Bull. Korean Math. Soc. **59** (2022), No. 5, pp. 1093–1103
https://doi.org/10.4134/BKMS.b210542
pISSN: 1015-8634 / eISSN: 2234-3016

EVALUATION FORMULA FOR WIENER INTEGRAL OF POLYNOMIALS IN TERMS OF NATURAL DUAL PAIRINGS ON ABSTRACT WIENER SPACES

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Abstract. In this paper, we establish an evaluation formula to calculate the Wiener integral of polynomials in terms of natural dual pairings on abstract Wiener spaces (H,B,ν). To do this we first derive a translation theorem for the Wiener integral of functionals associated with operators in $\mathcal{L}(B)$, the Banach space of bounded linear operators from B to itself. We then apply the translation theorem to establish an integration by parts formula for the Wiener integral of functionals combined with operators in $\mathcal{L}(B)$. We finally apply this parts formula to evaluate the Wiener integral of certain polynomials in terms of natural dual pairings.

1. Introduction

Let H be a real infinite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and associated norm $|\cdot|$, and let B be a real separable Banach space with norm $\|\cdot\|$. It is assumed that H is continuously, linearly, and densely embedded in B. The natural injection (i.e., embedding) is denoted by $\iota : H \hookrightarrow B$. Let ν be a centered Gaussian probability measure on $(B,\mathcal{B}(B))$, where $\mathcal{B}(B)$ is the Borel σ-field of B. The triple (H,B,ν) is called an abstract Wiener space if

$$\int_B \exp\big[i(h,x)\big]d\nu(x)=\exp\bigg[-\frac{1}{2}|\iota^*(h)|^2\bigg]=\exp\bigg[-\frac{1}{2}|h|^2\bigg]$$

for any $h\in B^*$, where $(\cdot,\cdot)$ denotes the natural dual pairing (B^*–B pairing) and $\iota^* : B^*\to H^*$ is the dual map to the natural injection $\iota : H\hookrightarrow B$, and where B^* and H^* are the topological duals of B and H, respectively. The space B^* is identified as a dense subspace of $H^*\approx H$ in the sense that, for all

Received July 22, 2021; Revised March 19, 2022; Accepted July 15, 2022.
2020 *Mathematics Subject Classification.* Primary 28C20, 46G12; Secondary 46B09, 46B20.
Key words and phrases. Abstract Wiener space, natural dual pairing, translation theorem, integration by parts formula.
The second author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1F1A1062770).

$y \in B^*$ and $x \in H$,

$$\langle y, x\rangle = (y, x). \tag{1.1}$$

Thus we have the triple

$$B^* \subset H^* \approx H \subset B. \tag{1.2}$$

The Hilbert space H is called the Cameron–Martin space in the abstract Wiener space B. For more details, see [8, 12, 15].

Given a nonnegative integer m, let $\mathsf{M}_m(B^*)$ denote the set of all monomials F, in terms of natural dual pairings on the abstract Wiener space (H, B, ν), defined as follows:

$$\begin{aligned} F \in \mathsf{M}_m(B^*) \iff & F(x) = c(\theta_1, x)^{k_1}(\theta_2, x)^{k_2} \cdots (\theta_m, x)^{k_m} \\ & \text{for some } c \in \mathbb{R}, \text{ a finite subset } \{\theta_1, \theta_2, \ldots, \theta_m\} \text{ of } B^*, \\ & \text{and a finite subset } \{k_1, k_2, \ldots, k_m\} \text{ of } \{0\} \cup \mathbb{N}. \end{aligned}$$

Let $\mathsf{M}_0 = \mathbb{R}$ for notational convenience, let $\Sigma_{\mathbf{f}}(\mathsf{M}_m)$ denote the class of all linear combinations (with real coefficients) of the monomials in M_m, and let $\mathcal{P}(B^*) = \cup_{m=0}^{\infty} \Sigma_{\mathbf{f}}(\mathsf{M}_m)$. Then $\mathcal{P}(B^*)$ is the set of all conventional polynomials in a finite number of linear functionals (natural dual pairings) on the abstract Wiener space (H, B, ν), see, [1, pp. 43–45] and [25, p. 119]. It is well known that $\mathcal{P}(B^*)$ is a dense subset of the space $L_2(B)$, the space of square-integrable functionals on B, see [11, Chapter 1]. From this fact, many mathematicians have studied the Wiener integral of the polynomials in $\mathcal{P}(B^*)$, the structures of cylinder functionals (rather than L_2-functionals) on Wiener spaces, and the related topics. See, for instance, [9, 10, 18, 20, 26].

Based on those historical background, we will study the Wiener integral of the polynomials in terms of natural dual pairings on an abstract Wiener space B. In this paper, we establish an evaluation formula to calculate the Wiener integral of monomials in terms of natural dual pairings on B. To do this, we first derive a translation theorem for the Wiener integral of functionals associated with operators in $\mathcal{L}(B)$, the Banach space of bounded operators from B to itself. We then apply the translation theorem to establish an integration by parts formula for the Wiener integral of functionals combined with operators in $\mathcal{L}(B)$. We finally apply this parts formula to evaluate the Wiener integral of the monomials. Precisely speaking, we provide an evaluation formula for the Winer integral of monomials given by

$$F(x) = \prod_{\alpha \in \mathfrak{D}} (A_\alpha^* g, x), \quad x \in B,$$

where $\mathfrak{D}$ is a finite index set and for each $\alpha \in \mathfrak{D}$, A_α^* is the Banach space dual operator of the bounded operator A_α in $\mathcal{L}(B)$.

2. Background

In order to present our evaluation formula for the Wiener integral, we follow the exposition of [6, 12, 13, 15]. Let (H, B, ν) be an abstract Wiener space, and let $\{e_n\}$ be a complete orthonormal set in H such that e_j's are in B^*. For each $h \in H$ and $x \in B$, a stochastic inner product $(h, x)^\sim$ is defined by

$$(h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (e_j, x), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

By the definition of the stochastic inner product $(\cdot,\cdot)^\sim$ and (1.1), it is clear that $(\theta, x)^\sim = (\theta, x)$ for all $\theta \in B^*$ and $x \in B$. It is well known [6, 12, 13, 15, 25] that for every non-zero h in H, $(h, x)^\sim$ is a Gaussian random variable on B with mean 0 and variance $|h|^2$. The stochastic inner product $(h, x)^\sim$ given by (2.1) is essentially independent of the choice of the complete orthonormal set used in its definition. Also, if both h and x are in H, then Parseval's identity gives $(h, x)^\sim = \langle h, x\rangle$. Furthermore, $(h, \lambda x)^\sim = (\lambda h, x)^\sim = \lambda(h, x)^\sim$ for any $\lambda \in \mathbb{R}$, $h \in H$ and $x \in B$. We also see that if $\{h_1, \ldots, h_n\}$ is an orthogonal set in H, then the random variables $(h_j, x)^\sim$'s are independent.

By the concept of the Banach space adjoint operator, given an operator $A \in \mathcal{L}(B)$, there exists a bounded linear operator $A^* : B^* \to B^*$ such that for all $\theta \in B^*$ and $x \in B$,

$$(A^*\theta)x = \theta(Ax). \tag{2.2}$$

By the structure of the dual pairing and the triple (1.2) (i.e., in the sense of Riesz representation theorem), equation (2.2) can be rewritten by

$$(A^*\theta, x) = (\theta, Ax).$$

The Cameron–Martin translation theorem describes how abstract Wiener measure changes under translation by certain elements of the Cameron–Martin space H.

Theorem 2.1 ([14])**.** *Let (H, B, ν) be an abstract Wiener space, let $F \in L_1(B)$ and let $x_0 \in H$. Then it follows that*

$$\int_B F(x)d\nu(x) = \exp\Big[-\frac{1}{2}|x_0|^2\Big] \int_B F(x + x_0) \exp\big[-(x_0, x)^\sim\big] d\nu(x). \tag{2.3}$$

From equation (2.3), one can obtain the following theorem. For a simple proof, we refer the reader to [19].

Theorem 2.2. *Let F and x_0 be as in Theorem 2.1. Then it follows that*

$$\int_B F(x + x_0)d\nu(x) = \exp\Big[-\frac{1}{2}|x_0|^2\Big] \int_B F(x) \exp\big[(x_0, x)^\sim\big] d\nu(x). \tag{2.4}$$

We finish this section by stating the definition of the first variation associated with bounded operators on B. This definition comes from the definition of the first variation studied in [2, 5].

Definition 2.3. Let F be a measurable functional on B and let $w \in B$. Then given two bounded operators A_1 and A_2 in $\mathcal{L}(B)$,

$$\delta_{A_1,A_2}F(x|w) = \frac{\partial}{\partial\mu}F(A_1x + \mu A_2 w)\bigg|_{\mu=0} \tag{2.5}$$

(if it exists) is called the first variation of F associated with the operators A_1 and A_2.

Remark 2.4. (i) Setting $A_1 = A_2 \equiv I$ (the identity operator) on B, our definition of the first variation reduces to the first variation studied in [2, 5, 23, 24]. That is,

$$\delta_{I,I}F(x|w) = \delta F(x|w).$$

In this case $\delta_{I,I}F(x|w)$ acts like a directional derivative of F in the direction of w.

(ii) Given any three operators A_1, A_2 and A_3 in $\mathcal{L}(B)$ and $\theta \in B^*$, one can observe that

$$\delta_{A_1,A_2A_3}F(x|\theta) = \delta_{A_1,A_2}F(x|A_3\theta).$$

3. Preliminary results: Translation theorems associated with bounded operators

The translation theorem was initiated by Cameron and Martin [4]. Notice in equations (2.3) and (2.4) that the Radon–Nikodym derivatives involve two contributions. Of these, the stochastic inner product $(x_0, x)^\sim$ exactly corresponds to the Paley–Wiener–Zygmund stochastic integral [21,22] in the original Cameron–Martin theorems, see [4, Equation (1.3)] and [3, Equation (1.2)], and the term $-\frac{1}{2}|x_0|^2$ is the direct analog of the corresponding term in the original theorems. Therefore, it is necessary to develop a better understanding of those behavior. In this section, we develop the translation theorem to the abstract Wiener integral associated with bounded operators.

Theorem 3.1. *Let A_1 and A_2 be bounded operators in $\mathcal{L}(B)$ and let $\theta \in B^*$. Let F be a functional on B such that $F(A_1x)$ is ν-integrable over B. Then it follows that*

$$\begin{aligned}&\int_B F(A_1x)d\nu(x)\\ &= \exp\Big[-\frac{1}{2}|A_2^*\theta|^2\Big]\int_B F(A_1x + A_1A_2^*\theta)\exp\big[-(\theta, A_2x)\big]d\nu(x).\end{aligned} \tag{3.1}$$

Proof. Letting $G(x) = F(A_1x)$ and using equation (2.3) with F and x_0 replaced with G and $A_2^*\theta$, it follows that

$$\begin{aligned}
&\int_B F(A_1x)d\nu(x)\\
&=\int_B G(x)d\nu(x)\\
&=\exp\Big[-\frac{1}{2}|A_2^*\theta|^2\Big]\int_B G(x+A_2^*\theta)\exp\big[-(A_2^*\theta,x)\big]d\nu(x)\\
&=\exp\Big[-\frac{1}{2}|A_2^*\theta|^2\Big]\int_B F(A_1x+A_1A_2^*\theta)\exp\big[-(\theta,A_2x)\big]d\nu(x)
\end{aligned}$$

as desired. □

Remark 3.2. In the proof of [19, Lemma 1.4], equation (2.4) was derived by equation (2.3). One can also verify equation (2.3) by use of (2.4). But equation (3.2) below can not be derived by equation (3.1) with the techniques as those used in the proof of [19, Lemma 1.4].

Theorem 3.3. *Let A_1, A_2, θ, and F be as in Theorem 3.1. Then it follows that*

$$\begin{aligned}
&\int_B F(A_1x+A_1A_2^*\theta)d\nu(x)\\
&=\exp\Big[-\frac{1}{2}|A_2^*\theta|^2\Big]\int_B F(A_1x)\exp\big[(\theta,A_2x)\big]d\nu(x).
\end{aligned}\tag{3.2}$$

Proof. Letting $G(x) = F(A_1x)$ and using equation (2.4) with F and x_0 replaced with G and $A_2^*\theta$, it follows that

$$\begin{aligned}
&\int_B F(A_1x+A_1A_2^*\theta)d\nu(x)\\
&=\int_B G(x+A_2^*\theta)d\nu(x)\\
&=\exp\Big[-\frac{1}{2}|A_2^*\theta|^2\Big]\int_B G(x)\exp\big[(A_2^*\theta,x)\big]d\nu(x)\\
&=\exp\Big[-\frac{1}{2}|A_2^*\theta|^2\Big]\int_B F(A_1x)\exp\big[(\theta,A_2x)\big]d\nu(x)
\end{aligned}$$

as desired. □

4. Integration by parts formula

In this section, we provide an integration by parts formula for the Wiener integral associated with bounded operators on abstract Wiener spaces.

Theorem 4.1. *Let A_1, A_2, θ, and F be as in Theorem 3.1. Furthermore assume that*

$$\int_B \big|\delta_{A_1,A_1A_2^*}F(x|\theta)\big|d\nu(x) < +\infty. \tag{4.1}$$

Then it follows that

$$\int_B \delta_{A_1,A_1A_2^*}F(x|\theta)d\nu(x) = \int_B (\theta, A_2x)F(A_1x)d\nu(x). \tag{4.2}$$

Proof. Using (2.5) and (3.2), it follows that

$$\begin{aligned}
&\int_B \delta_{A_1,A_1A_2^*}F(x|\theta)d\nu(x)\\
&= \int_B \frac{\partial}{\partial\mu}F(A_1x+\mu A_1A_2^*\theta)\bigg|_{\mu=0} d\nu(x)\\
&=\frac{\partial}{\partial\mu}\bigg(\int_B F(A_1x+\mu A_1A_2^*\theta)d\nu(x)\bigg)\bigg|_{\mu=0}\\
&=\frac{\partial}{\partial\mu}\bigg(\exp\bigg[-\frac{\mu^2}{2}|A_2^*\theta|^2\bigg]\int_B F(A_1x)\exp\big[\mu(\theta,A_2x)\big]d\nu(x)\bigg)\bigg|_{\mu=0}\\
&= \int_B (\theta, A_2x)F(A_1x)d\nu(x).
\end{aligned} \tag{4.3}$$

The second equality of (4.3) follows from (4.1) and Theorem 2.27 in [7]. □

Remark 4.2. Other study of integration by part formulas for various kind of functionals on abstract Wiener spaces can be found in [16, 17].

5. Evaluation formulas for the Wiener integral of monomials in terms of natural dual pairings

In this section, as suggested in Section 1, we will provide evaluation formulas for the Winer integral of monomials in terms of the natural dual pairings, given by

$$F(x) = \prod_{j=1}^m (A_j^*g, x), \quad x \in B, \tag{5.1}$$

where for each $j \in \{1, \dots, m\}$, A_j^* is the Banach space dual operator of the bounded operator A_j in $\mathcal{L}(B)$.

When we evaluate the Wiener integral

$$\int_B \prod_{j=1}^m (A_j^*g, x)d\nu(x), \tag{5.2}$$

we might not be able to use the change of variables theorem of the usual measure theory, because the set of Gaussian random variables (A_j^*g, x), $j \in \{1, \dots, m\}$, is generally not independent.

Throughout the remainder of this paper, we will present interesting formulas to calculate the Wiener integral of functionals F given by (5.1). Using equation (4.2), we indeed see that the Wiener integral of functionals having the form (5.1) can be calculated very explicitly. We now show that the integration by parts formula (namely, equation (4.2)) can be used to calculate the Wiener integral (5.2).

Example 5.1. Let A_1 and A_2 be operators in $\mathcal{L}(B)$. Given a non-zero element g in B^*, set $F(x) = (g, x)$. Then using equation (2.5), it follows that for any θ in B^*,

$$\begin{aligned}\delta_{A_1,A_1A_2^*}F(x|\theta) &= \frac{\partial}{\partial\mu}\big\{(g, A_1x) + \mu(g, A_1A_2^*\theta)\big\}\bigg|_{\mu=0}\\ &= (g, A_1A_2^*\theta)\\ &= (A_1^*g, A_2^*\theta).\end{aligned}$$

In particular, we obtain that

$$\delta_{A_1,A_1A_2^*}F(x|g) = (A_1^*g, A_2^*g). \tag{5.3}$$

Next applying equation (4.2) with $F(x) = (g, x)$ and using equation (5.3), we obtain the formula

$$\begin{aligned}\int_B (g, A_1x)(g, A_2x)d\nu(x) &= \int_B (g, A_2x)F(A_1x)d\nu(x)\\ &= \int_B \delta_{A_1,A_1A_2^*}F(x|\theta)d\nu(x) = (A_1^*g, A_2^*g).\end{aligned} \tag{5.4}$$

Remark 5.2. Frankly speaking, calculating the covariance of the two random variables (A_1^*g, x) and (A_2^*g, x) (or, calculating the variance of the random variable $(A_1^*g+A_2^*g, x)$, merely), one can obtain the Wiener integration formula (5.4). But to calculate the Wiener integral

$$\int_B \prod_{j=1}^{m}(A_j^*g, x)d\nu(x) = \int_B \prod_{j=1}^{m}(g, A_jx)d\nu(x) \text{ with } m \geq 3,$$

we may apply the Gram–Schmidt process to the subset $\{A_1^*g, \ldots, A_n^*g\}$ of B^* and use a well-known Wiener integration theorem (see, [19, Equation (1.3)]), if the set $\{A_1^*g, \ldots, A_n^*g\}$ is not orthogonal in H.

In our next example, for any positive integer $m \in \{3, 4, \ldots\}$, we obtain a recurrence relation for the Wiener integral of the monomials $\prod_{j=1}^{m}(A_j^*g, x)$ of the natural dual pairings. To do this, we just apply equation (4.2).

Example 5.3. Let $m \geq 3$ be a positive integer and let g be a non-zero element in B^*. Let $\{A_1, \ldots, A_{m-1}, A_m\}$ be a finite sequence of operators in $\mathcal{L}(B)$. Set

$$F(x) = \prod_{j=1}^{m-1}(A_j^*g, x) = \prod_{j=1}^{m-1}(g, A_jx).$$

First, using equation (2.5), it follows that for all $\theta \in B^*$,

$$\begin{aligned}
\delta_{I,A_m^*}F(x|\theta) &= \frac{\partial}{\partial\mu}\prod_{j=1}^{m-1}\big\{(A_j^*g,x)+\mu(A_j^*g,A_m^*\theta)\big\}\Big|_{\mu=0} \\
&= \sum_{l=1}^{m-1}\prod_{\substack{j=1\\ j\neq l}}^{m-1}(A_j^*g,x)(A_l^*g,A_m^*\theta) \\
&= \sum_{l=1}^{m-1}(A_l^*g,A_m^*\theta)\prod_{\substack{j=1\\ j\neq l}}^{m-1}(A_j^*g,x),
\end{aligned} \tag{5.5}$$

where I denotes the identity operator on B. Then in particular, replacing θ with g in (5.5), it follows that

$$\delta_{I,A_m^*}F(x|g) = \sum_{l=1}^{m-1}(A_l^*g,A_m^*g)\prod_{\substack{j=1\\ j\neq l}}^{m-1}(A_j^*g,x). \tag{5.6}$$

Hence, using equation (4.2) with $F(x) = \prod_{j=1}^{m-1}(A_j^*g,x) = \prod_{j=1}^{m-1}(g,A_jx)$ and applying equation (5.6), we obtain the formula

$$\begin{aligned}
w_m &\equiv \int_B (g,A_mx)\prod_{j=1}^{m-1}(g,A_jx)d\nu(x) \\
&= \int_B (g,A_mx)F(Ix)d\nu(x) \\
&= \int_B \delta_{I,IA_m^*}F(x|g)d\nu(x) \\
&= \int_B \delta_{I,A_m^*}F(x|g)d\nu(x) \\
&= \sum_{l=1}^{m-1}(A_l^*g,A_m^*g)\int_B\prod_{\substack{j=1\\ j\neq l}}^{m-1}(A_j^*g,x)d\nu(x).
\end{aligned} \tag{5.7}$$

Letting $m = 3$ in equation (5.7) and applying equation (5.4) allow us to easily and completely evaluate the Wiener integral

$$w_3 \equiv \int_B (A_1^*g,x)(A_2^*g,x)(A_3^*g,x)d\nu(x).$$

Then setting $m = 4$ in equation (5.7) allows us to completely evaluate the Wiener integral

$$w_4 \equiv \int_B (A_1^*g,x)(A_2^*g,x)(A_3^*g,x)(A_4^*g,x)d\nu(x),$$

since we already have complete evaluation formulas for

$$\int_B \prod_{j=1}^{l} (A_j^* g, x) d\nu(x), \quad l = 1, 2 \text{ and } 3.$$

Then we can evaluate

$$w_5 \equiv \int_B \prod_{j=1}^{5} (A_j^* g, x) d\nu(x),$$

since we have already evaluated

$$\int_B \prod_{j=1}^{l} (A_j^* g, x) d\nu(x)$$

for $l = 1, 2, 3$ and 4; etc.

Using those calculations, it follows $w_3 = w_5 = 0$ and

$$w_4 = (A_1^* g, A_4^* g)(A_2^* g, A_3^* g) + (A_2^* g, A_4^* g)(A_1^* g, A_3^* g) + (A_3^* g, A_4^* g)(A_1^* g, A_2^* g).$$

Remark 5.4. (i) Applying equations (5.7) and (5.4) and the linearity of the Wiener integral, we can calculate the Wiener integral of the polynomials P having the form

$$P(x) = \sum_{\mathcal{S} \in \mathfrak{F}} c_{\mathcal{S}} \prod_{A \in \mathcal{S}} (A^* g, x) + c_0, \tag{5.8}$$

where $\mathfrak{F}$ is any finite family of finite sequences $\mathcal{S}$ of $\mathcal{L}(B)$, $c_{\mathcal{S}} \in \mathbb{C}$ for each $\mathcal{S} \in \mathfrak{F}$, and $c_0 \in \mathbb{C}$. The polynomials P having the form (5.8) are in the class $\mathcal{P}(B^*)$, see Section 1 above.

(ii) Using the recursive formula (5.7) and a tedious calculation, we conclude that

$$\int_B \prod_{j=1}^{m} (A_j^* g, x) d\nu(x) = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \sum \prod_k (A_{i_k}^* g, A_{j_k}^* g), & \text{if } m \text{ is even,} \end{cases}$$

where the sum is over all partitions of $\{1, 2, \ldots, m\}$ into disjoint pairs $\{i_k, j_k\}$. This result subsumes Wick's theorem (see [11, Theorem 1.28]) in quantum field theory.

Acknowledgment. The authors would like to express their gratitude to the editor and the referees for their valuable comments and suggestions which have improved the original paper.

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