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BASIC FORMULAS FOR THE DOUBLE INTEGRAL TRANSFORM OF FUNCTIONALS ON ABSTRACT WIENER SPACE

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Abstract. In this paper, we establish several basic formulas among the double-integral transforms, the double-convolution products, and the inverse double-integral transforms of cylinder functionals on abstract Wiener space. We then discuss possible relationships involving the double-integral transform.

1. Introduction

Let H be a real separable infinite-dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$. Let $\|\cdot\|_0$ be a measurable norm on H with respect to the Gaussian cylinder set measure ν_0 on H. Let B denote the completion of H with respect to $\|\cdot\|_0$. Let i denote the natural injection from H to B. The adjoint operator i^* of i is one to one and maps B^* continuously onto a dense subset H^*, where B^* and H^* are topological duals of B and H, respectively. By identifying H^* with H and B^* with i^*B^*, we have a triple $B^* \subset H^* \approx H \subset B$ with $\langle x, y\rangle = (x, y)$ for all x in H and y in B^*, where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. By a well known result of Gross [11], $\nu_0 \circ i^{-1}$ has a unique countably additive extension ν to the Borel σ-algebra $\mathcal{B}(B)$ of B. The triple (B, H, ν) is called an abstract Wiener space. For more details, see [5, 11–13, 17–19].

Let $\{\alpha_j\}_{j=1}^\infty$ be a complete orthonormal set in H with α_j's are in B^*. For each $h \in H$ and for $x \in B$, we define a stochastic inner product $(h, x)^\sim$ by

$$(h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum_{j=1}^n \langle h, \alpha_j\rangle (x, \alpha_j), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

Then for every $h(\neq 0)$ in H, $(h, x)^\sim$ exists for all $x \in B$, $(h,\cdot)^\sim$ is a Gaussian random variable on B with mean zero and variance $|h|^2$ and is essentially independent of the choice of the complete orthonormal set used in its definition.

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Also, if both h and x are in H, then Parseval's identity gives $(h,x)^\sim = \langle h,x\rangle$. Furthermore, $(h,\lambda x)^\sim = (\lambda h,x)^\sim = \lambda(h,x)^\sim$ for all $\lambda \in \mathbb{R}$, $h \in H$ and for all $x \in B$. We also see that if $\{h_1,\ldots,h_n\}$ is an orthonormal set in H, then the random variables $(h_j,x)^\sim$'s are independent, see [6].

In [18], Lee defined an integral transform

$$\mathcal{F}_{\gamma,\beta}(F)(y) = \int_B F(\gamma x + \beta y)d\nu(x)$$

of analytic functionals on abstract Wiener space. For an appropriate functional $u(x)$ on B, let N_c be an operator defined by the formula

$$N_c u(x) = -Tr_H D^2 u(x) + c(x, Du(x))^\sim, \qquad x \in B,\ c \in \mathbb{C}/\{0\},$$

where D^2 denotes the second Fréchet derivative and Tr_H denotes the trace of an operator. He showed that the integral transform $\mathcal{F}_{1/c,i}$, $c \in \mathbb{C}/\{0\}$ forms the solution of a differential equation which is called a Cauchy problem

$$\begin{cases} u_t(x,t) = \mathcal{P}(N_c)u(x,t), & x \in B,\ t > 0 \\ u(x,0) = F(x), \end{cases} \tag{1.1}$$

where $\mathcal{P}(\eta) = a_m\eta^m + \cdots + a_1\eta + a_0$ is an m-dimensional polynomial function with respect to η. In addition, let $\mathcal{P}(\eta) = -\eta$ and $c = 1$ in equation (1.1) above. Then the solution of the Cauchy problem is given by formula

$$u(x,t) = \int_B F(e^{-t}x + \sqrt{1-e^{-2t}}y)d\nu(y);$$

or equivalently,

$$u(x,t) = \int_B F(y)o_t(x,dy),$$

where $o_t(x,dy) = \nu_{1-e^{-2t}}(e^{-t}x,dy)$ and ν_t is the Wiener measure which is generated by the Gauss Cylinder set measure μ_t with variance t. This showed that the family of measures $\{o_t(x,dy)\}$ serves as the "fundamental solution" of the operator $\partial/\partial t + N_1$, for more details see [12, 18].

One can see that many transforms: the Fourier-Wiener transform [1], the modified Fourier-Wiener transform [2], the Fourier-Feynman transform [14] and the Gauss transform: are special cases of Lee's integral transform $\mathcal{F}_{\gamma,\beta}$. Since then the integral transform $\mathcal{F}_{\gamma,\beta}$ was introduced by Lee, many mathematicians have studied integral transforms in conjunction with related topics involving functionals in various classes. In particular, the authors obtained basic formulas for integral transforms and convolution products of functionals in several classes, [4, 5, 7–9, 15, 16]. In [10], the authors introduced a double integral transform, a double convolution product and a Banach algebra, and established various basic formulas.

In this paper, we prove the existence of the double-integral transform, the double-convolution product, and the inverse double-integral transform of cylinder functionals on abstract Wiener space. We then obtain various basic formulas involving them, and provide additional relations involving the double-integral transforms.

2. Definitions and preliminaries

We begin this section by stating some definitions and previous results.

Let $[B]$ denote the complexification of B. We note that for $h \in H$ and $x \in [B]$,

$$(h,x)^{\sim} = (h,x_1)^{\sim} + i(h,x_2)^{\sim},\ x = x_1 + ix_2 \in [B].$$

For any orthonormal set $\{\alpha_1, \dots, \alpha_n\}$ in H and $x \in [B]$, let

$$(\vec{\alpha}, x)^{\sim} = ((\alpha_1, x)^{\sim}, \dots, (\alpha_n, x)^{\sim})$$

and let $\vec{\lambda_k} = (\lambda_{k,1}, \dots, \lambda_{k,n}) \in \mathbb{C}^n,\ k = 1, 2, \dots$. Also, for $\vec{\gamma} = (\gamma_1, \gamma_2) \in \mathbb{C}^2/\{(0,0)\}$ and $\vec{x} = (x_1, x_2) \in [B]^2$, let

$$\vec{\gamma} \circ \vec{x} = (\gamma_1 x_1, \gamma_2 x_2).$$

We note that for $\vec{\gamma} = (\gamma_1, \gamma_2) \in \mathbb{C}^2/\{(0,0)\}$ and $\vec{x} = (x_1, x_2) \in B^2$, $\vec{\gamma} \circ \vec{x}$ is well-defined because $B \subset [B]$.

We now recall the definitions of double integral transform (DIT) and the double convolution product (DCP) of functionals on $[B]^2$, see [10].

Definition 1. Let F and G be functionals on $[B]^2$. Then a DIT $\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)$ of F is defined by the formula (if it exists)

$$\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\vec{y}) = \int_{B^2} F(\vec{\gamma} \circ \vec{x} + \vec{\beta} \circ \vec{y})\, d\vec{\nu}(\vec{x}), \qquad \vec{y} \in [B]^2 \tag{2.1}$$

and a DCP $(F * G)_{\vec{\gamma}}$ of F and G is defined by the formula (if it exists)

$$(F * G)_{\vec{\gamma}}(\vec{y}) = \int_{B^2} F\left(\frac{\vec{y} + \vec{\gamma} \circ \vec{x}}{\sqrt{2}}\right) G\left(\frac{\vec{y} - \vec{\gamma} \circ \vec{x}}{\sqrt{2}}\right) d\vec{\nu}(\vec{x}), \qquad \vec{y} \in [B]^2. \tag{2.2}$$

We next state an integration formula which we use several times in this paper. Let $\{\alpha_1, \dots, \alpha_n\}$ be any orthonormal set in H and $f : \mathbb{C}^n \to \mathbb{C}$ be Borel measurable. Then for each complex number γ,

$$\begin{aligned} &\int_B f(\gamma(\alpha_1, x)^{\sim}, \dots, \gamma(\alpha_n, x)^{\sim})\, d\nu(x) \\ &\doteq \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} f(\gamma \vec{u}) \exp\left\{-\sum_{j=1}^n \frac{u_j^2}{2}\right\} d\vec{u}, \end{aligned} \tag{2.3}$$

where $\doteq$ means that if either side of (2.3) exists, both sides exist and equality holds.

We finish this section by describing the class of functionals that we work with in this paper. This class is a more generalized class used in [5, 15]. Let

$\{\alpha_1, \ldots, \alpha_n\}$ be any orthonormal set in H and let $\mathcal{A}_0^2 \equiv \mathcal{A}_0 \times \mathcal{A}_0$ be the space of all functionals $F : [B]^2 \to \mathbb{C}$ of the form

$$F(x_1, x_2) = f((\vec{\alpha}, x_1)^\sim, (\vec{\alpha}, x_2)^\sim) \tag{2.4}$$

for some positive integer n, $f(\vec{\lambda_1}, \vec{\lambda_2})$ is an entire function of exponential type; that is to say,

$$|f(\vec{\lambda_1}, \vec{\lambda_2})| \le L_f \exp\bigg\{ M_f \sum_{j=1}^{n} (|\lambda_{1,j}| + |\lambda_{2,j}|) \bigg\}$$

for some positive constants L_f and M_f. We note that the restriction functional $F|_{[B]}$ of F is an element of the class used in [5, 15].

Remark 2.1. (1) Note that $\mathcal{A}_0^2$ is a very rich class of functionals because $\mathcal{A}_0^2$ contains many unbounded functionals. In fact, if F is given by (2.4), then the function f is bounded if and only if it is a constant function.

(2) When F and G in $\mathcal{A}_0^2$, we can always express F by (2.4) and G by

$$G(x_1, x_2) = g((\vec{\alpha}, x_1)^\sim, (\vec{\alpha}, x_2)^\sim) \tag{2.5}$$

using the same positive integer n, where g is an entire function of exponential type.

3. Existence theorems

In this section we obtain the existence of the DIT, DCP and inverse DIT (IDIT). In Theorem 3.1 below, we obtain a formula for the DIT of functionals from $\mathcal{A}_0^2$ to $\mathcal{A}_0^2$.

Theorem 3.1. *Let $F \in \mathcal{A}_0^2$ be given by equation* (2.4). *Then for each $\vec{\gamma} = (\gamma_1, \gamma_2)$ and $\vec{\beta} = (\beta_1, \beta_2)$, the DIT $\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)$ of F exists, belongs to $\mathcal{A}_0^2$ and is given by the formula*

$$\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\vec{y}) = \Gamma_1((\vec{\alpha}, y_1)^\sim, (\vec{\alpha}, y_2)^\sim) \tag{3.1}$$

for $\vec{y} \in [B]^2$, where

$$\begin{aligned} \Gamma_1(\vec{v_1}, \vec{v_2}) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} & f(\gamma_1 \vec{u_1} + \beta_1 \vec{v_1}, \gamma_2 \vec{u_2} + \beta_2 \vec{v_2}) \\ & \times \exp\bigg\{ -\sum_{j=1}^{n} \frac{u_{1,j}^2 + u_{2,j}^2}{2} \bigg\} d\vec{u_1} d\vec{u_2}. \end{aligned} \tag{3.2}$$

Proof. First, let $\vec{v_i} = (\vec{\alpha}, y_i)$ for $i = 1, 2$. Using equations (2.1) and (2.3) it follows that for $\vec{y} \in [B]^2$,

$$\begin{aligned} \mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\vec{y}) &= \int_{B^2} F(\vec{\gamma} \circ \vec{x} + \vec{\beta} \circ \vec{y}) d\vec{\nu}(\vec{x}) \\ &= \int_{B^2} f(\gamma_1 (\vec{\alpha}, x_1)^\sim + \beta_1 (\vec{\alpha}, y_1)^\sim, \gamma_2 (\vec{\alpha}, x_2)^\sim + \beta_2 (\vec{\alpha}, y_2)^\sim) d\vec{\nu}(\vec{x}) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\gamma_1\vec{u_1}+\beta_1\vec{v_1}, \gamma_2\vec{u_2}+\beta_2\vec{v_2}) \\
&\qquad\qquad \times \exp\bigg\{-\sum_{j=1}^{n}\frac{u_{1,j}^2+u_{2,j}^2}{2}\bigg\} d\vec{u_1}d\vec{u_2} \\
&= \Gamma_1(\vec{v_1},\vec{v_2}),
\end{aligned}
$$

where Γ_1 is given by equation (3.2). Using the similar method used in [15] and by Morera's theorem,

$$
\begin{aligned}
&\int_\Lambda \Gamma_1(\vec{v_1},\vec{v_2}) d\vec{v_1}d\vec{v_2} \\
&= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_\Lambda f(\gamma_1\vec{u_1}+\beta_1\vec{v_1}, \gamma_2\vec{u_2}+\beta_2\vec{v_2}) \\
&\qquad\qquad \times \exp\bigg\{-\sum_{j=1}^{n}\frac{u_{1,j}^2+u_{2,j}^2}{2}\bigg\} d\vec{v_1}d\vec{v_2}d\vec{u_1}d\vec{u_2} \\
&= 0
\end{aligned}
$$

and so the function $\Gamma_1(\vec{v_1},\vec{v_2})$ is an entire function on $\mathbb{C}^{2n}$ for any simple closed contour Λ in $\mathbb{C}^{2n}$ because f is an entire function. Also, we have

$$
\begin{aligned}
&|\Gamma_1(\vec{v_1},\vec{v_2})| \\
&\le L_f\left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\bigg\{M_f\sum_{j=1}^{n}(|\gamma_1 u_{1,j}+\beta_1 v_{1,j}|+|\gamma_2 u_{2,j}+\beta_2 v_{2,j}|)\bigg\} \\
&\qquad\qquad \times \exp\bigg\{-\sum_{j=1}^{n}\frac{u_{1,j}^2+u_{2,j}^2}{2}\bigg\} d\vec{u_1}d\vec{u_2} \\
&\le L_f\left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\bigg\{\sum_{j=1}^{n}\Big(M_f|\gamma_1 u_{1,j}|-\frac{u_{1,j}^2}{2}\Big)+\sum_{j=1}^{n}\Big(M_f|\gamma_2 u_{2,j}|-\frac{u_{2,j}^2}{2}\Big)\bigg\} \\
&\qquad\qquad \times \exp\bigg\{M_f\sum_{j=1}^{n}(|\beta_1 v_{1,j}|+|\beta_2 v_{2,j}|)\bigg\} d\vec{u_1}d\vec{u_2} \\
&\le L_{\Gamma_1}\exp\bigg\{M_{\Gamma_1}\sum_{j=1}^{n}(|v_{1,j}|+|v_{2,j}|)\bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
L_{\Gamma_1} = L_f\left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\bigg\{&\sum_{j=1}^{n}\Big(M_f|\gamma_1||u_{1,j}|-\frac{u_{1,j}^2}{2}\Big) \\
&+\sum_{j=1}^{n}\Big(M_f|\gamma_2||u_{2,j}|-\frac{u_{2,j}^2}{2}\Big)\bigg\} d\vec{u_1}d\vec{u_2} < \infty,
\end{aligned}
$$

$M_{\Gamma_1} = M_f\beta^0$ and $\beta^0 = \max\{|\beta_1|,|\beta_2|\}$. Hence $\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)$ is an element of $\mathcal{A}_0^2$. $\square$

In the next theorem, we obtain a formula for the DCP of functionals from $\mathcal{A}_0^2$ to $\mathcal{A}_0^2$.

Theorem 3.2. *Let F and f be as in Theorem 3.1. Let $G \in \mathcal{A}_0^2$ and g be given by equation* (2.5). *Then for each $\vec{\gamma} = (\gamma_1, \gamma_2)$, the DCP $(F * G)_{\vec{\gamma}}$ of F and G exists, belongs to $\mathcal{A}_0^2$ and is given by the formula*

$$(F * G)_{\vec{\gamma}}(\vec{y}) = \Gamma_2((\vec{\alpha}, y_1)^{\sim}, (\vec{\alpha}, y_2)^{\sim}) \tag{3.3}$$

for $\vec{y} \in [B]^2$, where

$$\begin{aligned} &\Gamma_2(\vec{v_1}, \vec{v_2}) \\ &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f\left(\frac{1}{\sqrt{2}}(\vec{v_1} + \gamma_1 \vec{u_1}), \frac{1}{\sqrt{2}}(\vec{v_2} + \gamma_2 \vec{u_2})\right) \\ &\quad \times g\left(\frac{1}{\sqrt{2}}(\vec{v_1} - \gamma_1 \vec{u_1}), \frac{1}{\sqrt{2}}(\vec{v_2} - \gamma_2 \vec{u_2})\right) \exp\left\{-\sum_{j=1}^n \frac{u_{1,j}^2 + u_{2,j}^2}{2}\right\} d\vec{u_1} d\vec{u_2}. \end{aligned} \tag{3.4}$$

Proof. Let $\vec{v_i} = (\vec{\alpha}, y_i)$ for $i = 1, 2$. Using equations (2.2) and (2.3) it follows that for $\vec{y} \in [B]^2$,

$$\begin{aligned} &(F * G)_{\vec{\gamma}}(\vec{y}) \\ &= \int_{B^2} f\left(\frac{1}{\sqrt{2}}(\vec{v_1} + \gamma_1 (\vec{\alpha}, x_1)^{\sim}), \frac{1}{\sqrt{2}}(\vec{v_2} + \gamma_2 (\vec{\alpha}, x_2)^{\sim})\right) \\ &\quad \times g\left(\frac{1}{\sqrt{2}}(\vec{v_1} - \gamma_1 (\vec{\alpha}, x_1)^{\sim}), \frac{1}{\sqrt{2}}(\vec{v_2} - \gamma_2 (\vec{\alpha}, x_2)^{\sim})\right) d\vec{\nu}(\vec{x}) \\ &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f\left(\frac{1}{\sqrt{2}}(\vec{v_1} + \gamma_1 \vec{u_1}), \frac{1}{\sqrt{2}}(\vec{v_2} + \gamma_2 \vec{u_2})\right) \\ &\quad \times g\left(\frac{1}{\sqrt{2}}(\vec{v_1} - \gamma_1 \vec{u_1}), \frac{1}{\sqrt{2}}(\vec{v_2} - \gamma_2 \vec{u_2})\right) \exp\left\{-\sum_{j=1}^n \frac{u_{1,j}^2 + u_{2,j}^2}{2}\right\} d\vec{u_1} d\vec{u_2} \\ &= \Gamma_2(\vec{v_1}, \vec{v_2}), \end{aligned}$$

where Γ_2 is given by equation (3.4). Furthermore, by using the same method in the proof of Theorem 3.1, the function $\Gamma_2(\vec{v_1}, \vec{v_2})$ is an entire function on $\mathbb{C}^{2n}$. Also, we have

$$\begin{aligned} &|\Gamma_2(\vec{v_1}, \vec{v_2})| \\ &\le L_f L_g \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left\{\frac{M_f + M_g}{\sqrt{2}} \sum_{j=1}^n (|\gamma_1 u_{1,j} + v_{1,j}| + |\gamma_2 u_{2,j} + v_{2,j}|)\right\} \\ &\qquad \times \exp\left\{-\sum_{j=1}^n \frac{u_{1,j}^2 + u_{2,j}^2}{2}\right\} d\vec{u_1} d\vec{u_2} \\ &\le L_f L_g \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left\{\sum_{j=1}^n \left(\frac{M_f + M_g}{\sqrt{2}} |\gamma_1 u_{1,j}| - \frac{u_{1,j}^2}{2}\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&\times \exp\bigg\{\sum_{j=1}^{n}\bigg(\frac{M_f+M_g}{\sqrt{2}}|\gamma_2 u_{2,j}|-\frac{u_{2,j}^2}{2}\bigg)\bigg\}\\
&\times \exp\bigg\{\frac{M_f+M_g}{\sqrt{2}}\sum_{j=1}^{n}\Big(|v_{1,j}|+|v_{2,j}|\Big)\bigg\}d\vec{u_1}d\vec{u_2}
\end{aligned}
$$

$$
\leq L_{\Gamma_2}\exp\bigg\{M_{\Gamma_2}\sum_{j=1}^{n}(|v_{1,j}|+|v_{2,j}|)\bigg\},
$$

where

$$
\begin{aligned}
L_{\Gamma_2}=L_fL_g\bigg(\frac{1}{2\pi}\bigg)^n\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}&\exp\bigg\{\sum_{j=1}^{n}\bigg(\frac{M_f+M_g}{\sqrt{2}}|\gamma_1||u_{1,j}|-\frac{u_{1,j}^2}{2}\bigg)\bigg\}\\
\times \exp&\bigg\{\sum_{j=1}^{n}\bigg(\frac{M_f+M_g}{\sqrt{2}}|\gamma_2||u_{2,j}|-\frac{u_{2,j}^2}{2}\bigg)\bigg\}d\vec{u_1}d\vec{u_2}<\infty,
\end{aligned}
$$

and $M_{\Gamma_2}=\frac{M_f+M_g}{\sqrt{2}}$. Hence $(F*G)_{\vec{\gamma}}$ is an element of $\mathcal{A}_0^2$. □

To establish the existence of the IDIT, we need the following lemma.

Lemma 3.3. *Let γ_1 and γ_2 be nonzero complex numbers. Let f be as in Theorem 3.1. Then for each $(\vec{v_1},\vec{v_2})\in\mathbb{C}^n\times\mathbb{C}^n$,*

$$
\begin{aligned}
&\bigg(\frac{1}{2\pi}\bigg)^{2n}\int_{\mathbb{R}^{4n}}f(\gamma_1\vec{u_1}+i\gamma_1\vec{w_1}+\vec{v_1},\gamma_2\vec{u_2}+i\gamma_2\vec{w_2}+\vec{v_2})\\
&\times\exp\bigg\{-\sum_{j=1}^{n}\frac{u_{1,j}^2+u_{2,j}^2+w_{1,j}^2+w_{2,j}^2}{2}\bigg\}d\vec{u_1}d\vec{u_2}d\vec{w_1}d\vec{w_2}=f(\vec{v_1},\vec{v_2}).
\end{aligned}\tag{3.5}
$$

Proof. First converting to polar coordinates with $u_{i,j}=r_{i,j}\cos\theta_{i,j}$ and $w_{i,j}=r_{i,j}\sin\theta_{i,j}$ for $i=1,2$, $j=1,\dots,n$ yields the expression

$$
\begin{aligned}
&\bigg(\frac{1}{2\pi}\bigg)^{2n}\int_0^\infty(2n)\int_0^\infty\exp\bigg\{-\sum_{j=1}^{n}\frac{r_{1,j}^2+r_{2,j}^2}{2}\bigg\}\\
&\times\int_0^{2\pi}(2n)\int_0^{2\pi}f(\gamma_1r_{1,1}e^{i\theta_{1,1}}+v_{1,1},\dots,\gamma_1r_{1,n}e^{i\theta_{1,n}}+v_{1,n},\\
&\qquad\gamma_2r_{2,1}e^{i\theta_{2,1}}+v_{2,1},\dots,\gamma_2r_{2,n}e^{i\theta_{2,n}}+v_{2,n})d\vec{\theta_1}d\vec{\theta_2}\vec{r_1}\vec{r_2}d\vec{r_1}d\vec{r_2}\\
&=f(\vec{v_1},\vec{v_2}),
\end{aligned}\tag{3.6}
$$

where $\vec{r_1}\vec{r_2}=r_{1,1}\cdots r_{1,n}r_{2,1}\cdots r_{2,n}$. Using Gauss's mean value theorem from complex variables to evaluate the integral (3.6) with respect to $\vec{\theta_1},\vec{\theta_2}$ and thereby carrying out the integration with respect to $\vec{r_1}$ and $\vec{r_2}$, the desired formula (3.5) now follows. □

In the next theorem, we establish the existence of the inverse DIT.

Theorem 3.4. *Let F be as in Theorem 3.1. Then*

$$\mathcal{F}_{\vec{\gamma_0},\vec{\beta_0}}(\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F))(\vec{y}) = F(\vec{y}) = \mathcal{F}_{\vec{\gamma},\vec{\beta}}(\mathcal{F}_{\vec{\gamma_0},\vec{\beta_0}}(F))(\vec{y}) \tag{3.7}$$

for $\vec{y} \in [B]^2$, where $\vec{\gamma_0} = (i\frac{\gamma_1}{\beta_1}, i\frac{\gamma_2}{\beta_2})$ and $\vec{\beta_0} = (\frac{1}{\beta_1}, \frac{1}{\beta_2})$.

Proof. From Theorem 3.1, all expressions in equation (3.7) exist as elements of $\mathcal{A}_0^2$. Let $v_{i,j} = (\alpha_j, y_i)^{\sim}$, $i = 1, 2$ and $j = 1, 2, \dots, n$. We shall show that equalities hold in equation (3.7). Using equations (2.1), (3.1) and (2.3), it follows that for $\vec{y} \in [B]^2$,

$$\begin{aligned}
&\mathcal{F}_{\vec{\gamma_0},\vec{\beta_0}}(\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F))(\vec{y}) \\
&= \int_{B^2} \mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\vec{\gamma_0} \circ \vec{x} + \vec{\beta_0} \circ \vec{y}) d\vec{\nu}(\vec{x}) \\
&= \left(\frac{1}{2\pi}\right)^n \int_{B^2} \int_{\mathbb{R}^{2n}} f(\gamma_1 \vec{u_1} + i\gamma_1(\vec{\alpha}, x_1)^{\sim} + \vec{v_1}, \gamma_2 \vec{u_2} + i\gamma_2(\vec{\alpha}, x_2)^{\sim} + \vec{v_2}) \\
&\qquad \times \exp\left\{ -\sum_{j=1}^n \frac{u_{1,j}^2 + u_{2,j}^2}{2} \right\} d\vec{u_1} d\vec{u_2} d\vec{\nu}(\vec{x}) \\
&= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbb{R}^{4n}} f(\gamma_1 \vec{u_1} + i\gamma_1 \vec{w_1} + \vec{v_1}, \gamma_2 \vec{u_2} + i\gamma_2 \vec{w_2} + \vec{v_2}) \\
&\qquad \times \exp\left\{ -\sum_{j=1}^n \frac{u_{1,j}^2 + u_{2,j}^2 + w_{1,j}^2 + w_{2,j}^2}{2} \right\} d\vec{u_1} d\vec{u_2} d\vec{w_1} d\vec{w_2}.
\end{aligned}$$

From Lemma 3.3, the first equation of (3.7) equals $F(y_1, y_2)$. Also, proceeding as above, we find that the third equation of (3.7) equals $F(y_1, y_2)$, and this concludes the proof of Theorem 3.4. □

4. Basic formulas

In this section we establish various basic formulas for the DIT, the DCP and the IDIT of functionals in $\mathcal{A}_0^2$. In our first theorem of this section, we show that the DIT of the DCP is the product of their DITs.

Theorem 4.1. *Let F and G be as in Theorem 3.2. Then for each $\vec{\gamma} = (\gamma_1, \gamma_2)$ and $\vec{\beta} = (\beta_1, \beta_2)$,*

$$\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F * G)_{\vec{\gamma}}(\vec{y}) = \mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\vec{y}/\sqrt{2}) \mathcal{F}_{\vec{\gamma},\vec{\beta}}(G)(\vec{y}/\sqrt{2}) \tag{4.1}$$

as elements of $\mathcal{A}_0^2$.

Proof. First, from Theorems 3.1 and 3.2, all expressions in equation (4.1) exist as elements of $\mathcal{A}_0^2$. Now, using equations of Definition 1, (2.2), (3.3), (3.2) and

(2.3) it follows that for $\vec{y} \in [B]^2$,

$$
\begin{aligned}
&\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F * G)_{\vec{\gamma}}(\vec{y}) \\
&= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbb{R}^{4n}} f\Big(\frac{\gamma_1}{\sqrt{2}}\vec{v_1} + \frac{\gamma_1}{\sqrt{2}}\vec{u_1} + \frac{\beta_1}{\sqrt{2}}(\vec{\alpha}, y_1)^{\sim}, \frac{\gamma_2}{\sqrt{2}}\vec{v_2} + \frac{\gamma_2}{\sqrt{2}}\vec{u_2} + \frac{\beta_2}{\sqrt{2}}(\vec{\alpha}, y_2)^{\sim}\Big) \\
&\qquad\qquad \times g\Big(\frac{\gamma_1}{\sqrt{2}}\vec{v_1} - \frac{\gamma_1}{\sqrt{2}}\vec{u_1} + \frac{\beta_1}{\sqrt{2}}(\vec{\alpha}, y_1)^{\sim}, \frac{\gamma_2}{\sqrt{2}}\vec{v_2} - \frac{\gamma_2}{\sqrt{2}}\vec{u_2} + \frac{\beta_2}{\sqrt{2}}(\vec{\alpha}, y_2)^{\sim}\Big) \\
&\qquad\qquad \times \exp\Big\{-\sum_{j=1}^{n} \frac{u_{1,j}^2 + u_{2,j}^2 + v_{1,j}^2 + v_{2,j}^2}{2}\Big\} d\vec{u_1}\, d\vec{u_2}\, d\vec{v_1}\, d\vec{v_2}.
\end{aligned}
$$

Next, letting $w_{i,j} = \frac{v_{i,j}+u_{i,j}}{\sqrt{2}}$ and $z_{i,j} = \frac{v_{i,j}-u_{i,j}}{\sqrt{2}}$ for $i = 1, 2$ and $j = 1, \ldots, n$, we obtain

$$
\begin{aligned}
&\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F * G)_{\vec{\gamma}}(\vec{y}) \\
&= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbb{R}^{4n}} f\Big(\gamma_1\vec{w_1} + \frac{\beta_1}{\sqrt{2}}(\vec{\alpha}, y_1)^{\sim}, \gamma_2\vec{w_2} + \frac{\beta_2}{\sqrt{2}}(\vec{\alpha}, y_2)^{\sim}\Big) \\
&\qquad\qquad \times g\Big(\gamma_1\vec{z_1} + \frac{\beta_1}{\sqrt{2}}(\vec{\alpha}, y_1)^{\sim}, \gamma_2\vec{z_2} + \frac{\beta_2}{\sqrt{2}}(\vec{\alpha}, y_2)^{\sim}\Big) \\
&\qquad\qquad \times \exp\Big\{-\sum_{j=1}^{n} \frac{w_{1,j}^2 + w_{2,j}^2 + z_{1,j}^2 + z_{2,j}^2}{2}\Big\} d\vec{w_1}\, d\vec{w_2}\, d\vec{z_1}\, d\vec{z_2} \\
&= \left(\frac{1}{2\pi}\right)^{n} \int_{\mathbb{R}^{2n}} f\Big(\gamma_1\vec{w_1} + \frac{\beta_1}{\sqrt{2}}(\vec{\alpha}, y_1)^{\sim}, \gamma_2\vec{w_2} + \frac{\beta_2}{\sqrt{2}}(\vec{\alpha}, y_2)^{\sim}\Big) \\
&\qquad\qquad\qquad\qquad \times \exp\Big\{-\sum_{j=1}^{n} \frac{w_{1,j}^2 + w_{2,j}^2}{2}\Big\} d\vec{w_1}\, d\vec{w_2} \\
&\qquad \times \left(\frac{1}{2\pi}\right)^{n} \int_{\mathbb{R}^{2n}} g\Big(\gamma_1\vec{z_1} + \frac{\beta_1}{\sqrt{2}}(\vec{\alpha}, y_1)^{\sim}, \gamma_2\vec{z_2} + \frac{\beta_2}{\sqrt{2}}(\vec{\alpha}, y_2)^{\sim}\Big) \\
&\qquad\qquad\qquad\qquad \times \exp\Big\{-\sum_{j=1}^{n} \frac{z_{1,j}^2 + z_{2,j}^2}{2}\Big\} d\vec{z_1}\, d\vec{z_2} \\
&= \int_{B^2} F\Big(\vec{\gamma} \circ \vec{x} + \frac{1}{\sqrt{2}}\vec{\beta} \circ \vec{y}\Big) d\vec{\nu}(\vec{x}) \int_{B^2} G\Big(\vec{\gamma} \circ \vec{x} + \frac{1}{\sqrt{2}}\vec{\beta} \circ \vec{y}\Big) d\vec{\nu}(\vec{x}) \\
&= 0\ \mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\vec{y}/\sqrt{2})\mathcal{F}_{\vec{\gamma},\vec{\beta}}(G)(\vec{y}/\sqrt{2}),
\end{aligned}
$$

which completes the proof of Theorem 4.1. □

Next, we obtain two basic formulas for the DCPs, whose proofs follow immediately from equations (4.1) and (3.7).

Theorem 4.2. *Let F and G be as in Theorem 4.1. Then*

$$
(F * G)_{\vec{\gamma}}(\vec{y}) = \mathcal{F}_{\vec{\gamma_0},\vec{\beta_0}}(\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\cdot/\sqrt{2})\mathcal{F}_{\vec{\gamma},\vec{\beta}}(G)(\cdot/\sqrt{2}))(\vec{y}) \tag{4.2}
$$

as elements of $\mathcal{A}_0^2$*, where* $\vec{\gamma_0}$ *and* $\vec{\beta_0}$ *are as in Theorem 3.4. Also, by interchanging the two pairs* $(\vec{\gamma},\vec{\beta})$ *and* $(\vec{\gamma_0},\vec{\beta_0})$ *in equation (4.2), we establish the basic formula*

$$(F*G)_{\vec{\gamma_0}}(\vec{y}) = \mathcal{F}_{\vec{\gamma},\vec{\beta}}(\mathcal{F}_{\vec{\gamma_0},\vec{\beta_0}}(F)(\vec{\cdot}/\sqrt{2})\mathcal{F}_{\vec{\gamma_0},\vec{\beta_0}}(G)(\vec{\cdot}/\sqrt{2}))(\vec{y})$$

as elements of $\mathcal{A}_0^2$.

In order to establish the Fubini theorem, we need the following lemma. The proof of lemma was established in [5] and used in [4, 9].

Lemma 4.3. *Let* ϕ *be an integrable functional on* $[B]$*. Then for all non-zero complex numbers* γ *and* β*,*

$$\int_{B^2}\phi(\gamma x+\beta y+z)d\nu(x)d\nu(y) = \int_B \phi(\sqrt{\gamma^2+\beta^2}w+z)d\nu(w), \tag{4.3}$$

where $z\in[B]$.

In our next theorem, we establish the Fubini theorem with respect to the DIT.

Theorem 4.4. *Let* F *be as in Theorem 4.1. Let* $\vec{\gamma}=(\gamma_1,\gamma_2)$*,* $\vec{\beta}=(\beta_1,\beta_2)$*,* $\vec{\eta}=(\eta_1,\eta_2)$ *and* $\vec{\delta}=(\delta_1,\delta_2)$ *with* $\gamma_i^2+\beta_i^2=1$ *and* $\eta_i^2+\delta_i^2=1$ *for* $i=1,2$*. Then*

$$\mathcal{F}_{\vec{\gamma},\vec{\beta}}(\mathcal{F}_{\vec{\eta},\vec{\delta}}(F))(\vec{y}) = \mathcal{F}_{\vec{\gamma}',\vec{\beta}'}(F)(\vec{y}) = \mathcal{F}_{\vec{\eta},\vec{\delta}}(\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F))(\vec{y}) \tag{4.4}$$

as elements of $\mathcal{A}_0^2$ *with* $\gamma_i'=\sqrt{\gamma_i^2+\beta_i^2\eta_i^2}$ *and* $\beta_i'=\beta_i\delta_i$ *for* $i=1,2$.

Proof. Using equations (2.1) and (2.3) it follows that for $\vec{y}\in[B]^2$,

$$\begin{aligned}
&\mathcal{F}_{\vec{\gamma},\vec{\beta}}(\mathcal{F}_{\vec{\eta},\vec{\delta}}F)(\vec{y})\\
&= \int_{B^2}\int_{B^2} f(\eta_1(\vec{\alpha},z_1)^\sim+\delta_1\gamma_1(\vec{\alpha},x_1)^\sim+\delta_1\beta_1(\vec{\alpha},y_1)^\sim,\\
&\qquad \eta_2(\vec{\alpha},z_2)^\sim+\delta_2\gamma_2(\vec{\alpha},x_2)^\sim+\delta_2\beta_2(\vec{\alpha},y_2)^\sim)d\vec{\nu}(\vec{z})d\vec{\nu}(\vec{x}).
\end{aligned}$$

Now applying formula (4.3) to the first and second components, respectively, we obtain

$$\begin{aligned}
&\mathcal{F}_{\vec{\gamma},\vec{\beta}}(\mathcal{F}_{\vec{\eta},\vec{\delta}}F)(\vec{y})\\
&= \int_{B^2} f(\sqrt{\eta_1^2+\delta_1^2\gamma_1^2}(\vec{\alpha},w_1)^\sim+\delta_1\beta_1(\vec{\alpha},y_1)^\sim,\\
&\qquad \sqrt{\eta_2^2+\delta_2^2\gamma_2^2}(\vec{\alpha},w_2)^\sim+\delta_2\beta_2(\vec{\alpha},y_2)^\sim)d\vec{\nu}(\vec{w}).
\end{aligned}$$

On the other hand, again using equations (2.1) and (2.3), it follows that for $\vec{y}\in[B]^2$,

$$\begin{aligned}
&\mathcal{F}_{\vec{\eta},\vec{\delta}}(\mathcal{F}_{\vec{\gamma},\vec{\beta}}F)(\vec{y})\\
&= \int_{B^2}\int_{B^2} f(\gamma_1(\vec{\alpha},z_1)^\sim+\beta_1\eta_1(\vec{\alpha},x_1)^\sim+\delta_1\beta_1(\vec{\alpha},y_1)^\sim,\\
&\qquad \gamma_2(\vec{\alpha},z_2)^\sim+\beta_2\eta_2(\vec{\alpha},x_2)^\sim+\delta_2\beta_2(\vec{\alpha},y_2)^\sim)d\vec{\nu}(\vec{z})d\vec{\nu}(\vec{x}).
\end{aligned}$$

Next again applying formula (4.3) to the first and second components, respectively, it follows that for $\vec{y} \in [B]^2$,

$$\begin{aligned}&\mathcal{F}_{\vec{\eta},\vec{\delta}}(\mathcal{F}_{\vec{\gamma},\vec{\beta}}F)(\vec{y})\\&= \int_{B^2} f(\sqrt{\gamma_1^2+\beta_1^2\eta_1^2}(\vec{\alpha},w_1)^\sim + \delta_1\beta_1(\vec{\alpha},y_1)^\sim,\\&\qquad \sqrt{\gamma_2^2+\beta_2^2\eta_2^2}(\vec{\alpha},w_2)^\sim + \delta_2\beta_2(\vec{\alpha},y_2)^\sim)d\vec{\nu}(\vec{w}).\end{aligned}$$

Note that $\gamma_i^2+\beta_i^2\eta_i^2 = \eta_i^2+\gamma_i^2\delta_i^2$ because $\gamma_i^2+\beta_i^2 = 1$ and $\eta_i^2+\delta_i^2 = 1$ for $i = 1, 2$ and so equation (4.4) is established. □

We obtain the following corollary by letting $\gamma_1 = \gamma_2 = \eta_1 = \eta_2 = \gamma$ and $\beta_1 = \beta_2 = \delta_1 = \delta_2 = \beta$, in equation (4.4).

Corollary 4.5. *Let F be as in Theorem 4.4. Let $\vec{\gamma} = (\gamma,\gamma)$ and $\vec{\beta} = (\beta,\beta)$ with $\gamma^2+\beta^2 = 1$ in Theorem 4.4. Then*

$$\mathcal{F}_{\vec{\gamma},\vec{\beta}}(\mathcal{F}_{\vec{\gamma},\vec{\beta}}F)(\vec{y}) = \mathcal{F}_{\vec{\gamma''},\vec{\beta''}}F(\vec{y})$$

as elements of $\mathcal{A}_0^2$, where $\vec{\gamma''} = (\gamma_1'',\gamma_2'')$ and $\vec{\beta''} = (\beta_1'',\beta_2'')$ with $\gamma_i'' = \sqrt{\gamma^2+\beta^2\gamma^2}$ and $\beta_i'' = \beta^2$, $i = 1, 2$.

5. Additional relationships

In this section, we give some additional results and applications with respect to the results in Sections 3 and 4.

(1) Let $f(\vec{\lambda_1},\vec{\lambda_2}) = f^*(\vec{\lambda_1}+\vec{\lambda_2})$, where $F^*(x) = f^*((\vec{\alpha},x)^\sim) \in \mathbb{E}_0$, where $\mathbb{E}_0$ is the class of functionals introduced in [15]. Then for $\gamma^* = \sqrt{\gamma_1^2+\gamma_2^2} \neq 0$ and $\beta^* = \beta_1\beta_2$,

$$\begin{aligned}\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\vec{y}) &= \int_{B^2} f(\gamma_1(\vec{\alpha},x_1)^\sim + \beta_1(\vec{\alpha},y_1)^\sim, \gamma_2(\vec{\alpha},x_2)^\sim + \beta_2(\vec{\alpha},y_2)^\sim)d\vec{\nu}(\vec{x})\\&= \int_{B^2} f^*(\gamma_1(\vec{\alpha},x_1)^\sim + \gamma_2(\vec{\alpha},x_2)^\sim + \beta_1(\vec{\alpha},y_1)^\sim + \beta_2(\vec{\alpha},y_2)^\sim)d\vec{\nu}(\vec{x})\\&= \int_{B} f^*(\sqrt{\gamma_1^2+\gamma_2^2}(\vec{\alpha},x)^\sim + \beta_1(\vec{\alpha},y_1)^\sim + \beta_2(\vec{\alpha},y_2)^\sim)d\nu(x)\\&= \mathcal{G}_{\gamma^*,\beta^*}(F^*)(y_1/\beta_2+y_2/\beta_1),\end{aligned}$$

where $\mathcal{G}_{\gamma,\beta}$ is the ordinary integral transform used in [5, 18].

Let $f(u,v)$ be analytic and bounded on all compact subsets of the region by intersection the product of two circles

$$\mathcal{R} = \{(u,v) : |u| < R_1 \le \infty, |v| < R_2 \le \infty\}$$

with the union of two hyper-planes $\mathcal{P} = \{(u,v)\,|\,Im(u)Im(v) = 0\}$. In [3], Cameron and Storvick showed that the function $f(u,v)$ on $\mathcal{R}$ can be expressed by the form

$$f(u,v) \approx \sum_{m,n=0}^{\infty} a_{(m,n)} P_m(u) P_n(v), \qquad u, v \in \mathbb{R} \tag{5.1}$$

in the sense of convergence absolutely where P_n is the Legendre polynomial. Since the Legendre polynomial has an analytic extension, equation (5.1) still holds for the complex numbers u and v. These facts indicate that the following formula has some meaning.

(2) Let $f(\vec{\lambda_1}, \vec{\lambda_2}) = f_1(\vec{\lambda_1}) f_2(\vec{\lambda_2})$, where $F_i(x) = f_i((\vec{\alpha}, x)^\sim) \in \mathbb{E}_0$ for $i = 1, 2$. Then

$$\begin{aligned}
\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\vec{y}) &= \int_{B^2} f(\gamma_1(\vec{\alpha}, x_1)^\sim + \beta_1(\vec{\alpha}, y_1)^\sim, \gamma_2(\vec{\alpha}, x_2)^\sim + \beta_2(\vec{\alpha}, y_2)^\sim) d\vec{\nu}(\vec{x}) \\
&= \left(\int_B f_1(\gamma_1(\vec{\alpha}, x_1)^\sim + \beta_1(\vec{\alpha}, y_1)^\sim) d\nu(x_1) \right) \\
&\quad \times \left(\int_B f_2(\gamma_2(\vec{\alpha}, x_2)^\sim + \beta_2(\vec{\alpha}, y_2)^\sim) d\nu(x_2) \right) \\
&= \mathcal{G}_{\gamma_1,\beta_1}(F_1)(y_1) \mathcal{G}_{\gamma_2,\beta_2}(F_2)(y_2),
\end{aligned}$$

where $\mathcal{G}_{\gamma,\beta}$ is the ordinary integral transform used in [5, 18]. Furthermore, using the mathematical induction, let $f(\vec{\lambda_1}, \ldots, \vec{\lambda_n}) = f_1(\vec{\lambda_1}) \cdots f_n(\vec{\lambda_n})$, where $F_i(x) = f_i((\vec{\alpha}, x)^\sim) \in \mathbb{E}_0$ for $i = 1, \ldots, n$. Then for $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$ and $\vec{\beta} = (\beta_1, \ldots, \beta_n)$,

$$\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\vec{y}) = \prod_{i=1}^{n} \mathcal{G}_{\gamma_i,\beta_i}(F_i)(y_i).$$

In particular, let $f(\vec{\lambda_1}, \vec{\lambda_2}) = f_1(\vec{\lambda_1})$ (or $f(\vec{\lambda_1}, \vec{\lambda_2}) = f_2(\vec{\lambda_2})$). Then

$$\mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\vec{y}) = \mathcal{G}_{\gamma_1,\beta_1}(F_1)(y_1) \quad \text{or} \quad \mathcal{F}_{\vec{\gamma},\vec{\beta}}(F)(\vec{y}) = \mathcal{G}_{\gamma_2,\beta_2}(F_2)(y_2).$$

(3) Let $F \in \mathcal{A}_0^2$ be given by equation (2.4) and let $\{(\gamma_{m,j}, \beta_{m,j})\}$ be a sequence in $\mathbb{C} \times \mathbb{C}$ with $\gamma_{m,j} \to \gamma_j \neq 0$ and $\beta_{m,j} \to \beta_j \neq 0$ as $m \to \infty$ for some $(\gamma_j, \beta_j) \in \mathbb{C} \times \mathbb{C}$, $j = 1, 2$. Then using the dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{m\to\infty} \mathcal{F}_{\vec{\gamma}_m,\vec{\beta}_m} F(\vec{y}) \\
&= \lim_{m\to\infty} \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\gamma_{m,1}\vec{u_1} + \beta_{m,1}\vec{v_1}, \gamma_{m,2}\vec{u_2} + \beta_{m,2}\vec{v_2}) \\
&\qquad\qquad \times \exp\left\{ -\sum_{j=1}^{n} \frac{u_{1,j}^2 + u_{2,j}^2}{2} \right\} d\vec{u_1} d\vec{u_2} \\
&= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\gamma_1\vec{u_1} + \beta_1\vec{v_1}, \gamma_2\vec{u_2} + \beta_2\vec{v_2})
\end{aligned}$$

$$\times \exp\bigg\{-\sum_{j=1}^{n}\frac{u_{1,j}^2+u_{2,j}^2}{2}\bigg\}d\vec{u_1}d\vec{u_2}$$

$$= \mathcal{F}_{\vec{\gamma},\vec{\beta}}F(\vec{y}).$$

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