Bull. Korean Math. Soc. **59** (2022), No. 5, pp. 1215–1235
https://doi.org/10.4134/BKMS.b210688
pISSN: 1015-8634 / eISSN: 2234-3016

SZEGÖ PROJECTIONS FOR HARDY SPACES IN QUATERNIONIC CLIFFORD ANALYSIS

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Abstract. In this paper we study Szegö kernel projections for Hardy spaces in quaternionic Clifford analysis. At first we introduce the matrix Szegö projection operator for the Hardy space of quaternionic Hermitean monogenic functions by the characterization of the matrix Hilbert transform in the quaternionic Clifford analysis. Then we establish the Kerzman-Stein formula which closely connects the matrix Szegö projection operator with the Hardy projection operator onto the Hardy space, and we get the matrix Szegö projection operator in terms of the Hardy projection operator and its adjoint. At last, we construct the explicit matrix Szegö kernel function for the Hardy space on the sphere as an example, and get the solution to a Diriclet boundary value problem for matrix functions.

1. Introduction

The study of the Szegö kernel and the Szegö projection is a classical subject in several complex variables, which were first introduced in [19] and have significant importance in the development of complex analysis.

The Szegö kernel was expressed in terms of the Cauchy-Fantappiè kernels for planar domains and smooth bounded strictly pseudoconvex domains in $\mathbb{C}^n$ by Kerzman and Stein in [14, 15]. It reveals the properties of the holomorphic map between two domains, and the conformal mappings onto the canonical domains, the classical functions and other important objects of potential theory can be simply expressed in virtue of the Szegö kernels (see Refs. e.g. [6, 7, 14]). The Szegö projection operator associated with smooth boundary of a domain is of fundamental interest in the complex analysis. Since its action can often be expressed as an integration against a distribution, known as the Szegö kernel, it is natural to introduce the space of square integrable function onto Hardy space defined on the boundaries of a domain (see Refs. e.g. [8, 14, 19]). [13] is

Received September 16, 2021; Revised November 10, 2021; Accepted December 6, 2021.
2020 *Mathematics Subject Classification.* Primary 30G35, 15A66, 30C40, 31A25, 31B10.
Key words and phrases. Szegö projections, quaternionic Clifford analysis, Hardy space, matrix function.
This work was financially supported by National Natural Science Foundation of China (11601525, 12071485), Natural Science Foundation of Hunan Province (2020JJ4105).

devoted to studying the matrix Szegö projection operator for the Hardy space of Hermitian monogenic functions defined on a bounded sub-domain of even dimensional Euclidean space, and established the Kerzman-Stein formula which is closely connected with the matrix Szegö projection operator. In [10] D. Constales and R. S. Krausshar have considered half-space domains (semi-infinite in one of the dimensions) and strip domains (finite in one of the dimensions) in real Euclidean spaces of dimension at least 2.

Classical Clifford analysis nowadays has been a well established mathematical subject which is closely related to harmonic analysis but complements on each other. It has gradually developed into a comprehensive theory, which provides a direct, natural and concise generalization for the high-dimensional theory of holomorphic functions on the complex plane [9, 11, 12]. In the simplest but still useful settings, it focuses on the zero solutions of various special partial differential operators naturally generated in Clifford algebraic language, the most important of which is the so-called Dirac operator.

Recently, numbers of papers [1–4,18] further generalized the classical Clifford analysis by considering functions on $\mathbb{R}^{4n}$ which take values in a quaternionic Clifford algebra (called quaternionic Clifford analysis). D. Peña-Peña, I. Sabadini and F. Sommen [18] introduced the quaternionic Witt basis in $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}_{4n}$ and they not only studied the resolutions associated to quaternionic Hermitian systems but also proved a Bochner-Martnelli type formula. In what follows, by following a (4×4) circulant matrix approach the authors in [2] addressd the problem of establishing a quaternionic Hermitian Clifford Cauchy integral formula. Later in [4] R. Abreu Blaya, J. Bory Reyes, F. Brackx, H. De Schepper and F. Sommen studied matrix Cauchy and Hilbert transform in Hermitian quaternionic Clifford analysis. While the boundary value problems for the quaternionic Hermitian system in $\mathbb{R}^{4n}$ was investigated in [3], then the authors in [1] studied the boundary value problems on fractal hypersurfaces for the quaternionic Hermitian system in $\mathbb{R}^{4n}$. In [5] R. Abreu Blaya and L. De la Cruz Toranzo have generalized the classical Hardy decomposition of Hölder continuous functions on the boundary of a domain.

We will consider the Szegö projections for Hardy spaces in quaternionic Clifford analysis. First we will give a well-defined definition of inner product on the space of square integral circulant (4×4) matrix functions which are defined on the boundary of a bounded sub-domain in $4n$ dimensional Euclidean space, and introduce the matrix Szegö projection operator to be orthogonal for the Hardy space of quaternionic Hermitian monogenic functions defined on a bounded sub-domain of $4n$ dimensional Euclidean space. Then we will establish the Kerzman-Stein formula, which is closely related to the matrix Szegö projection operator and the Hardy projection operator onto the Hardy space of quaternionic Hermitian monogenic functions defined on a bounded sub-domain as well as present the matrix Szegö projection operator in terms of the Hardy projection operator and its adjoint, explicitly.

The remaining part of the paper proceeds as follows. In Section 2, we briefly recall some basic facts about quaternionic Hermitian Clifford analysis settings which will be needed in the sequel. Section 3 is devoted to studying the matrix Szegö projection for the Hardy space of quaternionic Hermitian monogenic functions defined on a bounded sub-domain and establish the Kerzman-Stein formula, which is closely related to the matrix Szegö projection operator. In Section 4, we will give the explicit matrix Szegö kernel for the Hardy space and we get the solution to a quaternionic Hermitian Dirichlet problem for matrix functions.

2. Preliminaries

In this section, we introduce some basic facts about quaternionic Hermitian Clifford analysis settings and some related theorems which will be needed in the sequel and more details can be found in [1–4, 18].

Let $(e_1, \ldots, e_{4n})$ be the standard orthogonal basis of the real orthogonal space $\mathbb{R}^{4n}$ which is endowed with the symmetric real-bilinear form $B_{\mathbb{R}}(\cdot,\cdot)$ of signature (0, $4n$), i.e., with $B_{\mathbb{R}}(e_i, e_j) = -\delta_{ij},\ i, j = 1, 2, \ldots, 4n$. The Clifford algebra $\mathbb{R}_{0,4n}$ is constructed over $\mathbb{R}^{4n}$, and its geometric multiplication is governed by the rules

$$e_i e_j + e_j e_i = -2\delta_{ij},\ i, j = 1, \ldots, 4n.$$

A basis for $\mathbb{R}_{0,4n}$ then consists of the elements $e_A = e_{i_1} e_{i_2} \cdots e_{i_k}$, where $A = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, 4n\}$, $i_1 < i_2 < \cdots < i_k$. For $A = \emptyset$, we put $e_\emptyset = 1$, the identity element of $\mathbb{R}_{0,4n}$.

Consider the algebra of quaternions which is often denoted by

$$\mathbb{H} := \{q = q_0 + q_1 i + q_2 j + q_3 k,\ q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where i,j,k satisfy the multiplication table formed by

$$i^2 = j^2 = k^2 = ijk = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j,$$

and it is clearly that $\mathbb{H}$ can be identified with the Clifford algebra $\mathbb{R}_{0,2}$.

The $\mathbb{H}$-conjugation of q is

$$\overline{q} = q_0 - q_1 i - q_2 j - q_3 k,$$

and its three $\mathbb{H}$-involutions are defined by

$$\begin{aligned} q^\alpha &= q_0 + q_1 i - q_2 j - q_3 k, \\ q^\beta &= q_0 - q_1 i + q_2 j - q_3 k, \\ q^\gamma &= q_0 - q_1 i - q_2 j + q_3 k. \end{aligned}$$

In the following study, we should consider the Clifford algebra $\mathbb{H}_{4n} = \mathbb{H} \otimes \mathbb{R}_{0,4n}$ whose element has the form $\lambda = \sum_A e_A \lambda_A, \lambda_A \in \mathbb{H}$.

The quaternionic Hermitian conjugate of $\lambda \in \mathbb{H}_{4n}$ is

$$\lambda^\dagger = \sum_A \overline{e_A}\,\overline{\lambda_A},$$

we may also define the norm of $\mathbb{H}_{4n}$

$$\|\lambda\| = [\lambda\lambda^{\dagger}]_0 = \sum_A |\lambda_A|^2,$$

where $[\cdot]_0$ stands for the scalar part of quaternionic Clifford elements.

Definition 2.1. The quaternionic Witt basis of $\mathbb{H}_{4n} = \mathbb{H} \otimes \mathbb{R}_{0,4n}$ is given by $\left\{f_l, f_l^{\alpha}, f_l^{\beta}, f_l^{\gamma}\right\}$, $l = 1, \ldots, n$, where

$$\begin{aligned}
f_l &= e_{1+4(l-1)} - ie_{2+4(l-1)} - je_{3+4(l-1)} - ke_{4+4(l-1)},\\
f_l^{\alpha} &= e_{1+4(l-1)} - ie_{2+4(l-1)} + je_{3+4(l-1)} + ke_{4+4(l-1)},\\
f_l^{\beta} &= e_{1+4(l-1)} + ie_{2+4(l-1)} - je_{3+4(l-1)} + ke_{4+4(l-1)},\\
f_l^{\gamma} &= e_{1+4(l-1)} + ie_{2+4(l-1)} + je_{3+4(l-1)} - ke_{4+4(l-1)}.
\end{aligned}$$

We define the real Clifford vectors associated to an element $(x_1, x_2, \ldots, x_{4n})$ in $\mathbb{R}^{4n}$ as follows (see Refs. e.g. [1–4, 18])

$$\begin{aligned}
\underline{X} = \underline{X}_0 &= \sum_{l=1}^{n}(e_{4l-3}x_{4l-3} + e_{4l-2}x_{4l-2} + e_{4l-1}x_{4l-1} + e_{4l}x_{4l}),\\
\underline{X}_1 &= \sum_{l=1}^{n}(e_{4l-3}x_{4l-2} - e_{4l-2}x_{4l-3} - e_{4l-1}x_{4l} + e_{4l}x_{4l-1}),\\
\underline{X}_2 &= \sum_{l=1}^{n}(e_{4l-3}x_{4l-1} + e_{4l-2}x_{4l} - e_{4l-1}x_{4l-3} - e_{4l}x_{4l-2}),\\
\underline{X}_3 &= \sum_{l=1}^{n}(e_{4l-3}x_{4l} - e_{4l-2}x_{4l-1} + e_{4l-1}x_{4l-2} - e_{4l}x_{4l-3}).
\end{aligned}$$

By this definition and direct calculation, we can get

$$\underline{X}_0^2 = \underline{X}_1^2 = \underline{X}_2^2 = \underline{X}_3^2 = -|\underline{X}^2|,$$

$$\{\underline{X}_r, \underline{X}_s\} = \underline{X}_r\underline{X}_s + \underline{X}_s\underline{X}_r = 0,\ r, s = 0, 1, 2, 3,\ r \neq s.$$

Then we define the differential operators

$$\begin{aligned}
\partial_{\underline{X}} = \partial_{\underline{X}_0} &= \sum_{l=1}^{n}(e_{4l-3}\partial_{x_{4l-3}} + e_{4l-2}\partial_{x_{4l-2}} + e_{4l-1}\partial_{x_{4l-1}} + e_{4l}\partial_{x_{4l}}),\\
\partial_{\underline{X}_1} &= \sum_{l=1}^{n}(e_{4l-3}\partial_{x_{4l-2}} - e_{4l-2}\partial_{x_{4l-3}} - e_{4l-1}\partial_{x_{4l}} + e_{4l}\partial_{x_{4l-1}}),\\
\partial_{\underline{X}_2} &= \sum_{l=1}^{n}(e_{4l-3}\partial_{x_{4l-1}} + e_{4l-2}\partial_{x_{4l}} - e_{4l-1}\partial_{x_{4l-3}} - e_{4l}\partial_{x_{4l-2}}),\\
\partial_{\underline{X}_3} &= \sum_{l=1}^{n}(e_{4l-3}\partial_{x_{4l}} - e_{4l-2}\partial_{x_{4l-1}} + e_{4l-1}\partial_{x_{4l-2}} - e_{4l}\partial_{x_{4l-3}}).
\end{aligned}$$

From this we have that

$$\partial^2_{\underline{X}_0} = \partial^2_{\underline{X}_1} = \partial^2_{\underline{X}_2} = \partial^3_{\underline{X}_0} = -\Delta_{4n},$$

$$\left\{\partial_{\underline{X}_r}, \partial_{\underline{X}_s}\right\} = \partial_{\underline{X}_r}\partial_{\underline{X}_s} + \partial_{\underline{X}_s}\partial_{\underline{X}_r} = 0,\ r, s = 0, 1, 2, 3,\ r \neq s.$$

We can find that $\partial_{\underline{X}_0}$ corresponds to the usual Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}_1}$, $\partial_{\underline{X}_2}$, $\partial_{\underline{X}_3}$ are consistent with the twisted Dirac operator $\partial_{\underline{X}|}$ in the complex Hermitian setting.

Similar with complex Hermitian Clifford variables, the quaternionic Hermitian variables are

$$\underline{Z} = \underline{Z}_0 = \underline{X}_0 + i\underline{X}_1 + j\underline{X}_2 + k\underline{X}_3 = \sum_{l=1}^{n} f_l(x_{4l-3} + ix_{4l-2} + jx_{4l-1} + kx_{4l}),$$

$$\underline{Z}_1 = \underline{X}_0 + i\underline{X}_1 - j\underline{X}_2 - k\underline{X}_3 = \sum_{l=1}^{n} f_l^{\alpha}(x_{4l-3} + ix_{4l-2} - jx_{4l-1} - kx_{4l}),$$

$$\underline{Z}_2 = \underline{X}_0 - i\underline{X}_1 + j\underline{X}_2 - k\underline{X}_3 = \sum_{l=1}^{n} f_l^{\beta}(x_{4l-3} - ix_{4l-2} + jx_{4l-1} - kx_{4l}),$$

$$\underline{Z}_3 = \underline{X}_0 - i\underline{X}_1 - j\underline{X}_2 + k\underline{X}_3 = \sum_{l=1}^{n} f_l^{\gamma}(x_{4l-3} - ix_{4l-2} - jx_{4l-1} + kx_{4l}).$$

We note that

$$\underline{Z}_0\underline{Z}_0^{\dagger} + \underline{Z}_1\underline{Z}_1^{\dagger} + \underline{Z}_2\underline{Z}_2^{\dagger} + \underline{Z}_3\underline{Z}_3^{\dagger} = \underline{Z}_0^{\dagger}\underline{Z}_0 + \underline{Z}_1^{\dagger}\underline{Z}_1 + \underline{Z}_2^{\dagger}\underline{Z}_2 + \underline{Z}_3^{\dagger}\underline{Z}_3 = 16|\underline{X}|^2.$$

The Hermitian Dirac operators are given as follows:

$$\partial_{\underline{Z}_0} = \frac{1}{16}(\partial_{\underline{X}_0} + i\partial_{\underline{X}_1} + j\partial_{\underline{X}_2} + k\partial_{\underline{X}_3}),$$

$$\partial_{\underline{Z}_1} = \frac{1}{16}(\partial_{\underline{X}_0} + i\partial_{\underline{X}_1} - j\partial_{\underline{X}_2} - k\partial_{\underline{X}_3}),$$

$$\partial_{\underline{Z}_2} = \frac{1}{16}(\partial_{\underline{X}_0} - i\partial_{\underline{X}_1} + j\partial_{\underline{X}_2} - k\partial_{\underline{X}_3}),$$

$$\partial_{\underline{Z}_3} = \frac{1}{16}(\partial_{\underline{X}_0} - i\partial_{\underline{X}_1} - j\partial_{\underline{X}_2} + k\partial_{\underline{X}_3}).$$

We also have that

$$\begin{aligned} &16(\partial_{\underline{Z}_0}\partial^{\dagger}_{\underline{Z}_0} + \partial_{\underline{Z}_1}\partial^{\dagger}_{\underline{Z}_1} + \partial_{\underline{Z}_2}\partial^{\dagger}_{\underline{Z}_2} + \partial_{\underline{Z}_3}\partial^{\dagger}_{\underline{Z}_3}) \\ &= 16(\partial^{\dagger}_{\underline{Z}_0}\partial_{\underline{Z}_0} + \partial^{\dagger}_{\underline{Z}_1}\partial_{\underline{Z}_1} + \partial^{\dagger}_{\underline{Z}_2}\partial_{\underline{Z}_2} + \partial^{\dagger}_{\underline{Z}_3}\partial_{\underline{Z}_3}) \\ &= \Delta^2_{4n}. \end{aligned}$$

Now we can define the quaternionic Hermitian monogenic functions (see Refs. e.g. [1–4, 18]).

Definition 2.2. Let Ω be an open set in $\mathbb{R}^{4n}$. We say a continuously differentiable function $f : \Omega \mapsto \mathbb{H}_{4n}$ is quaternionic Hermitian monogenic (or for short q-Hermitian monogenic) in Ω if and only if it satisfies

$$\partial_{\underline{Z}_0} f = \partial_{\underline{Z}_1} f = \partial_{\underline{Z}_2} f = \partial_{\underline{Z}_3} f = 0,$$

or equivalently,

$$\partial_{\underline{X}_0} f = \partial_{\underline{X}_1} f = \partial_{\underline{X}_2} f = \partial_{\underline{X}_3} f = 0.$$

The fundamental solutions of the Dirac operators $\partial_{\underline{X}_r}$, $r = 0, 1, 2, 3$, are

$$E_r(\underline{X}) = \frac{1}{\omega_{4n}} \frac{\overline{\underline{X}_r}}{|\underline{X}_r|^{4n}}, \quad r = 0, 1, 2, 3, \quad \underline{X} \in \mathbb{R}^{4n} \setminus \{0\},$$

where ω_{4n} denotes the surface area of the unit sphere in $\mathbb{R}^{4n}$.

We introduce the following Hermitian Cauchy kernels:

$$\begin{aligned}
\mathcal{E}_0 &= E_0 - iE_1 - jE_2 - kE_3, \\
\mathcal{E}_1 &= E_0 - iE_1 + jE_2 + kE_3, \\
\mathcal{E}_2 &= E_0 + iE_1 - jE_2 + kE_3, \\
\mathcal{E}_3 &= E_0 + iE_1 + jE_2 - kE_3.
\end{aligned}$$

They can also be written as

$$\mathcal{E}_r = \frac{1}{\omega_{4n}} \frac{\underline{Z}_r^{\dagger}}{|\underline{Z}|^{4n}}, \quad r = 0, 1, 2, 3.$$

Moreover, we should consider the functions which are defined on an open subset Ω of $\mathbb{R}^{4n}$ and take values in the Clifford algebra $\mathbb{H}_{4n}$. They have the form $f = \Sigma_A f_A e_A$, where the functions f_A are $\mathbb{H}$-valued. We introduce the corresponding circulant (4×4) matrix function

$$\boldsymbol{G} = \begin{pmatrix} g_0 & g_3 & g_2 & g_1 \\ g_1 & g_0 & g_3 & g_2 \\ g_2 & g_1 & g_0 & g_3 \\ g_3 & g_2 & g_1 & g_0 \end{pmatrix} = \operatorname{circ} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix},$$

and we say the circulant (4×4) matrix function $\boldsymbol{G} \in \boldsymbol{C}^k(\Omega, \mathbb{H}_{4n})$, $\boldsymbol{H}^{\mu}(\Omega, \mathbb{H}_{4n})$, $\boldsymbol{L}_p(\Omega, \mathbb{H}_{4n})$ which means each entry of $\boldsymbol{G}$ belongs to $C^k(\Omega, \mathbb{H}_{4n})$, $H^{\mu}(\Omega, \mathbb{H}_{4n})$, $L_p(\Omega, \mathbb{H}_{4n})$.

Similarly we introduce the following circulant (4×4) matrices:

$$\boldsymbol{\mathcal{D}} = \begin{pmatrix} \partial_{\underline{Z}_0} & \partial_{\underline{Z}_3} & \partial_{\underline{Z}_2} & \partial_{\underline{Z}_1} \\ \partial_{\underline{Z}_1} & \partial_{\underline{Z}_0} & \partial_{\underline{Z}_3} & \partial_{\underline{Z}_2} \\ \partial_{\underline{Z}_2} & \partial_{\underline{Z}_1} & \partial_{\underline{Z}_0} & \partial_{\underline{Z}_3} \\ \partial_{\underline{Z}_3} & \partial_{\underline{Z}_2} & \partial_{\underline{Z}_1} & \partial_{\underline{Z}_0} \end{pmatrix} = \operatorname{circ} \begin{pmatrix} \partial_{\underline{Z}_0} \\ \partial_{\underline{Z}_1} \\ \partial_{\underline{Z}_2} \\ \partial_{\underline{Z}_3} \end{pmatrix},$$

$$\boldsymbol{\mathcal{E}} = \begin{pmatrix} \mathcal{E}_0 & \mathcal{E}_3 & \mathcal{E}_2 & \mathcal{E}_1 \\ \mathcal{E}_1 & \mathcal{E}_0 & \mathcal{E}_3 & \mathcal{E}_2 \\ \mathcal{E}_2 & \mathcal{E}_1 & \mathcal{E}_0 & \mathcal{E}_3 \\ \mathcal{E}_3 & \mathcal{E}_2 & \mathcal{E}_1 & \mathcal{E}_0 \end{pmatrix} = \operatorname{circ} \begin{pmatrix} \mathcal{E}_0 \\ \mathcal{E}_1 \\ \mathcal{E}_2 \\ \mathcal{E}_3 \end{pmatrix}$$

and

$$\boldsymbol{\delta} = \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix},$$

where δ is the Dirac distribution in $\mathbb{R}^{4n}$, and we have

$$\boldsymbol{\mathcal{D}}^T \boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}} \boldsymbol{\mathcal{D}}^T = \boldsymbol{\delta},$$

i.e., $\boldsymbol{\mathcal{E}}$ is the fundamental solution of $\boldsymbol{\mathcal{D}}$.

Definition 2.3. $\boldsymbol{G} \in \boldsymbol{C}^1(\Omega, \mathbb{H}_{4n})$ is called (left) $\boldsymbol{Q}$-Hermitian monogenic if and only if it satisfies the system

$$\boldsymbol{\mathcal{D}}^T \boldsymbol{G} = \mathbf{0},$$

where $\mathbf{0}$ denotes the (4×4) matrix with zero entries. We denote the space of $\boldsymbol{Q}$-Hermitian monogenic as $\boldsymbol{\mathcal{M}}(\Omega, \mathbb{H}_{4n})$.

The unit normal vector $\underline{v}(\underline{X}) \equiv \underline{v}_0(\underline{X})$ on $\partial\Omega$ at point $\underline{X} \in \partial\Omega$ is given by

$$\underline{v}_0 = \sum_{l=1}^{n} (e_{4l-3}v_{4l-3} + e_{4l-2}v_{4l-2} + e_{4l-1}v_{4l-1} + e_{4l}v_{4l}),$$

and

$$\underline{v}_1 = \sum_{l=1}^{n} (e_{4l-3}v_{4l-2} - e_{4l-2}v_{4l-3} - e_{4l-1}v_{4l} + e_{4l}v_{4l-1}),$$

$$\underline{v}_2 = \sum_{l=1}^{n} (e_{4l-3}v_{4l-1} + e_{4l-2}v_{4l} - e_{4l-1}v_{4l-3} - e_{4l}v_{4l-2}),$$

$$\underline{v}_3 = \sum_{l=1}^{n} (e_{4l-3}v_{4l} - e_{4l-2}v_{4l-1} + e_{4l-1}v_{4l-2} - e_{4l}v_{4l-3}).$$

Then we can have their Hermitian counterparts:

$$\mathcal{V}_0 = \frac{1}{16}(\underline{v}_0 + i\underline{v}_1 + j\underline{v}_2 + k\underline{v}_3),$$

$$\mathcal{V}_1 = \frac{1}{16}(\underline{v}_0 + i\underline{v}_1 - j\underline{v}_2 - k\underline{v}_3),$$

$$\mathcal{V}_2 = \frac{1}{16}(\underline{v}_0 - i\underline{v}_1 + j\underline{v}_2 - k\underline{v}_3),$$

$$\mathcal{V}_3 = \frac{1}{16}(\underline{v}_0 - i\underline{v}_1 - j\underline{v}_2 + k\underline{v}_3).$$

For any $\boldsymbol{G} \in \boldsymbol{L}_p(\partial\Omega, \mathbb{H}_{4n})$, $1 < p < \infty$, $i = 0, 1, 2, 3$, we define the Cauchy type integrals as

$$\boldsymbol{\mathcal{C}}[\boldsymbol{G}](\underline{Y}) = \int_{\partial\Omega} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}) \boldsymbol{\mathcal{V}}^T \boldsymbol{G}(\underline{X}) dS(\underline{X}), \quad \underline{Y} \notin \partial\Omega,$$

where

$$\boldsymbol{\mathcal{V}} = \text{circ}\begin{pmatrix}\mathcal{V}_0\\ \mathcal{V}_1\\ \mathcal{V}_2\\ \mathcal{V}_3\end{pmatrix}$$

and

$$\mathcal{C}_{rs}[g](\underline{Y}) = 2\int_{\partial\Omega} E_r(\underline{X}-\underline{Y})\underline{v}_s g(\underline{X})dS(\underline{X}), \quad \underline{Y}\notin\partial\Omega,\ r,s=0,1,2,3.$$

Therefore, it can also be written as

$$\boldsymbol{\mathcal{C}}[\boldsymbol{G}] = \frac{1}{4}\text{circ}\begin{pmatrix}\mathcal{C}_{00}+\mathcal{C}_{11}+\mathcal{C}_{22}+\mathcal{C}_{33}\\ \mathcal{C}_{00}-\mathcal{C}_{22}+j(\mathcal{C}_{13}+\mathcal{C}_{31})\\ \mathcal{C}_{00}-\mathcal{C}_{11}+\mathcal{C}_{22}-\mathcal{C}_{33}\\ \mathcal{C}_{00}-\mathcal{C}_{22}-j(\mathcal{C}_{13}+\mathcal{C}_{31})\end{pmatrix}[\boldsymbol{G}].$$

3. Matrix Sezgö projection

In this section, we will mainly study the matrix Sezgö projection operator for the Hardy space of quaternionic Hermitian monogenic functions which is defined on a bounded sub-domain, we also establish the Kerzman-Stein formula, and give the matrix Sezgö projection operator in terms of the Hardy projection operator and its adjoint.

First we recall the inner product $\langle\cdot,\cdot\rangle_{L_2}$ on $L_2(\partial\Omega,\mathbb{H}_{4n})$ which is defined as follows:

$$\langle f_1,f_2\rangle = \left[\int_{\partial\Omega} f_1^{\dagger}(\underline{X})f_2(\underline{X})dS_{\underline{X}}\right]_0,\ \forall f_1,f_2\in L_2(\partial\Omega,\mathbb{H}_{4n}),$$

where $[\cdot]_0$ denotes its scalar part in $\mathbb{H}_{2n}$. Analogously we introduce the following bi-linear form in the space $\boldsymbol{L}_2(\partial\Omega,\mathbb{H}_{4n})$

$$\langle\cdot,\cdot\rangle_{\boldsymbol{L}_2} : \boldsymbol{L}_2\times\boldsymbol{L}_2\to\mathbb{H},$$

$$\langle\boldsymbol{F},\boldsymbol{G}\rangle_{\boldsymbol{L}_2}\mapsto\langle f_0,g_0\rangle_{L_2}+\langle f_1,g_1\rangle_{L_2}+\langle f_2,g_2\rangle_{L_2}+\langle f_3,g_3\rangle_{L_2}.$$

What's more, for any $\boldsymbol{F},\boldsymbol{G},\boldsymbol{H}\in\boldsymbol{L}_2(\partial\Omega,\mathbb{H}_{4n})$ and $\lambda\in\mathbb{H}$, we have that

(i) $\langle\lambda\boldsymbol{F}+\boldsymbol{G},\boldsymbol{H}\rangle_{\boldsymbol{L}_2}=\lambda\langle\boldsymbol{F},\boldsymbol{H}\rangle_{\boldsymbol{L}_2}+\langle\boldsymbol{G},\boldsymbol{H}\rangle_{\boldsymbol{L}_2}$,

(ii) $\langle\boldsymbol{F},\mu\boldsymbol{G}+\boldsymbol{H}\rangle_{\boldsymbol{L}_2}=\langle\boldsymbol{F},\boldsymbol{G}\rangle_{\boldsymbol{L}_2}\overline{\mu}+\langle\boldsymbol{F},\boldsymbol{H}\rangle_{\boldsymbol{L}_2}$,

(iii) $(\langle\boldsymbol{F},\boldsymbol{G}\rangle_{\boldsymbol{L}_2})^{\dagger}=\langle\boldsymbol{G},\boldsymbol{F}\rangle_{\boldsymbol{L}_2}$,

(iv) $\langle\boldsymbol{G},\boldsymbol{G}\rangle_{\boldsymbol{L}_2}\geq 0$ and $\langle\boldsymbol{G},\boldsymbol{G}\rangle_{\boldsymbol{L}_2}=0$ if and only if $\boldsymbol{G}=0$.

This implies that $\langle\cdot,\cdot\rangle_{\boldsymbol{L}_2}$ is an inner product and its norm is given by

$$\|\boldsymbol{F}\| = \sqrt{\langle f_0,f_0\rangle_{L_2}+\langle f_1,f_1\rangle_{L_2}+\langle f_2,f_2\rangle_{L_2}+\langle f_3,f_3\rangle_{L_2}}.$$

Therefore $(\boldsymbol{L}_2(\partial\Omega),\|\cdot\|)$ is a Hilbert space, which is different from the space of $\mathbf{L}_2(\partial\Omega)$ in Refs. e.g. [13,16,17]. Under this setting, we have the following Theorem without proof, which was also stated in [13,16,17] in the sense of different topology. For convenience without confusion and ambiguity, $(\boldsymbol{L}_2(\partial\Omega), \|\cdot\|)$ still denotes by $\boldsymbol{L}_2(\partial\Omega)$.

Theorem 3.1. *Let Ω be a non-empty open and bounded subset of $\mathbb{R}^{4n}$ with smooth boundary $\partial\Omega$, $\boldsymbol{\mathcal{C}}[\boldsymbol{G}](\underline{X})$ is defined as above. If $\boldsymbol{G}(\underline{X}) \in \boldsymbol{L}_p(\partial\Omega, \mathbb{H}_{4n})$ $(1 < p < \infty)$, then for arbitrary $\underline{Z} \in \partial\Omega$, we have*

(i) $\forall \underline{X} \in \mathbb{R}^{4n} \backslash \partial\Omega, \boldsymbol{\mathcal{D}}^T \boldsymbol{G} = 0$, *i.e., $\boldsymbol{G}$ is quaternionic Hermitian monogenic in $\mathbb{R}^{4n} \backslash \partial\Omega$;*

(ii)
$$(\boldsymbol{\mathcal{C}}[\boldsymbol{G}])^{\pm}(\underline{Z}) \triangleq \lim_{\Omega^{\pm} \ni \underline{X} \to \underline{Z}} (\boldsymbol{\mathcal{C}}[\boldsymbol{G}])^{\pm}(\underline{X}) = (-1)^{\frac{n(n+1)}{2}} \frac{(2i)^{2n}}{2} (\pm \boldsymbol{G}(\underline{Z}) + \boldsymbol{\mathcal{H}}[\boldsymbol{G}](\underline{Z}));$$

(iii) $(\boldsymbol{\mathcal{C}}[\boldsymbol{G}])^{\pm}(\underline{Z}) \in \boldsymbol{L}_p(\partial\Omega, \mathbb{H}_{4n})$,

where the limits is the non-tangential limits, which is the same in the following context, and

$$\boldsymbol{\mathcal{H}}[\boldsymbol{G}] = \frac{1}{4} circ \begin{pmatrix} \mathcal{H}_{00} + \mathcal{H}_{11} + \mathcal{H}_{22} + \mathcal{H}_{33} \\ \mathcal{H}_{00} - \mathcal{H}_{22} + j(\mathcal{H}_{13} + \mathcal{H}_{31}) \\ \mathcal{H}_{00} - \mathcal{H}_{11} + \mathcal{H}_{22} - \mathcal{H}_{33} \\ \mathcal{H}_{00} - \mathcal{H}_{22} - j(\mathcal{H}_{13} + \mathcal{H}_{31}) \end{pmatrix} [\boldsymbol{G}],$$

and

$$\mathcal{H}_{rs}[g](\underline{T}) = 2 \int_{\partial\Omega} E_r(\underline{X} - \underline{T}) \underline{v}_s g(\underline{X}) dS(\underline{X}), \quad \underline{T} \in \partial\Omega, \ r, s = 0, 1, 2, 3$$

are Cauchy principal values.

Next, we consider the Hardy space

$$\mathbf{H}^2(\Omega) = \{\boldsymbol{G} \in \boldsymbol{M}(\Omega, \mathbb{H}_{4n}) \,|\, \boldsymbol{G} \text{ has non-tangential } \boldsymbol{L}_2(\partial\Omega)\text{-boundary values}\}.$$

Associating the definition of the above $\mathbb{H}$-valued inner product on $\boldsymbol{L}_2(\partial\Omega)$, we have the following Lemma which is only stated without proof (see Refs. e.g. [2, 13, 16]).

Lemma 3.1. *Suppose that $\boldsymbol{\mathcal{H}}$ are the same as above. Then we have*

(i) $\boldsymbol{\mathcal{H}}^2 = \boldsymbol{I}$,

(ii) $\boldsymbol{\mathcal{H}}^* = \underline{\boldsymbol{v}} \boldsymbol{\mathcal{H}} \underline{\boldsymbol{v}}$,

(iii) *for arbitrary $\boldsymbol{G} \in \boldsymbol{L}_2(\partial\Omega)$, $\boldsymbol{\mathcal{H}}[\boldsymbol{G}] = \boldsymbol{G}$ if and only if $\boldsymbol{G} \in \mathbf{H}^2(\partial\Omega)$,*

(iv) $\boldsymbol{L}_2(\partial\Omega) = \mathbf{H}^2(\Omega) \oplus \underline{\boldsymbol{v}} \mathbf{H}^2(\partial\Omega)$,

where $\boldsymbol{I}$ denotes the (4×4) identity matrix operator, $\boldsymbol{\mathcal{H}}^*$ is the adjoint operator of $\boldsymbol{\mathcal{H}}$ on $\boldsymbol{L}_2(\partial\Omega)$ and

$$\underline{\boldsymbol{v}} = \frac{1}{4} circ \begin{pmatrix} \underline{v}_0 + \underline{v}_1 + \underline{v}_2 + \underline{v}_3 \\ \underline{v}_0 + \underline{v}_1 - \underline{v}_2 - \underline{v}_3 \\ \underline{v}_0 - \underline{v}_1 + \underline{v}_2 - \underline{v}_3 \\ \underline{v}_0 - \underline{v}_1 - \underline{v}_2 + \underline{v}_3 \end{pmatrix}.$$

Now we define the matrix orthogonal projection operator $\boldsymbol{\mathcal{S}}$ from $\boldsymbol{L}_2(\partial\Omega)$ onto $\mathsf{H}^2(\Omega)$, also called the matrix Szegö projection operator, it can be $\boldsymbol{Q}$-Hermitian monogenically extended to $\mathsf{H}^2(\Omega)$

$$\boldsymbol{\mathcal{S}}[\boldsymbol{G}](\underline{X}) = \int_{\partial\Omega} \underline{\boldsymbol{\mathcal{S}}_{X}}(\underline{Y})\boldsymbol{G}(\underline{Y})dS_{\underline{Y}},$$

where $\underline{\boldsymbol{\mathcal{S}}_{X}}(\underline{Y})$ is so-called the matrix Szegö kernel and

$$\boldsymbol{\mathcal{S}}[\boldsymbol{G}](\underline{X}) = \boldsymbol{G} \quad \text{for arbitrary } \underline{\mathrm{X}} \in \Omega.$$

Particularly, when $\Omega = B(1)$ stands for the unit ball in $\mathbb{R}^{4n}$, $\partial\Omega = S^{4n-1}$ is the unit sphere of $\mathbb{R}^{4n}$ and $\underline{v}(\underline{W}) = \underline{W}$ for arbitrary $\underline{W} \in S^{4n-1}$, then

$$\boldsymbol{L}_2(S^{4n-1}) = \mathsf{H}^2(S^{4n-1}) \oplus \underline{\boldsymbol{v}}|_{S^{4n-1}}\mathsf{H}^2(S^{4n-1}),$$

where

$$\underline{\boldsymbol{v}}|_{S^{4n-1}} = \frac{1}{4}\text{circ}\begin{pmatrix} \underline{W}_0 + \underline{W}_1 + \underline{W}_2 + \underline{W}_3 \\ \underline{W}_0 + \underline{W}_1 - \underline{W}_2 - \underline{W}_3 \\ \underline{W}_0 - \underline{W}_1 + \underline{W}_2 - \underline{W}_3 \\ \underline{W}_0 - \underline{W}_1 - \underline{W}_2 + \underline{W}_3 \end{pmatrix}.$$

Now we introduce the matrix Kerzman operator on $\mathbb{L}_2(\partial\Omega)$ by

$$\boldsymbol{\mathcal{A}}[\boldsymbol{G}] = \frac{1}{4}\text{circ}\begin{pmatrix} \mathcal{A}_{00} + \mathcal{A}_{11} + \mathcal{A}_{22} + \mathcal{A}_{33} \\ \mathcal{A}_{00} - \mathcal{A}_{22} + j(\mathcal{A}_{13} + \mathcal{A}_{31}) \\ \mathcal{A}_{00} - \mathcal{A}_{11} + \mathcal{A}_{22} - \mathcal{A}_{33} \\ \mathcal{A}_{00} - \mathcal{A}_{22} - j(\mathcal{A}_{13} + \mathcal{A}_{31}) \end{pmatrix}[\boldsymbol{G}],$$

where $\mathcal{A}_{rs} = \mathcal{C}_{rs} - \mathcal{C}^*_{rs}$ $(r, s = 0, 1, 2, 3)$ are both well-defined, and $\mathcal{C}^*_{rs}$ mean the adjoint operators of $\mathcal{C}_{rs}$, where

$$\mathcal{C}^*_{rs} = \frac{1}{2}(1 + \underline{v}_r\mathcal{H}_{rs}\underline{v}_s).$$

Associated with Lemma 3.1, we have the following Lemma.

Lemma 3.2. *One has*

$$\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{C}} - \boldsymbol{\mathcal{C}}^* = \frac{1}{2}(\boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{H}}^*),$$

where $\boldsymbol{\mathcal{H}}^*$ *and* $\boldsymbol{\mathcal{C}}^* = \frac{1}{2}(\boldsymbol{I} + \boldsymbol{\mathcal{H}}^*)$ *mean the adjoint operators of* $\boldsymbol{\mathcal{C}}$ *and* $\boldsymbol{\mathcal{H}}$.

Proof. Since $\mathcal{A}_{rs} = \mathcal{C}_{rs} - \mathcal{C}^*_{rs}$ $(r, s = 0, 1, 2, 3)$, it follows immediately that $\boldsymbol{\mathcal{A}} = \boldsymbol{\mathcal{C}} - \boldsymbol{\mathcal{C}}^*$, also since $\mathcal{C}^*_{rs} = \frac{1}{2}(1 + \underline{v}_r\mathcal{H}_{rs}\underline{v}_s)$, $\boldsymbol{\mathcal{C}}^* = \frac{1}{2}(\boldsymbol{I} + \boldsymbol{\mathcal{H}}^*)$, by direct calculation we can get the desired result. □

Theorem 3.2. $\boldsymbol{\mathcal{S}}(\boldsymbol{I} + \boldsymbol{\mathcal{A}}) = \boldsymbol{\mathcal{C}}$, *where* $\boldsymbol{I}$ *is the* (4×4) *identity matrix operator.*

Proof. Since the matrix operator $\boldsymbol{\mathcal{S}}$ is an orthogonal projection on the Hilbert space $\boldsymbol{L}_2(\partial\Omega)$, we have $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}^*$. And $\boldsymbol{\mathcal{S}}$, $\boldsymbol{\mathcal{C}}$ are orthogonal and skew projection operators from $\boldsymbol{L}_2(\partial\Omega)$ to $\mathsf{H}^2(\partial\Omega)$, then $\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{S}}$ are both operators from

$\boldsymbol{L}_2(\partial\Omega)$ to $\mathbf{H}^2(\partial\Omega)$. What's more, since $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{C}}$ are identical operators, then we can get that:

$$\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}},\ \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}.$$

It follows that

$$\boldsymbol{\mathcal{C}}^*\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{C}}^*\boldsymbol{\mathcal{S}}^* = (\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{C}})^* = \boldsymbol{\mathcal{C}}^*, \quad \boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{C}}^* = \boldsymbol{\mathcal{S}}^*\boldsymbol{\mathcal{C}}^* = (\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{S}})^* = \boldsymbol{\mathcal{S}}^* = \boldsymbol{\mathcal{S}},$$

therefore we have that

$$\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{C}} - \boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{C}}^* = \boldsymbol{\mathcal{C}} - \boldsymbol{\mathcal{S}}.$$

This has completed the proof. □

Theorem 3.3. *Let $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{C}}$ be as the same as above. Then*

$$\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{C}}(\boldsymbol{I} + \boldsymbol{\mathcal{A}})^{-1},$$

where $\boldsymbol{I}$ denotes the identity matrix operator.

Proof. From Lemma 3.1 we have that $\mathcal{A}$ is anti-self conjugate. This implies that the spectra of operator $\mathcal{A}$ are pure imaginary numbers. Therefore, $\boldsymbol{I} + \boldsymbol{\mathcal{A}}$ is invertible. And from Theorem 3.2, we get the desired result. □

4. Szegö Kernel and its application

In this section, we introduce the Szegö Kernel for the Hardy space $\mathbf{H}^2(S^{4n-1})$. First we introduce the functions

$$K_0(\underline{X}_0, \underline{Y}_0) = -\frac{1}{\omega_{4n}}\frac{1+\overline{\underline{X}_0\underline{Y}_0}}{|1+\underline{X}_0\underline{Y}_0|^{4n}}, \quad K_1(\underline{X_1}, \underline{Y_1}) = -\frac{1}{\omega_{4n}}\frac{1+\overline{\underline{X}_1\underline{Y}_1}}{|1+\underline{X}_1\underline{Y}_1|^{4n}},$$

$$K_2(\underline{X_2}, \underline{Y_2}) = -\frac{1}{\omega_{4n}}\frac{1+\overline{\underline{X}_2\underline{Y}_2}}{|1+\underline{X}_2\underline{Y}_2|^{4n}}, \quad K_3(\underline{X_3}, \underline{Y_3}) = -\frac{1}{\omega_{4n}}\frac{1+\overline{\underline{X}_3\underline{Y}_3}}{|1+\underline{X}_3\underline{Y}_3|^{4n}},$$

where $\underline{X} \neq \underline{Y}$ and ω_{4n} denotes the surface area of the unit sphere S^{4n-1} in $\mathbb{R}^{4n}$.

Theorem 4.1. *For arbitrary $\underline{\mathcal{S}}_{\underline{X}}(\underline{Y})$, $\underline{X} \in B(1)$, $\underline{Y} \in S^{4n-1}$, the reproducing Szegö kernel has the expression*

$$\underline{\boldsymbol{\mathcal{S}}}_{\underline{X}}(\underline{Y}) = circ\begin{pmatrix}\mathcal{K}_0\\ \mathcal{K}_1\\ \mathcal{K}_2\\ \mathcal{K}_3\end{pmatrix},$$

where

$$\mathcal{K}_0 = K_0 - iK_1 - jK_2 - kK_3,$$
$$\mathcal{K}_1 = K_0 - iK_1 + jK_2 + kK_3,$$
$$\mathcal{K}_2 = K_0 + iK_1 - jK_2 + kK_3,$$
$$\mathcal{K}_3 = K_0 + iK_1 + jK_2 - kK_3.$$

Proof. For arbitrary $\underline{Y}_i \in S^{4n-1}$, $i = 0, 1, 2, 3$, we have

$$K_0(\underline{X}_0, \underline{Y}_0) = -\frac{1}{\omega_{4n}} \frac{1 + \overline{\underline{X}_0 \underline{Y}_0}}{|1 + \underline{X}_0 \underline{Y}_0|^{4n}} = -\frac{1}{\omega_{4n}} \frac{\overline{\underline{Y}_0} - \overline{\underline{X}_0}}{|\overline{\underline{Y}_0} - \overline{\underline{X}_0}|^{4n}} \underline{Y}_0,$$

$$K_1(\underline{X}_1, \underline{Y}_1) = -\frac{1}{\omega_{4n}} \frac{1 + \overline{\underline{X}_1 \underline{Y}_1}}{|1 + \underline{X}_1 \underline{Y}_1|^{4n}} = -\frac{1}{\omega_{4n}} \frac{\overline{\underline{Y}_1} - \overline{\underline{X}_1}}{|\overline{\underline{Y}_1} - \overline{\underline{X}_1}|^{4n}} \underline{Y}_1,$$

$$K_2(\underline{X}_2, \underline{Y}_2) = -\frac{1}{\omega_{4n}} \frac{1 + \overline{\underline{X}_2 \underline{Y}_2}}{|1 + \underline{X}_2 \underline{Y}_2|^{4n}} = -\frac{1}{\omega_{4n}} \frac{\overline{\underline{Y}_2} - \overline{\underline{X}_2}}{|\overline{\underline{Y}_2} - \overline{\underline{X}_2}|^{4n}} \underline{Y}_2,$$

$$K_3(\underline{X}_3, \underline{Y}_3) = -\frac{1}{\omega_{4n}} \frac{1 + \overline{\underline{X}_3 \underline{Y}_3}}{|1 + \underline{X}_3 \underline{Y}_3|^{4n}} = -\frac{1}{\omega_{4n}} \frac{\overline{\underline{Y}_3} - \overline{\underline{X}_3}}{|\overline{\underline{Y}_3} - \overline{\underline{X}_3}|^{4n}} \underline{Y}_3,$$

we have

$$\operatorname{circ}\begin{pmatrix} \mathcal{K}_0 \\ \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} = \begin{pmatrix} \mathcal{E}_0 & \mathcal{E}_3 & \mathcal{E}_2 & \mathcal{E}_1 \\ \mathcal{E}_1 & \mathcal{E}_0 & \mathcal{E}_3 & \mathcal{E}_2 \\ \mathcal{E}_2 & \mathcal{E}_1 & \mathcal{E}_0 & \mathcal{E}_3 \\ \mathcal{E}_3 & \mathcal{E}_2 & \mathcal{E}_1 & \mathcal{E}_0 \end{pmatrix} \operatorname{circ}\begin{pmatrix} \mathcal{Y}_0 \\ \mathcal{Y}_1 \\ \mathcal{Y}_2 \\ \mathcal{Y}_3 \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{Y}_0 &= \underline{Y}_0 + \underline{Y}_1 + \underline{Y}_2 + \underline{Y}_3, \\ \mathcal{Y}_1 &= \underline{Y}_0 + \underline{Y}_1 - \underline{Y}_2 - \underline{Y}_3, \\ \mathcal{Y}_2 &= \underline{Y}_0 - \underline{Y}_1 + \underline{Y}_2 - \underline{Y}_3, \\ \mathcal{Y}_3 &= \underline{Y}_0 - \underline{Y}_1 - \underline{Y}_2 + \underline{Y}_3. \end{aligned}$$

Hence we have

$$\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{S}}_{\underline{X}}(\underline{Y}) = \mathbf{0}, \underline{X} \in B(1).$$

By the Cauchy formula in [17], we get that for arbitrary $\boldsymbol{G}(\underline{Y}) \in \boldsymbol{L}_2(S^{4n-1})$,

$$\boldsymbol{G}(\underline{X}) = \int_{S^{4n-1}} \boldsymbol{\mathcal{S}}_{\underline{X}}(\underline{Y}) \boldsymbol{G}(\underline{Y}) dS_{\underline{Y}}, \quad \underline{X} \in B(1).$$

□

We now turn our attention towards the Dirichlet problem. In the sequel we denote the open unit ball centered at the origin by $B(1)$ whose closure is $\overline{B}(1)$. We introduce the following functions:

$$\alpha_j(\underline{X}_j) = \frac{1}{2}(1 + i\underline{X}_j), \quad j = 0, 1, 2, 3,$$

$$\beta_j(\underline{X}_j) = \frac{1}{2}(1 - i\underline{X}_j), \quad j = 0, 1, 2, 3.$$

By direct caculation we have the following Lemma.

Lemma 4.1. *Let $\alpha_j(\underline{X}_j)$ and $\beta_j(\underline{X}_j)$ be the same as above. Then*

(i) $\alpha_j(\underline{X}_j)\beta_j(\underline{X}_j) = \frac{1-|\underline{X}|^2}{4}$,
(ii) $\alpha_j^\dagger(\underline{X}_j) = \alpha_j(\underline{X}_j)$, $\beta_j^\dagger(\underline{X}_j) = \beta_j(\underline{X}_j)$,
(iii) $\alpha_j(\underline{X}_j) + \beta_j(\underline{X}_j) = 1$, $j = 0, 1, 2, 3$,
(iv) *If* $\underline{X}|_{S^{4n-1}} = \underline{W}$, *then*

$$\alpha_j^2(\underline{W}) = \alpha_j(\underline{W}), \quad \beta_j^2(\underline{W}) = \beta_j(\underline{W}).$$

Then we introduce the matrix functions

$$\underline{\boldsymbol{\alpha}} = \frac{1}{4}\text{circ}\begin{pmatrix} \underline{\alpha}_0 + \underline{\alpha}_1 + \underline{\alpha}_2 + \underline{\alpha}_3 \\ \underline{\alpha}_0 + \underline{\alpha}_1 - \underline{\alpha}_2 - \underline{\alpha}_3 \\ \underline{\alpha}_0 - \underline{\alpha}_1 + \underline{\alpha}_2 - \underline{\alpha}_3 \\ \underline{\alpha}_0 - \underline{\alpha}_1 - \underline{\alpha}_2 + \underline{\alpha}_3 \end{pmatrix},$$

$$\underline{\boldsymbol{\beta}} = \frac{1}{4}\text{circ}\begin{pmatrix} \underline{\beta}_0 + \underline{\beta}_1 + \underline{\beta}_2 + \underline{\beta}_3 \\ \underline{\beta}_0 + \underline{\beta}_1 - \underline{\beta}_2 - \underline{\beta}_3 \\ \underline{\beta}_0 - \underline{\beta}_1 + \underline{\beta}_2 - \underline{\beta}_3 \\ \underline{\beta}_0 - \underline{\beta}_1 - \underline{\beta}_2 + \underline{\beta}_3 \end{pmatrix}.$$

By Lemma 4.1 we can get the following Lemma.

Lemma 4.2. *Suppose the matrix functions $\underline{\boldsymbol{\alpha}}$ and $\underline{\boldsymbol{\beta}}$ are the same as above. Then*

(i) $\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}} = circ\begin{pmatrix} \frac{2-|\underline{X}|^2}{8} \\ 0 \\ \frac{-|\underline{X}|^2}{8} \\ 0 \end{pmatrix}$ *and* $\underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}} = \frac{1-|\underline{X}|^2}{4}\boldsymbol{E}$ *if and only if* $\underline{\alpha}_1 = \underline{\alpha}_3$,

(ii) $\underline{\boldsymbol{\alpha}} = \underline{\boldsymbol{\alpha}}^\dagger$, $\underline{\boldsymbol{\beta}} = \underline{\boldsymbol{\beta}}^\dagger$,

(iii) $\underline{\boldsymbol{\alpha}} + \underline{\boldsymbol{\beta}} = \boldsymbol{E}$,

(iv) $\underline{\boldsymbol{\alpha}}^2 = \underline{\boldsymbol{\alpha}} + \frac{1}{4}circ\begin{pmatrix} -2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$, $\underline{\boldsymbol{\beta}}^2 = \underline{\boldsymbol{\beta}} + \frac{1}{4}circ\begin{pmatrix} -2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$, *where $\boldsymbol{E}$ denotes the* (4×4) *identity matrix.*

Particularly when $\underline{X}|_{S^{4n-1}} = \underline{W}$ and $\underline{\alpha}_1 = \underline{\alpha}_3$, then we have

$$\underline{\boldsymbol{\alpha}}^2 = \underline{\boldsymbol{\alpha}}, \quad \underline{\boldsymbol{\beta}}^2 = \underline{\boldsymbol{\beta}} \text{ and } \underline{\boldsymbol{\alpha}}\underline{\boldsymbol{\beta}} = \boldsymbol{0}.$$

In the following study, we only consider the case $\underline{X}|_{S^{4n-1}} = \underline{W}$ and $\underline{\alpha}_1 = \underline{\alpha}_3$. The half Dirichlet problems with respect to the matrix functions $\underline{\boldsymbol{\alpha}}$ and $\underline{\boldsymbol{\beta}}$ are formulated as follows:

For the given boundary data $\boldsymbol{G} \in \boldsymbol{L}_p(S^{4n-1}, \mathbb{H}_{4n})$, find the function $\boldsymbol{F}$ such that

$$\text{(1)} \qquad \begin{cases} \boldsymbol{\mathcal{D}}^T\boldsymbol{F}(\underline{X}) = \boldsymbol{0}, & \underline{X} \in B(1), \\ \underline{\boldsymbol{\alpha}}\boldsymbol{F}(\underline{W}) = \underline{\boldsymbol{\alpha}}\boldsymbol{G}(\underline{W}), & \underline{W} \in S^{4n-1}, \end{cases}$$

$$\text{(2)} \qquad \begin{cases} \boldsymbol{\mathcal{D}}^T\boldsymbol{F}(\underline{X}) = \boldsymbol{0}, & \underline{X} \in B(1), \\ \underline{\boldsymbol{\beta}}\boldsymbol{F}(\underline{W}) = \underline{\boldsymbol{\beta}}\boldsymbol{G}(\underline{W}), & \underline{W} \in S^{4n-1}, \end{cases}$$

where the matrix function $\boldsymbol{F} = \text{circ}\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{pmatrix}$ is defined similarly to $\boldsymbol{G}$.

Theorem 4.2 ([3])**.** *The Dirichlet problem*

$$\begin{cases} \boldsymbol{\mathcal{D}}^{\boldsymbol{T}}\boldsymbol{F}(\underline{X}) = \boldsymbol{0}, \\ \boldsymbol{F}(\underline{W}) = \boldsymbol{G}(\underline{W}) \end{cases}$$

has a solution if and only $\boldsymbol{\mathcal{H}}[\boldsymbol{G}] = \boldsymbol{G}$.

Theorem 4.3. *For the above two half Dirichlet problems* (1) *and* (2), *there exist the unique solutions and their solutions are given respectively by*

$$\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\alpha}}} \triangleq \boldsymbol{\mathcal{C}}[2\underline{\boldsymbol{\alpha}}\boldsymbol{G}](\underline{Y}) = \int_{S^{4n-1}} \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V})\boldsymbol{\mathcal{V}}^T 2\underline{\boldsymbol{\alpha}}\widetilde{\boldsymbol{G}}(\underline{X})dS(\underline{X}), \quad \underline{X} \in \overline{B}(1),$$

$$\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\beta}}} \triangleq \boldsymbol{\mathcal{C}}[2\underline{\boldsymbol{\beta}}\boldsymbol{G}](\underline{Y}) = \int_{S^{4n-1}} \boldsymbol{\mathcal{E}}(\underline{Z}-\underline{V})\boldsymbol{\mathcal{V}}^T 2\underline{\boldsymbol{\beta}}\widetilde{\boldsymbol{G}}(\underline{X})dS(\underline{X}), \quad \underline{X} \in \overline{B}(1),$$

where $\widetilde{\boldsymbol{G}} = (-1)^{\frac{n(n+1)}{2}}\frac{(2i)^{-2n}}{2}\boldsymbol{G}$.

Proof. When $\underline{X} \in B(1)$, from (i) in Theorem 3.1, we have $\boldsymbol{\mathcal{D}}^T\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\alpha}}} = 0$, $\mathcal{D}^T\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\beta}}} = 0$, hence it suffices to consider (1).

For arbitrary $\underline{W} \in S^{4n-1}$, put $\underline{X} = r\underline{W}$, $\underline{W} \in S^{4n-1}$ $(0 < r < 1)$. Let

$$\boldsymbol{F}(\underline{W}) = \lim_{r\to 1^-} \boldsymbol{F}(r\underline{W})_{\underline{\boldsymbol{\alpha}}},\ \underline{W} \in S^{4n-1}.$$

By (ii) in Theorem 3.1, we get

$$\boldsymbol{F}(\underline{W}) = \underline{\boldsymbol{\alpha}}\boldsymbol{G}(\underline{W}) + \boldsymbol{\mathcal{H}}[\underline{\boldsymbol{\alpha}}\boldsymbol{G}](\underline{W}).$$

Then from Lemma 4.1 we have that

$$\lim_{r\to 1^-} \underline{\boldsymbol{\alpha}}\boldsymbol{F}(r\underline{W})_{\underline{\boldsymbol{\alpha}}} = \underline{\boldsymbol{\alpha}}^2\boldsymbol{G}(\underline{W}) + \underline{\boldsymbol{\alpha}}\boldsymbol{\mathcal{H}}[\underline{\boldsymbol{\alpha}}\boldsymbol{G}](\underline{W}) = \underline{\boldsymbol{\alpha}}\boldsymbol{G}(\underline{W}) + \underline{\boldsymbol{\alpha}}\boldsymbol{\mathcal{H}}[\underline{\boldsymbol{\alpha}}\boldsymbol{G}](\underline{W}),$$

where

$$\underline{\boldsymbol{\alpha}}\boldsymbol{\mathcal{H}}[\underline{\boldsymbol{\alpha}}\boldsymbol{G}](\underline{W}) = \frac{1}{16}\text{circ}\begin{pmatrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_3\\ \mathcal{H}_4\end{pmatrix} \times \text{circ}\begin{pmatrix}g_0\\ g_1\\ g_2\\ g_3\end{pmatrix}$$

and

$$\begin{aligned}
\mathcal{H}_1 &= 4\underline{\alpha}_0\mathcal{H}_{00}\underline{\alpha}_0 - 4\underline{\alpha}_2\mathcal{H}_{22}\underline{\alpha}_1 + 4\underline{\alpha}_1(\mathcal{H}_{11}+\mathcal{H}_{33})\underline{\alpha}_1\\
&\quad - \underline{\alpha}_0 j(\mathcal{H}_{13}+\mathcal{H}_{31})(\underline{\alpha}_0+\underline{\alpha}_1) - \underline{\alpha}_1 j(\mathcal{H}_{13}+\mathcal{H}_{31})\underline{\alpha}_2,\\
\mathcal{H}_2 &= 4\underline{\alpha}_0\mathcal{H}_{00}\underline{\alpha}_0 - 4\underline{\alpha}_2\mathcal{H}_{22}\underline{\alpha}_1 - \underline{\alpha}_0 j(\mathcal{H}_{13}+\mathcal{H}_{31})(\underline{\alpha}_0+\underline{\alpha}_1)\\
&\quad - \underline{\alpha}_1 j(\mathcal{H}_{13}+\mathcal{H}_{31})(\underline{\alpha}_0+2\underline{\alpha}_1+\underline{\alpha}_2),\\
\mathcal{H}_3 &= 4[\underline{\alpha}_0\mathcal{H}_{00}\underline{\alpha}_0 + \underline{\alpha}_2\mathcal{H}_{22}\underline{\alpha}_1 - \underline{\alpha}_1(\mathcal{H}_{11}+\mathcal{H}_{33})\underline{\alpha}_1]\\
&\quad - \underline{\alpha}_0 j(\mathcal{H}_{13}+\mathcal{H}_{31})(\underline{\alpha}_0-\underline{\alpha}_1) + \underline{\alpha}_1 j(\mathcal{H}_{13}+\mathcal{H}_{31})(\underline{\alpha}_0-\underline{\alpha}_2),\\
\mathcal{H}_4 &= 4\underline{\alpha}_0\mathcal{H}_{00}\underline{\alpha}_0 - 4\underline{\alpha}_2\mathcal{H}_{22}\underline{\alpha}_1 + \underline{\alpha}_0 j(\mathcal{H}_{13}+\mathcal{H}_{31})(\underline{\alpha}_1-\underline{\alpha}_2)\\
&\quad + \underline{\alpha}_1 j(\mathcal{H}_{13}+\mathcal{H}_{31})(\underline{\alpha}_0-2\underline{\alpha}_1+\underline{\alpha}_2).
\end{aligned}$$

Since $\underline{W}, \underline{\xi} \in S^{4n-1}$, then $\underline{W}_j, \underline{\xi}_j \in S^{4n-1}$, $j = 1, 2, 3$. By direct calculation we have $\underline{\alpha}_r\mathcal{H}_{rs}\underline{\alpha}_r = 0$, $r = 0, 1, 2, 3$. And from Theorem 4.2 we know if $\boldsymbol{G}$ is the solution, then $\boldsymbol{\mathcal{H}}[\boldsymbol{G}] = \boldsymbol{G}$, i.e.,

$$\boldsymbol{\mathcal{H}}_{00} = \boldsymbol{\mathcal{H}}_{22},\ 2\boldsymbol{\mathcal{H}}_{00} = \boldsymbol{\mathcal{H}}_{11} + \boldsymbol{\mathcal{H}}_{33},\ \boldsymbol{\mathcal{H}}_{13} = -\boldsymbol{\mathcal{H}}_{31}.$$

Hence $\underline{\boldsymbol{\alpha}}\boldsymbol{\mathcal{H}}[\underline{\boldsymbol{\alpha}}\boldsymbol{G}](\underline{W}) = \boldsymbol{0}$, $\underline{W} \in S^{4n-1}$, therefore we get

$$\lim_{r\to 1^-} \underline{\boldsymbol{\alpha}}\boldsymbol{F}(r\underline{W})_{\underline{\boldsymbol{\alpha}}} = \underline{\boldsymbol{\alpha}}\boldsymbol{G}(\underline{W}), \quad \underline{W} \in S^{4n-1}. \tag{3}$$

Now we prove the uniqueness of the solution. Suppose that $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ are both the solutions of (1), where $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ are the matrix functions defined similarly to $\boldsymbol{F}$. Let $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{A}} - \boldsymbol{\mathcal{B}}$, we have

$$\begin{cases} \boldsymbol{\mathcal{D}}^T\boldsymbol{\mathcal{U}}(\underline{X}) = \mathbf{0}, & \underline{X} \in B(1), \\ \underline{\boldsymbol{\alpha}}\boldsymbol{\mathcal{U}}(\underline{W}) = \mathbf{0}, & \underline{W} \in S^{4n-1}. \end{cases}$$

Since

$$\boldsymbol{\mathcal{D}} = \begin{pmatrix} \partial_{\underline{Z}_0} & \partial_{\underline{Z}_3} & \partial_{\underline{Z}_2} & \partial_{\underline{Z}_1} \\ \partial_{\underline{Z}_1} & \partial_{\underline{Z}_0} & \partial_{\underline{Z}_3} & \partial_{\underline{Z}_2} \\ \partial_{\underline{Z}_2} & \partial_{\underline{Z}_1} & \partial_{\underline{Z}_0} & \partial_{\underline{Z}_3} \\ \partial_{\underline{Z}_3} & \partial_{\underline{Z}_2} & \partial_{\underline{Z}_1} & \partial_{\underline{Z}_0} \end{pmatrix}$$

and

$$16(\boldsymbol{\mathcal{D}}^T\boldsymbol{\mathcal{D}}^\dagger) = \begin{pmatrix} \Delta_{4n} & 0 & 0 & 0 \\ 0 & \Delta_{4n} & 0 & 0 \\ 0 & 0 & \Delta_{4n} & 0 \\ 0 & 0 & 0 & \Delta_{4n} \end{pmatrix},$$

$$\begin{cases} \partial_{\underline{Z}_0}\mathcal{U}_0 + \partial_{\underline{Z}_1}\mathcal{U}_1 + \partial_{\underline{Z}_2}\mathcal{U}_2 + \partial_{\underline{Z}_3}\mathcal{U}_3 = 0, & \underline{X} \in B(1), \\ \partial_{\underline{Z}_3}\mathcal{U}_0 + \partial_{\underline{Z}_0}\mathcal{U}_1 + \partial_{\underline{Z}_1}\mathcal{U}_2 + \partial_{\underline{Z}_2}\mathcal{U}_3 = 0, & \underline{X} \in B(1), \\ \partial_{\underline{Z}_2}\mathcal{U}_0 + \partial_{\underline{Z}_3}\mathcal{U}_1 + \partial_{\underline{Z}_0}\mathcal{U}_2 + \partial_{\underline{Z}_1}\mathcal{U}_3 = 0, & \underline{X} \in B(1), \\ \partial_{\underline{Z}_1}\mathcal{U}_0 + \partial_{\underline{Z}_2}\mathcal{U}_1 + \partial_{\underline{Z}_3}\mathcal{U}_2 + \partial_{\underline{Z}_0}\mathcal{U}_3 = 0, & \underline{X} \in B(1), \\ \underline{\alpha}_0(\mathcal{U}_0 + \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3) = 0, & \underline{W} \in S^{4n-1}, \\ \underline{\alpha}_1(\mathcal{U}_0 + \mathcal{U}_1 - \mathcal{U}_2 - \mathcal{U}_3) = 0, & \underline{W} \in S^{4n-1}, \\ \underline{\alpha}_2(\mathcal{U}_0 - \mathcal{U}_1 + \mathcal{U}_2 - \mathcal{U}_3) = 0, & \underline{W} \in S^{4n-1}, \\ \underline{\alpha}_3(\mathcal{U}_0 - \mathcal{U}_1 - \mathcal{U}_2 + \mathcal{U}_3) = 0, & \underline{W} \in S^{4n-1}. \end{cases}$$

Also,

$$\partial_{\underline{Z}_0} = \frac{1}{16}(\partial_{\underline{X}_0} + i\partial_{\underline{X}_1} + j\partial_{\underline{X}_2} + k\partial_{\underline{X}_3}),$$

$$\partial_{\underline{Z}_1} = \frac{1}{16}(\partial_{\underline{X}_0} + i\partial_{\underline{X}_1} - j\partial_{\underline{X}_2} - k\partial_{\underline{X}_3}),$$

$$\partial_{\underline{Z}_2} = \frac{1}{16}(\partial_{\underline{X}_0} - i\partial_{\underline{X}_1} + j\partial_{\underline{X}_2} - k\partial_{\underline{X}_3}),$$

$$\partial_{\underline{Z}_3} = \frac{1}{16}(\partial_{\underline{X}_0} - i\partial_{\underline{X}_1} - j\partial_{\underline{X}_2} + k\partial_{\underline{X}_3}),$$

and

$$\partial^2_{\underline{X}_0} = \partial^2_{\underline{X}_1} = \partial^2_{\underline{X}_2} = \partial^3_{\underline{X}_0} = -\Delta_{4n},$$

then we have

$$\begin{cases} \partial_{\underline{X}_0}(\mathcal{U}_0 + \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3) = 0, & \underline{X} \in B(1), \\ \partial_{\underline{X}_1}(\mathcal{U}_0 + \mathcal{U}_1 - \mathcal{U}_2 - \mathcal{U}_3) = 0, & \underline{X} \in B(1), \\ \partial_{\underline{X}_2}(\mathcal{U}_0 - \mathcal{U}_1 + \mathcal{U}_2 - \mathcal{U}_3) = 0, & \underline{X} \in B(1), \\ \partial_{\underline{X}_3}(\mathcal{U}_0 - \mathcal{U}_1 - \mathcal{U}_2 + \mathcal{U}_3) = 0, & \underline{X} \in B(1), \end{cases}$$

and

$$\begin{cases} \Delta_{4n}(\mathcal{U}_0+\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)=0, & \underline{X}\in B(1), \\ \Delta_{4n}(\mathcal{U}_0+\mathcal{U}_1-\mathcal{U}_2-\mathcal{U}_3)=0, & \underline{X}\in B(1), \\ \Delta_{4n}(\mathcal{U}_0-\mathcal{U}_1+\mathcal{U}_2-\mathcal{U}_3)=0, & \underline{X}\in B(1), \\ \Delta_{4n}(\mathcal{U}_0-\mathcal{U}_1-\mathcal{U}_2+\mathcal{U}_3)=0, & \underline{X}\in B(1). \end{cases}$$

And by direct caculation, we have

$$\begin{cases} \Delta_{4n}[\underline{X}_0(\mathcal{U}_0+\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)]=0, & \underline{X}\in B(1), \\ \Delta_{4n}[\underline{X}_1(\mathcal{U}_0+\mathcal{U}_1-\mathcal{U}_2-\mathcal{U}_3)]=0, & \underline{X}\in B(1), \\ \Delta_{4n}[\underline{X}_2(\mathcal{U}_0-\mathcal{U}_1+\mathcal{U}_2-\mathcal{U}_3)]=0, & \underline{X}\in B(1), \\ \Delta_{4n}[\underline{X}_3(\mathcal{U}_0-\mathcal{U}_1-\mathcal{U}_2+\mathcal{U}_3)]=0, & \underline{X}\in B(1). \end{cases}$$

Hence we get

$$\begin{cases} \Delta_{4n}[\underline{\alpha}_0(\mathcal{U}_0+\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)]=0, & \underline{X}\in B(1), \\ \underline{\alpha}_0(\mathcal{U}_0+\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)\equiv 0, & \underline{W}\in S^{4n-1}, \\ \Delta_{4n}[\underline{\alpha}_1(\mathcal{U}_0+\mathcal{U}_1-\mathcal{U}_2-\mathcal{U}_3)]=0, & \underline{X}\in B(1), \\ \underline{\alpha}_1(\mathcal{U}_0+\mathcal{U}_1-\mathcal{U}_2-\mathcal{U}_3\equiv 0, & \underline{W}\in S^{4n-1}, \\ \Delta_{4n}[\underline{\alpha}_2(\mathcal{U}_0-\mathcal{U}_1+\mathcal{U}_2-\mathcal{U}_3)]=0, & \underline{X}\in B(1), \\ \underline{\alpha}_2(\mathcal{U}_0-\mathcal{U}_1+\mathcal{U}_2-\mathcal{U}_3)\equiv 0, & \underline{W}\in S^{4n-1}, \\ \Delta_{4n}[\underline{\alpha}_3(\mathcal{U}_0-\mathcal{U}_1-\mathcal{U}_2+\mathcal{U}_3)]=0, & \underline{X}\in B(1), \\ \underline{\alpha}_3(\mathcal{U}_0-\mathcal{U}_1-\mathcal{U}_2+\mathcal{U}_3)\equiv 0, & \underline{W}\in S^{4n-1}. \end{cases}$$

Therefore we get

$$\begin{cases} \underline{\alpha}_0(\mathcal{U}_0+\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)\equiv 0, & \underline{W}\in B(1), \\ \underline{\alpha}_1(\mathcal{U}_0+\mathcal{U}_1-\mathcal{U}_2-\mathcal{U}_3)\equiv 0, & \underline{W}\in B(1), \\ \underline{\alpha}_2(\mathcal{U}_0-\mathcal{U}_1+\mathcal{U}_2-\mathcal{U}_3)\equiv 0, & \underline{W}\in B(1), \\ \underline{\alpha}_3(\mathcal{U}_0-\mathcal{U}_1-\mathcal{U}_2+\mathcal{U}_3)\equiv 0, & \underline{W}\in B(1). \end{cases}$$

Therefore, we have

$$\begin{cases} \underline{\beta}_0\underline{\alpha}_0(\mathcal{U}_0+\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)=\dfrac{1-|\underline{X}|^2}{4}(\mathcal{U}_0+\mathcal{U}_1+\mathcal{U}_2+\mathcal{U}_3)\equiv 0, \\ \underline{\beta}_1\underline{\alpha}_1(\mathcal{U}_0+\mathcal{U}_1-\mathcal{U}_2-\mathcal{U}_3)=\dfrac{1-|\underline{X}|^2}{4}(\mathcal{U}_0+\mathcal{U}_1-\mathcal{U}_2-\mathcal{U}_3)\equiv 0, \\ \underline{\beta}_2\underline{\alpha}_2(\mathcal{U}_0-\mathcal{U}_1+\mathcal{U}_2-\mathcal{U}_3)=\dfrac{1-|\underline{X}|^2}{4}(\mathcal{U}_0-\mathcal{U}_1+\mathcal{U}_2-\mathcal{U}_3)\equiv 0, \\ \underline{\beta}_3\underline{\alpha}_3(\mathcal{U}_0-\mathcal{U}_1-\mathcal{U}_2+\mathcal{U}_3)=\dfrac{1-|\underline{X}|^2}{4}(\mathcal{U}_0-\mathcal{U}_1-\mathcal{U}_2+\mathcal{U}_3)\equiv 0, \end{cases}$$

from this we can get $\mathcal{U}_0=\mathcal{U}_1=\mathcal{U}_2=\mathcal{U}_3\equiv 0$, hence the uniqueness of the solution has been proved. □

Corollary 4.1 (Dirichlet problem)**.** *Given the boundary data* $\boldsymbol{G}\in\boldsymbol{L}_p(S^{4n-1},\mathbb{H}_{4n})$, *find the function* $\boldsymbol{K}$ *such that*

$$\begin{cases} \underline{\Delta}\boldsymbol{K}(\underline{X})=\boldsymbol{0}, & \underline{X}\in B(1), \\ \boldsymbol{K}(\underline{W})=\boldsymbol{G}(\underline{W}), & \underline{W}\in S^{4n-1}, \end{cases} \tag{4}$$

where

$$\underline{\Delta} = \begin{pmatrix} \Delta_{4n} & 0 & 0 & 0 \\ 0 & \Delta_{4n} & 0 & 0 \\ 0 & 0 & \Delta_{4n} & 0 \\ 0 & 0 & 0 & \Delta_{4n} \end{pmatrix}$$

and (4) *is equivalent to the system*

$$(5) \quad \begin{cases} \Delta_{4n} k_0(\underline{X}) = \Delta_{4n} k_1(\underline{X}) = \Delta_{4n} k_2(\underline{X}) & \\ \qquad = \Delta_{4n} k_3(\underline{X}) = 0, & \underline{X} \in B(1), \\ k_0(\underline{W}) = g_0(\underline{W}),\ k_1(\underline{W}) = g_1(\underline{W}),\ k_2(\underline{W}) = g_2(\underline{W}), & \\ k_3(\underline{W}) = g_3(\underline{W}), & \underline{W} \in S^{4n-1}. \end{cases}$$

Then there exists a unique solution and it has the form of

$$\boldsymbol{K}(\underline{X}) = \underline{\boldsymbol{\alpha}} \boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\alpha}}} + \underline{\boldsymbol{\beta}} \boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\beta}}}, \quad \underline{X} \in \overline{B}(1),$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\alpha}}}, \boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\beta}}}$ *are as above.*

Proof. For arbitrary $\underline{X} \in \overline{B}(1)$

$$\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\alpha}}} \triangleq \boldsymbol{\mathcal{C}}[2\underline{\boldsymbol{\alpha}}\boldsymbol{G}](\underline{X}) = \int_{S^{4n-1}} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}) \boldsymbol{\mathcal{V}}^T 2\underline{\boldsymbol{\alpha}} \widetilde{\boldsymbol{G}}(\underline{X}) dS(\underline{Y}),$$

$$\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\beta}}} \triangleq \boldsymbol{\mathcal{C}}[2\underline{\boldsymbol{\beta}}\boldsymbol{G}](\underline{X}) = \int_{S^{4n-1}} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V}) \boldsymbol{\mathcal{V}}^T 2\underline{\boldsymbol{\beta}} \widetilde{\boldsymbol{G}}(\underline{X}) dS(\underline{Y}),$$

where $\widetilde{\boldsymbol{G}} = (-1)^{\frac{n(n+1)}{2}} \frac{(2i)^{-2n}}{2} \boldsymbol{G} = \operatorname{circ}\begin{pmatrix} \widetilde{g_0} \\ \widetilde{g_1} \\ \widetilde{g_2} \\ \widetilde{g_3} \end{pmatrix}$.

Since $\partial_{\underline{X}_r}(\mathcal{C}_{rs}[\widetilde{g_i}](\underline{X}_r)) = 0$, $r, s, i = 0, 1, 2, 3$, then

$$\mathcal{C}_{rs}[\widetilde{g_i}](\underline{X}_r) \text{ and } \underline{X}_r \mathcal{C}_{rs}[\widetilde{g_i}](\underline{X}_r)$$

are harmonic $r, s, i = 0, 1, 2, 3$.

By direct calculation, we get

$$\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\alpha}}} = \frac{1}{4} \operatorname{circ}\begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \\ \mathcal{C}_3 \\ \mathcal{C}_4 \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{C}_1 &= \mathcal{C}_{00}(\widetilde{g_0} + \widetilde{g_1} + \widetilde{g_2} + \widetilde{g_3}) + \mathcal{C}_{22}(\widetilde{g_0} - \widetilde{g_1} + \widetilde{g_2} - \widetilde{g_3}) \\ &\quad + (\mathcal{C}_{11} + \mathcal{C}_{33})(\widetilde{g_0} - \widetilde{g_2}) - j(\mathcal{C}_{13} + \mathcal{C}_{31})(\widetilde{g_1} - \widetilde{g_3}), \\ \mathcal{C}_2 &= \mathcal{C}_{00}(\widetilde{g_0} + \widetilde{g_1} + \widetilde{g_2} + \widetilde{g_3}) - \mathcal{C}_{22}(\widetilde{g_0} - \widetilde{g_1} + \widetilde{g_2} - \widetilde{g_3}) \\ &\quad + (\mathcal{C}_{11} + \mathcal{C}_{33})(\widetilde{g_0} - \widetilde{g_2}) + j(\mathcal{C}_{13} + \mathcal{C}_{31})(\widetilde{g_0} - \widetilde{g_2}), \\ \mathcal{C}_3 &= \mathcal{C}_{00}(\widetilde{g_0} + \widetilde{g_1} + \widetilde{g_2} + \widetilde{g_3}) + \mathcal{C}_{22}(\widetilde{g_0} - \widetilde{g_1} + \widetilde{g_2} - \widetilde{g_3}) \\ &\quad - (\mathcal{C}_{11} + \mathcal{C}_{33})(\widetilde{g_0} - \widetilde{g_2}) + j(\mathcal{C}_{13} + \mathcal{C}_{31})(\widetilde{g_1} - \widetilde{g_3}), \\ \mathcal{C}_4 &= \mathcal{C}_{00}(\widetilde{g_0} + \widetilde{g_1} + \widetilde{g_2} + \widetilde{g_3}) - \mathcal{C}_{22}(\widetilde{g_0} - \widetilde{g_1} + \widetilde{g_2} - \widetilde{g_3}) \end{aligned}$$

$$-(\mathcal{C}_{11}+\mathcal{C}_{33})(\widetilde{g_0}-\widetilde{g_2})-j(\mathcal{C}_{13}+\mathcal{C}_{31})(\widetilde{g_0}-\widetilde{g_2}).$$

And we also obtain

$$\underline{\boldsymbol{\alpha}}\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\alpha}}}=\frac{1}{16}\text{circ}\begin{pmatrix}\mathfrak{A}\\ \mathfrak{B}\\ \mathfrak{C}\\ \mathfrak{D}\end{pmatrix},$$

where

$$\begin{aligned}\mathfrak{A}=&\ 4\underline{\alpha}_0\mathcal{C}_{00}(\widetilde{g_0}+\widetilde{g_1}+\widetilde{g_2}+\widetilde{g_3})+4\underline{\alpha}_2\mathcal{C}_{22}(\widetilde{g_0}-\widetilde{g_1}+\widetilde{g_2}-\widetilde{g_3})\\ &+2\underline{\alpha}_1(\mathcal{C}_{11}+\mathcal{C}_{33})(\widetilde{g_0}-\widetilde{g_2}-\widetilde{g_1}+\widetilde{g_3})+2\underline{\alpha}_3(\mathcal{C}_{11}+\mathcal{C}_{33})(\widetilde{g_0}-\widetilde{g_2}+\widetilde{g_1}-\widetilde{g_3})\\ &-2j\underline{\alpha}_1(\mathcal{C}_{13}+\mathcal{C}_{31})(\widetilde{g_1}-\widetilde{g_3}+\widetilde{g_0}-\widetilde{g_2})+2j\underline{\alpha}_3(\mathcal{C}_{13}+\mathcal{C}_{31})(\widetilde{g_0}-\widetilde{g_2}-\widetilde{g_1}+\widetilde{g_3}),\\ \mathfrak{B}=&\ 4\underline{\alpha}_0\mathcal{C}_{00}(\widetilde{g_0}+\widetilde{g_1}+\widetilde{g_2}+\widetilde{g_3})-4\underline{\alpha}_2\mathcal{C}_{22}(\widetilde{g_0}-\widetilde{g_1}+\widetilde{g_2}-\widetilde{g_3})\\ &+2\underline{\alpha}_1(\mathcal{C}_{11}+\mathcal{C}_{33})(\widetilde{g_0}-\widetilde{g_2}+\widetilde{g_1}-\widetilde{g_3})+2\underline{\alpha}_3(\mathcal{C}_{11}+\mathcal{C}_{33})(\widetilde{g_1}-\widetilde{g_3}+\widetilde{g_2}-\widetilde{g_0})\\ &+2j\underline{\alpha}_1(\mathcal{C}_{13}+\mathcal{C}_{31})(\widetilde{g_0}+\widetilde{g_1}-\widetilde{g_2}-\widetilde{g_3})+2j\underline{\alpha}_3(\mathcal{C}_{13}+\mathcal{C}_{31})(\widetilde{g_0}-\widetilde{g_2}+\widetilde{g_1}-\widetilde{g_3}),\\ \mathfrak{C}=&\ 4\underline{\alpha}_0\mathcal{C}_{00}(\widetilde{g_0}+\widetilde{g_1}+\widetilde{g_2}+\widetilde{g_3})+4\underline{\alpha}_2\mathcal{C}_{22}(\widetilde{g_0}-\widetilde{g_1}+\widetilde{g_2}-\widetilde{g_3})\\ &+2\underline{\alpha}_1(\mathcal{C}_{11}+\mathcal{C}_{33})(\widetilde{g_0}-\widetilde{g_2}+\widetilde{g_1}-\widetilde{g_3})-2\underline{\alpha}_3(\mathcal{C}_{11}+\mathcal{C}_{33})(\widetilde{g_0}-\widetilde{g_2}+\widetilde{g_1}-\widetilde{g_3})\\ &+2j\underline{\alpha}_1(\mathcal{C}_{13}+\mathcal{C}_{31})(\widetilde{g_1}-\widetilde{g_3}+\widetilde{g_0}-\widetilde{g_2})+2j\underline{\alpha}_3(\mathcal{C}_{13}+\mathcal{C}_{31})(\widetilde{g_1}+\widetilde{g_2}-\widetilde{g_0}-\widetilde{g_3}),\\ \mathfrak{D}=&\ 4\underline{\alpha}_0\mathcal{C}_{00}(\widetilde{g_0}+\widetilde{g_1}+\widetilde{g_2}+\widetilde{g_3})-4\underline{\alpha}_2\mathcal{C}_{22}(\widetilde{g_0}-\widetilde{g_1}+\widetilde{g_2}-\widetilde{g_3})\\ &-2\underline{\alpha}_1(\mathcal{C}_{11}+\mathcal{C}_{33})(\widetilde{g_0}-\widetilde{g_2}+\widetilde{g_1}-\widetilde{g_3})+2\underline{\alpha}_3(\mathcal{C}_{11}+\mathcal{C}_{33})(\widetilde{g_0}-\widetilde{g_1}-\widetilde{g_2}+\widetilde{g_3})\\ &+2j\underline{\alpha}_1(\mathcal{C}_{13}+\mathcal{C}_{31})(\widetilde{g_1}-\widetilde{g_3}-\widetilde{g_0}+\widetilde{g_2})-2j\underline{\alpha}_3(\mathcal{C}_{13}+\mathcal{C}_{31})(\widetilde{g_0}+\widetilde{g_1}-\widetilde{g_2}-\widetilde{g_3}).\end{aligned}$$

Since we have supposed $\underline{\alpha}_1=\underline{\alpha}_3$ and $\underline{X}_r\mathcal{C}_{rs}[\widetilde{g_i}](\underline{X}_r)$ are harmonic, $r,s,i=0,1,2,3$, then we can have that

$$\underline{\Delta}[\text{circ}\begin{pmatrix}\mathfrak{A}\\ \mathfrak{B}\\ \mathfrak{C}\\ \mathfrak{D}\end{pmatrix}]=\mathbf{0},$$

i.e.,

$$\underline{\Delta}[\underline{\boldsymbol{\alpha}}\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\alpha}}}]=\mathbf{0}.$$

With the same argument, we can get

$$\underline{\Delta}[\underline{\boldsymbol{\beta}}\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\alpha}}}]=\mathbf{0}.$$

Then we have that

$$\underline{\Delta}\boldsymbol{F}(\underline{X})=\underline{\Delta}[\underline{\boldsymbol{\alpha}}\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\alpha}}}+\underline{\boldsymbol{\beta}}\boldsymbol{F}(\underline{X})_{\underline{\boldsymbol{\alpha}}}]=\mathbf{0},\quad \underline{X}\in S^{4n-1},$$

and by means of (iii) in Lemma 4.2 and the term of (3) we get that

$$\begin{aligned}\boldsymbol{K}(\underline{W})=\lim_{r\to1^-}\boldsymbol{K}(r\underline{W})&=\lim_{r\to1^-}(\underline{\boldsymbol{\alpha}}\boldsymbol{F}(\underline{W})_{\underline{\boldsymbol{\alpha}}}+\underline{\boldsymbol{\beta}}\boldsymbol{F}(\underline{W})_{\underline{\boldsymbol{\beta}}})\\ &=\underline{\boldsymbol{\alpha}}\boldsymbol{G}(\underline{W})+\underline{\boldsymbol{\beta}}\boldsymbol{G}(\underline{W})=\boldsymbol{G}(\underline{W}).\end{aligned}$$

This has completed the proof. □

Specially, if $p = 2$, as an application of Szegö kernel we can get the following Theorem.

Theorem 4.4. *If* $\boldsymbol{G} \in L_2(S^{4n-1})$*, then the solution of system* (5) *is formulated by*

$$\boldsymbol{F}(\underline{X}) = \int_{S^{4n-1}} (\boldsymbol{\mathcal{S}}_{\underline{X}}(\underline{Y}) + \underline{\boldsymbol{v}}|_{S^{4n-1}}\boldsymbol{\mathcal{S}}_{\underline{X}}(\underline{Y}))\boldsymbol{G}(\underline{Y})dS_{\underline{Y}}.$$

Also, when $p = 2$, from Lemma 3.1(iv) we have

$$\boldsymbol{K} = \boldsymbol{L} + \underline{\boldsymbol{v}}\boldsymbol{M},$$

where $\boldsymbol{L}, \boldsymbol{M} \in \mathsf{H}^2(S^{4n-1})$. And the above Dirichlet problem exists the unique solution, the solution has the form

$$\boldsymbol{K}(\underline{X}) = \widetilde{\boldsymbol{L}} + \underline{\boldsymbol{\mathcal{X}}}\widetilde{\boldsymbol{M}}, \quad \underline{X} \in B(1),$$

where $\widetilde{\boldsymbol{L}}, \widetilde{\boldsymbol{M}} \in \mathsf{H}^2(B(1))$ are $\boldsymbol{Q}$-Hermitian monogenic extensions of $\boldsymbol{L}, \boldsymbol{M}$ and

$$\underline{\boldsymbol{\mathcal{X}}} = \frac{1}{4}\mathrm{circ}\begin{pmatrix} \underline{X}_0 + \underline{X}_1 + \underline{X}_2 + \underline{X}_3 \\ \underline{X}_0 + \underline{X}_1 - \underline{X}_2 - \underline{X}_3 \\ \underline{X}_0 - \underline{X}_1 + \underline{X}_2 - \underline{X}_3 \\ \underline{X}_0 - \underline{X}_1 - \underline{X}_2 + \underline{X}_3 \end{pmatrix}.$$

Now we consider the folllowing Dirichlet problem: For the given boundary data $\boldsymbol{G}_0 \in \boldsymbol{L}_p(S^{4n-1}, \mathbb{H}_{4n})$, find the function $\boldsymbol{F}_0$ such that

$$\text{(6)} \qquad \begin{cases} \boldsymbol{\mathcal{D}}^T\boldsymbol{F}_0(\underline{X}) = \boldsymbol{0}, & \underline{X} \in B(1), \\ \underline{\boldsymbol{\alpha}}\boldsymbol{F}_0(\underline{W}) = \underline{\boldsymbol{\alpha}}\boldsymbol{G}_0(\underline{W}), & \underline{W} \in S^{4n-1}, \end{cases}$$

$$\text{(7)} \qquad \begin{cases} \boldsymbol{\mathcal{D}}^T\boldsymbol{F}_0(\underline{X}) = \boldsymbol{0}, & \underline{X} \in B(1), \\ \underline{\boldsymbol{\beta}}\boldsymbol{F}_0(\underline{W}) = \underline{\boldsymbol{\beta}}\boldsymbol{G}_0(\underline{W}), & \underline{W} \in S^{4n-1}, \end{cases}$$

where the matrix function $\boldsymbol{F}_0 = \mathrm{circ}\begin{pmatrix} F_0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is defined similarly to $\boldsymbol{G}_0$.

As a special case of Theorem 4.3 we can directly give its solution to the above Dirichlet problems.

Theorem 4.5. *For the above two half Dirichlet problems* (6) *and* (7)*, there exist the unique solutions and their solutions are given respectively by*

$$\boldsymbol{F}_0(\underline{X})_{\underline{\boldsymbol{\alpha}}} \triangleq \boldsymbol{\mathcal{C}}[2\underline{\boldsymbol{\alpha}}\boldsymbol{G}_0](\underline{Y}) = \int_{S^{4n-1}} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\boldsymbol{\mathcal{V}}^T 2\underline{\boldsymbol{\alpha}}\widetilde{\boldsymbol{G}}_0(\underline{X})dS(\underline{X}),\ \underline{X} \in \overline{B}(1),$$

$$\boldsymbol{F}_0(\underline{X})_{\underline{\boldsymbol{\beta}}} \triangleq \boldsymbol{\mathcal{C}}[2\underline{\boldsymbol{\beta}}\boldsymbol{G}_0](\underline{Y}) = \int_{S^{4n-1}} \boldsymbol{\mathcal{E}}(\underline{Z} - \underline{V})\boldsymbol{\mathcal{V}}^T 2\underline{\boldsymbol{\beta}}\widetilde{\boldsymbol{G}}_0(\underline{X})dS(\underline{X}),\ \underline{X} \in \overline{B}(1),$$

where $\widetilde{\boldsymbol{G}}_0 = (-1)^{\frac{n(n+1)}{2}}\frac{(2i)^{-2n}}{2}\boldsymbol{G}_0$.

Corollary 4.2. *Given the boundary data* $\boldsymbol{G}_0 \in \boldsymbol{L}_p(S^{4n-1}, \mathbb{H}_{4n})$*, find the function* $\boldsymbol{K}_0$ *such that*

$$\text{(8)} \qquad \begin{cases} \Delta\boldsymbol{K}_0(\underline{X}) = \boldsymbol{0}, & \underline{X} \in B(1), \\ \boldsymbol{K}_0(\underline{W}) = \boldsymbol{G}_0(\underline{W}), & \underline{W} \in S^{4n-1}, \end{cases}$$

where

$$\underline{\Delta} = \begin{pmatrix} \Delta_{4n} & 0 & 0 & 0 \\ 0 & \Delta_{4n} & 0 & 0 \\ 0 & 0 & \Delta_{4n} & 0 \\ 0 & 0 & 0 & \Delta_{4n} \end{pmatrix},$$

and (8) *is equivalent to the system*

$$\begin{cases} \Delta_{4n} k_0(\underline{X}) = 0, & \underline{X} \in B(1), \\ k_0(\underline{W}) = g_0(\underline{W}), & \underline{W} \in S^{4n-1}. \end{cases}$$

Then there exists a unique solution and it has the form of

$$\boldsymbol{K}_0(\underline{X}) = \underline{\boldsymbol{\alpha}} \boldsymbol{F}_0(\underline{X})_{\underline{\boldsymbol{\alpha}}} + \underline{\boldsymbol{\beta}} \boldsymbol{F}_0(\underline{X})_{\underline{\boldsymbol{\beta}}}, \quad \underline{X} \in \overline{B}(1),$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{F}_0(\underline{X})_{\underline{\boldsymbol{\alpha}}}, \boldsymbol{F}_0(\underline{X})_{\underline{\boldsymbol{\beta}}}$ *are as above.*

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