathcal{H}\nabla_U \mathcal{X}$$

for any non-null vector fields $U, W \in \Gamma(ker\psi_*)$ and $\mathcal{X}, \mathcal{Y} \in \Gamma(ker\psi_*)^{\perp}$.

Now we can show

$$(\nabla_U t)W = \hat{\nabla}_U tW - t\hat{\nabla}_U W$$

$$(\nabla_U n)W = \mathcal{H}\nabla_U nW - n\hat{\nabla}_U W,$$

$$(\nabla_U B)\zeta = \hat{\nabla}_U B\zeta - B\mathcal{H}\nabla_U \zeta,$$

$$(\nabla_U C)\zeta = \mathcal{H}\nabla_U C\zeta - C\mathcal{H}\nabla_U \zeta$$

for any non-null vector fields $U, W \in ker\psi_*$ and $\zeta \in (ker\psi_*)^{\perp}$. Then, we can say that

- t is parallel $\iff \nabla t \equiv 0$.
- n is parallel $\iff \nabla n \equiv 0$.
- B is parallel $\iff \nabla B \equiv 0$.
- C is parallel $\iff \nabla C \equiv 0$.

Now, the equations we get below will help us in integrability, totally, and mixed geodesic for bi-slant submersions.

Lemma 3.12. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. If ψ is a bi-slant submersion of type ∼1, then, for any non-null vector fields $U, W \in \Gamma(D^{\varphi_1})$ and $\mathcal{X}, \mathcal{Y} \in \Gamma(D^{\varphi_2})$, we obtain;*

(33) $$g(\nabla_U W, \mathcal{X}) = (1 + \cosh^2 \varphi_1) g(\mathcal{T}_{\mathcal{X}} ntW - \mathcal{T}_{t\mathcal{X}} nW - \mathcal{A}_{n\mathcal{X}} nW, U),$$

(34) $$g(\nabla_{\mathcal{X}} \mathcal{Y}, U) = (1 + \cosh^2 \varphi_2) g(\mathcal{T}_U nt\mathcal{Y} - \mathcal{T}_{tU} n\mathcal{Y} - \mathcal{A}_{nU} n\mathcal{Y}, \mathcal{X}).$$

If ψ is a bi- slant submersion of type ∼2, then, for any non-null vector fields $U, W \in \Gamma(D^{\varphi_1})$ and $\mathcal{X}, \mathcal{Y} \in \Gamma(D^{\varphi_2})$, we obtain;

(35) $$g(\nabla_U W, \mathcal{X}) = (1 + \cos^2 \varphi_1) g(\mathcal{T}_{\mathcal{X}} ntW - \mathcal{T}_{t\mathcal{X}} nW - \mathcal{A}_{n\mathcal{X}} nW, U),$$

(36) $$g(\nabla_{\mathcal{X}} \mathcal{Y}, U) = (1 + \cos^2 \varphi_2) g(\mathcal{T}_U nt\mathcal{Y} - \mathcal{T}_{tU} n\mathcal{Y} - \mathcal{A}_{nU} n\mathcal{Y}, \mathcal{X}).$$

If ψ is a bi-slant submersion of type ∼3, then, for any non-null vector fields $U, W \in \Gamma(D^{\varphi_1})$ and $\mathcal{X}, \mathcal{Y} \in \Gamma(D^{\varphi_2})$, we obtain;

(37) $$g(\nabla_U W, \mathcal{X}) = (1 - \sinh^2 \varphi_1) g(\mathcal{T}_{\mathcal{X}} ntW - \mathcal{T}_{t\mathcal{X}} nW - \mathcal{A}_{n\mathcal{X}} nW, U),$$

$$g(\nabla_{\mathcal{X}}\mathcal{Y}, U) = (1 - \sinh^2\varphi_2) g(\mathcal{T}_U nt\mathcal{Y} - \mathcal{T}_{tU} n\mathcal{Y} - \mathcal{A}_{nU} n\mathcal{Y}, \mathcal{X}). \tag{38}$$

Proof. For any non-null vector fields $U, W \in \Gamma(D^{\varphi_1})$ and $\mathcal{X}, \mathcal{Y} \in \Gamma(D^{\varphi_2})$. Then, from (1), (2) and (23), we get

$$\begin{aligned} g(\nabla_U W, \mathcal{X}) &= g(\nabla_U PW, P\mathcal{X}) \\ &= g(\nabla_U tW, P\mathcal{X}) + g(\nabla_U nW, P\mathcal{X}). \end{aligned}$$

From (1) and (23), we get

$$\begin{aligned} g(\nabla_U W, \mathcal{X}) &= -g(\nabla_U t^2 W, \mathcal{X}) - g(\nabla_U ntW, \mathcal{X}) \\ &+ g(\nabla_U nW, t\mathcal{X}) + g(\nabla_U nW, n\mathcal{X}). \end{aligned}$$

Using Theorem 3.5-(a), (5), and (6), we get

$$\begin{aligned} g(\nabla_U W, \mathcal{X}) &= -\cosh^2\varphi_1 g(\nabla_U W, \mathcal{X}) - g(\mathcal{T}_U ntW, \mathcal{X}) \\ &+ g(\mathcal{T}_U nW, t\mathcal{X}) + g(\mathcal{A}_U nW, n\mathcal{X}). \end{aligned}$$

Therefore, with the help of (11) and (12), we obtain (33). Similarly, (34) is obtained. □

Moreover, the equations of type ∼2 and type ∼3 were obtained in a similar way.

Theorem 3.13. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper bi-slant pseudo-Riemannian submersion of type ∼1 from an almost para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. The slant distribution D^{φ_1} is integrable if and only if*

$$g(\mathcal{T}_{\mathcal{X}} ntW - \mathcal{T}_{t\mathcal{X}} nW - \mathcal{A}_{n\mathcal{X}} nW, U) = g(\mathcal{T}_{\mathcal{X}} ntU - \mathcal{T}_{t\mathcal{X}} nU - \mathcal{A}_{n\mathcal{X}} nU, W)$$

for any non-null vector fields $U, W \in \Gamma(D^{\varphi_1})$ and $\mathcal{X} \in \Gamma(D^{\varphi_2})$.

Proof. For any non-null vector fields $U, W \in \Gamma(D^{\varphi_1})$ and $\mathcal{X} \in \Gamma(D^{\varphi_2})$, using (33), we get:

$$\begin{aligned} g([U, W], \mathcal{X}) &= g(\nabla_U W, \mathcal{X}) - g(\nabla_W U, \mathcal{X}) \\ &= (1 + \cosh^2\varphi_1)\{g(\mathcal{T}_{\mathcal{X}} ntW - \mathcal{T}_{t\mathcal{X}} nW - \mathcal{A}_{n\mathcal{X}} nW, U) \\ &- g(\mathcal{T}_{\mathcal{X}} ntU - \mathcal{T}_{t\mathcal{X}} nU - \mathcal{A}_{n\mathcal{X}} nU, W)\}. \end{aligned}$$

So the proof is complete. □

Similarly, the following conclusion is obtained.

Theorem 3.14. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper bi-slant pseudo-Riemannian submersion of type ∼1 from an almost para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. The slant distribution D^{φ_2} is integrable if and only if*

$$g(\mathcal{T}_U nt\mathcal{Y} - \mathcal{T}_{tU} n\mathcal{Y} - \mathcal{A}_{nU} n\mathcal{Y}, \mathcal{X}) = g(\mathcal{T}_U nt\mathcal{X} - \mathcal{T}_{tU} n\mathcal{X} - \mathcal{A}_{nU} n\mathcal{X}, \mathcal{Y})$$

for any non-null vector fields $\mathcal{X}, \mathcal{Y} \in D^{\varphi_1}$ and $U \in D^{\varphi_2}$.

Now, let us investigate the cases where the fibres, vertical and horizontal distribution are totally geodesic.

Theorem 3.15. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper bi-slant pseudo-Riemannian submersion of type ∼1 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, the slant distribution D^{φ_1} describes a totally geodesic foliation on $(ker\psi_*)$ if and only if*

$$g(\mathcal{T}_{\mathcal{X}}ntW - \mathcal{T}_{t\mathcal{X}}nW - \mathcal{A}_{n\mathcal{X}}nW, U) = 0 \tag{39}$$

for any non-null vector fields $U, W \in D^{\varphi_1}$ and $\mathcal{X} \in D^{\varphi_2}$.

Proof. For any non-null vector fields $U, W \in D^{\varphi_1}$ and $\mathcal{X} \in D^{\varphi_2}$. Using (5) and (33) we get:

$$\begin{aligned} g(\hat{\nabla}_U W, \mathcal{X}) &= g(\nabla_U W, \mathcal{X}) \\ &= (1 + \cosh^2 \varphi_1) g(\mathcal{T}_{\mathcal{X}}ntW - \mathcal{T}_{t\mathcal{X}}nW - \mathcal{A}_{n\mathcal{X}}nW, U). \end{aligned}$$

Since the slant distribution D^{φ_1} describes a totally geodesic foliation on $(ker\psi_*)$, we show that $\hat{\nabla}_U W \in D^{\varphi_1}$. □

Note that the Theorem 3.15 is valid for proper bi-slant pseudo-Riemannian submersion of type ∼2.

Similarly, the following conclusion is obtained.

Theorem 3.16. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper bi-slant pseudo-Riemannian submersion of type ∼1 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, the slant distribution D^{φ_2} describes a totally geodesic foliation on $(ker\psi_*)$ if and only if*

$$g(\mathcal{T}_{\mathcal{X}}ntW - \mathcal{T}_{t\mathcal{X}}nW - \mathcal{A}_{n\mathcal{X}}nW, U) = 0 \tag{40}$$

for any non-null vector fields $\mathcal{X} \in D^{\varphi_1}$ and $U, W \in D^{\varphi_2}$.

Note that the Theorem 3.16 is valid for proper bi-slant pseudo-Riemannian submersion of type ∼2.

Proposition 3.17. *Assume that $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper bi-slant pseudo-Riemannian submersion of type ∼1 ,2 or 3 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, $(ker\psi_*)$ is a locally product $\mathcal{B}_{D^{\varphi_1}} \times \mathcal{B}_{D^{\varphi_2}}$ if and only if the equations (39) and (40) are hold where $\mathcal{B}_{D^{\varphi_1}}$ and $\mathcal{B}_{D^{\varphi_2}}$ integral manifolds of the distributions D^{φ_1} and D^{φ_2}, respectively.*

Theorem 3.18. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper bi-slant pseudo-Riemannian submersion of type ∼1 ,2 or 3 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, $(ker\psi_*)$ describes a totally geodesic foliation on if and only if*

$$Cg(\mathcal{T}_U tW + h\nabla_U nW) + n(\hat{\nabla}_U tW + \mathcal{T}_U nW) = 0 \tag{41}$$

for any non-null vector fields $U, W \in (ker\psi_*)$.

Proof. For any non-null vector fields $U, W \in (ker\psi_*)$. Using (1), (5), (6), (23) and (24), we get

$$\begin{aligned} \nabla_U W &= P\nabla_U PW = P(\nabla_U tW + \nabla_U nW) \\ &= P(\mathcal{T}_U tW + \hat{\nabla}_U tW + \mathcal{T}_U nW + h\nabla_U nW) \\ &= B\mathcal{T}_U tW + C\mathcal{T}_U tW + t\hat{\nabla}_U tW + n\hat{\nabla}_U tW \\ &+ t\mathcal{T}_U nW + n\mathcal{T}_U nW + Bh\nabla_U nW + Ch\nabla_U nW. \end{aligned}$$

□

Theorem 3.19. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper bi-slant pseudo-Riemannian submersion of type ∼1 ,2 or 3 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, $(ker\psi_*^{\perp})$ describes a totally geodesic foliation on if and only if*

$$Bg(\mathcal{A}_{\mathcal{X}} B\mathcal{Y} + h\nabla_{\mathcal{X}} C\mathcal{Y}) + t(\mathcal{A}_{\mathcal{X}} C\mathcal{Y} + v\nabla_{\mathcal{X}} B\mathcal{Y}) = 0 \tag{42}$$

for any non-null vector fields $\mathcal{X}, \mathcal{Y} \in (ker\psi_*^{\perp})$.

Proposition 3.20. *Assume that $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper bi-slant pseudo-Riemannian submersion of type ∼1 ,2 or 3 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. In this case, $(ker\psi_*)$ is a locally product $\mathcal{B}_{ker\psi_*} \times \mathcal{B}_{ker\psi_*^{\perp}}$ if and only if the equations (41) and (42) hold, where $\mathcal{B}_{ker\psi_*}$ and $\mathcal{B}_{ker\psi_*^{\perp}}$ are integral manifolds of the distributions $(ker\psi_*)$ and $(ker\psi_*)^{\perp}$, respectively.*

4. Curvature Relations

We now investigate the curvature relations between the base space, total space and the fibers of proper bi-slant pseudo-Riemannian submersions.

Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper bi-slant pseudo-Riemannian submersion from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. We first recall that the sectional curvature K is described by the following;

$$K(U, W) = \frac{R(U, W, W, U)}{g(U, U)g(W, W)} \tag{43}$$

for all pair of nonzero orthogonal vectors U, W [24].

Theorem 4.1. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper bi-slant pseudo-Riemannian submersion of type ∼1 or 2 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$*

onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$*. Then, we get*

$$\begin{aligned} K(\mathcal{X}_1,\mathcal{X}_2) &= \tilde{K}(t\mathcal{X}_1,t\mathcal{X}_2)||t\mathcal{X}_1||^{-2}||t\mathcal{X}_2||^{-2}+K^*(n\mathcal{X}_1,n\mathcal{X}_2)||n\mathcal{X}_1||^{-2}||n\mathcal{X}_2||^{-2} \\ &- g(\mathcal{T}_{t\mathcal{X}_1}t\mathcal{X}_1,\mathcal{T}_{t\mathcal{X}_2}t\mathcal{X}_2)+||\mathcal{T}_{t\mathcal{X}_1}t\mathcal{X}_2||^2-3||\mathcal{A}_{n\mathcal{X}_1}n\mathcal{X}_2||^2 \\ &+ g((\nabla_{n\mathcal{X}_2}\mathcal{T})_{t\mathcal{X}_1}t\mathcal{X}_1,n\mathcal{X}_2)+||\mathcal{A}_{n\mathcal{X}_2}t\mathcal{X}_1||^2-||\mathcal{T}_{t\mathcal{X}_1}n\mathcal{X}_2||^2 \\ &+ g((\nabla_{n\mathcal{X}_1}\mathcal{T})_{t\mathcal{X}_2}t\mathcal{X}_2,n\mathcal{X}_1)+||\mathcal{A}_{n\mathcal{X}_1}t\mathcal{X}_2||^2-||\mathcal{T}_{t\mathcal{X}_2}n\mathcal{X}_1||^2, \end{aligned} \tag{44}$$

$$\begin{aligned} K(\mathcal{X}_1,\mathcal{Y}_1) &= \tilde{K}(t\mathcal{X}_1,B\mathcal{Y}_1)||t\mathcal{X}_1||^{-2}||B\mathcal{Y}_1||^{-2}+K^*(n\mathcal{X}_1,C\mathcal{Y}_1)||n\mathcal{X}_1||^{-2}||C\mathcal{Y}_1||^{-2} \\ &- g(\mathcal{T}_{t\mathcal{X}_1}t\mathcal{X}_1,\mathcal{T}_{B\mathcal{Y}_1}B\mathcal{Y}_1)+||\mathcal{T}_{t\mathcal{X}_1}B\mathcal{Y}_1||^2-3||\mathcal{A}_{n\mathcal{X}_1}C\mathcal{Y}_1||^2 \\ &+ g((\nabla_{C\mathcal{Y}_1}\mathcal{T})_{t\mathcal{X}_1}t\mathcal{X}_1,C\mathcal{Y}_1)+||\mathcal{A}_{C\mathcal{Y}_1}t\mathcal{X}_1||^2-||\mathcal{T}_{t\mathcal{X}_1}C\mathcal{Y}_1||^2 \\ &+ g((\nabla_{n\mathcal{X}_1}\mathcal{T})_{B\mathcal{Y}_1}B\mathcal{Y}_1,n\mathcal{X}_1)+||\mathcal{A}_{n\mathcal{X}_1}B\mathcal{Y}_1||^2-||\mathcal{T}_{B\mathcal{Y}_1}n\mathcal{X}_1||^2, \end{aligned} \tag{45}$$

$$\begin{aligned} K(\mathcal{Y}_1,\mathcal{Y}_2) &= \tilde{K}(B\mathcal{Y}_1,B\mathcal{Y}_2)||B\mathcal{Y}_1||^{-2}||B\mathcal{Y}_2||^{-2}+K^*(C\mathcal{Y}_1,C\mathcal{Y}_2)||C\mathcal{Y}_1||^{-2}||C\mathcal{Y}_2||^{-2} \\ &- g(\mathcal{T}_{B\mathcal{Y}_1}B\mathcal{Y}_1,\mathcal{T}_{B\mathcal{Y}_2}B\mathcal{Y}_2)+||\mathcal{T}_{B\mathcal{Y}_1}B\mathcal{Y}_2||^2-3||\mathcal{A}_{C\mathcal{Y}_1}C\mathcal{Y}_2||^2 \\ &+ g((\nabla_{C\mathcal{Y}_2}\mathcal{T})_{B\mathcal{Y}_1}B\mathcal{Y}_1,C\mathcal{Y}_2)+||\mathcal{A}_{C\mathcal{Y}_2}B\mathcal{Y}_1||^2-||\mathcal{T}_{B\mathcal{Y}_1}C\mathcal{Y}_2||^2 \\ &+ g((\nabla_{C\mathcal{Y}_1}\mathcal{T})_{B\mathcal{Y}_2}B\mathcal{Y}_2,C\mathcal{Y}_1)+||\mathcal{A}_{C\mathcal{Y}_1}B\mathcal{Y}_2||^2-||\mathcal{T}_{B\mathcal{Y}_2}C\mathcal{Y}_1||^2. \end{aligned} \tag{46}$$

Proof. For every vertical vector fields $\mathcal{X}_1,\mathcal{X}_2$ and for every horizontal vector fields $\mathcal{Y}_1,\mathcal{Y}_2$ which are orthonormal vector fields, we have

$$K(\mathcal{X}_1,\mathcal{X}_2)=K(t\mathcal{X}_1,t\mathcal{X}_2)+K(t\mathcal{X}_1,n\mathcal{X}_2)+K(n\mathcal{X}_1,t\mathcal{X}_2)+K(n\mathcal{X}_1,n\mathcal{X}_2).$$

By using (14), (17) and (19), we get

$$\begin{aligned} K(\mathcal{X}_1,\mathcal{X}_2) &= \tilde{R}(t\mathcal{X}_1,t\mathcal{X}_2,t\mathcal{X}_2,t\mathcal{X}_1)-g(\mathcal{T}_{t\mathcal{X}_1}t\mathcal{X}_1,\mathcal{T}_{t\mathcal{X}_2}t\mathcal{X}_2)+||\mathcal{T}_{t\mathcal{X}_1}t\mathcal{X}_2||^2 \\ &+ g((\nabla_{n\mathcal{X}_2}\mathcal{T})_{t\mathcal{X}_1}t\mathcal{X}_1,n\mathcal{X}_2)+||\mathcal{A}_{n\mathcal{X}_2}t\mathcal{X}_1||^2-||\mathcal{T}_{t\mathcal{X}_1}n\mathcal{X}_2||^2 \\ &+ g((\nabla_{n\mathcal{X}_1}\mathcal{T})_{t\mathcal{X}_2}t\mathcal{X}_2,n\mathcal{X}_1)+||\mathcal{A}_{n\mathcal{X}_1}t\mathcal{X}_2||^2-||\mathcal{T}_{t\mathcal{X}_2}n\mathcal{X}_1||^2 \\ &+ R^*(n\mathcal{X}_1,n\mathcal{X}_2,n\mathcal{X}_2,n\mathcal{X}_1)-3||\mathcal{A}_{n\mathcal{X}_1}n\mathcal{X}_2||^2. \end{aligned}$$

Using the following equations,

$$\tilde{R}(t\mathcal{X}_1,t\mathcal{X}_2,t\mathcal{X}_2,t\mathcal{X}_1)=\tilde{K}(t\mathcal{X}_1,t\mathcal{X}_2)||t\mathcal{X}_1||^{-2}||t\mathcal{X}_2||^{-2}$$

and

$$R^*(n\mathcal{X}_1,n\mathcal{X}_2,n\mathcal{X}_2,n\mathcal{X}_1)=K^*(n\mathcal{X}_1,n\mathcal{X}_2)||n\mathcal{X}_1||^{-2}|nt\mathcal{X}_2||^{-2}.$$

we get (44) easily. Similarly, (45) and (46) can be obtained. □

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