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ENUMERATION OF RELAXED COMPLETE PARTITIONS AND DOUBLE-COMPLETE PARTITIONS

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Abstract. A partition of n is complete if every positive integer from 1 to n can be represented by the sum of its parts. The concept of complete partitions has been extended in several ways. In this paper, we consider the number of k-relaxed r-complete partitions of n and the number of double-complete partitions of n.

1. Introduction

A partition of a positive integer n is a finite sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_i > 0$ for all $i = 1, \ldots, \ell$ and $\sum_{i=1}^{\ell} \lambda_i = n$. Throughout this paper, we arrange λ_i in ascending order. We also write partitions in the form $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_t^{m_t})$, where the λ_i are strictly increasing, each m_i is the multiplicity of λ_i, and $\ell = \sum_{i=1}^{t} m_i$. Let $p(n)$ be the number of partitions of n.

MacMahon [4] introduced perfect partitions of n, which can represent every positive integer less than or equal to n by a unique sum of its parts. For example, $(1, 2, 4)$ is a perfect partition of 7 because $1 = 1$, $2 = 2$, $3 = 1 + 2$, $4 = 4$, $5 = 1 + 4$, $6 = 2 + 4$, and $7 = 1 + 2 + 4$.

One way of generalizing MacMahon's idea is to eliminate the uniqueness condition, which was done by Park [6]. A partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n is said to be complete if every integer m with $1 \le m \le n$ can be expressed as $\sum_{i=1}^{\ell} \alpha_i \lambda_i$, where $\alpha_i \in \{0, 1\}$. Note that O'Shea [5] independently defined the same notion, calling them weak M-partitions. The concept of complete partitions is further extended by Park [7], Lee and Park [3], and Andrews, Beck, and Hopkins [1]. First, Park introduced r-complete partitions.

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Definition 1.1 ([7])**.** Let r be a positive integer. An r-complete partition of n is a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that each integer m with $1 \le m \le rn$ can be expressed as $\sum_{i=1}^{\ell} \alpha_i \lambda_i$, where $\alpha_i \in \{0, 1, \ldots, r\}$.

For example, $(1, 1, 1, 1)$ and $(1, 1, 2)$ are complete partitions of 4 and $(1, 1, 1, 1)$, $(1, 1, 2)$, and $(1, 3)$ are 2-complete partitions of 4. Park found the following result on r-complete partitions.

Theorem 1.2 ([7, Theorem 2.2])**.** *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of n with $\lambda_1 = 1$. Then λ is an r-complete partition if and only if $\lambda_i \le 1 + r \sum_{j=1}^{i-1} \lambda_j$ for $i = 2, \ldots, \ell$.*

Lee and Park [3] studied complete partitions with more specified completeness, the double-complete partitions.

Definition 1.3 ([3])**.** A partition $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_t^{m_t})$ of n is said to be double-complete if each integer m with $2 \le m \le n - 2$ can be represented in at least two different ways as a sum $\sum_{i=1}^{t} \alpha_i \lambda_i$ with $\alpha_i \in \{0, 1, \ldots, m_i\}$.

For example, the partition $(1^4, 2^2)$ is a double-complete partition of 8 since $2 = 1+1 = 2$, $3 = 1+1+1 = 1+2$, $4 = 1+1+2 = 2+2$, $5 = 1+1+1+2 = 1+2+2$, and $6 = 1+1+1+1+2 = 1+1+2+2$. Note that all double-complete partitions must have at least two 1's and one 2 as its parts since all the partitions of 2 are $(1, 1)$ and (2). When $n \ge 5$, a double-complete partition of n must represent 3 at least twice as a sum of its parts, implying that it has either three 1's and one 2, or two 1's, one 2, and one 3 as its parts. Moreover, Lee and Park gave the following result.

Theorem 1.4 ([3, Theorem 2.4])**.** *A partition $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_t^{m_t})$ of a positive integer $n \ge 5$ is double-complete if and only if $\lambda_{i+1} \le -1 + \sum_{j=1}^{i} m_j \lambda_j$ for $2 \le i \le t - 1$ and λ should have at least three 1s and one 2, or two 1s, one 2, and one 3 as its parts.*

Recently, Andrews, Beck, and Hopkins [1] introduced k-step partitions. For a positive integer k, a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_1 \le k$ is called a k-step partition if $\lambda_i \le k + \sum_{j=1}^{i-1} \lambda_j$ for $i = 2, \ldots, \ell$. For example, $(2, 2)$ is not a complete partition of 4, but it is a 2-step partition.

Let $s(n, k)$ be the number of k-step partitions of n and $c(n)$ be the number of complete partitions of n. It is clear that $c(n) = s(n, 1)$ by definition. Andrews, Beck, and Hopkins found the following identity.

Theorem 1.5 ([1, Theorem 9])**.** *For every positive integer k,*

$$\sum_{n=0}^{\infty} s(n, k) q^n (1 - q)(1 - q^2) \cdots (1 - q^{n+k}) = 1.$$

In particular, $c(n)$ satisfies

$$\sum_{n=0}^{\infty} c(n) q^n (1 - q)(1 - q^2) \cdots (1 - q^{n+1}) = 1.$$

On the other hand, Bruno and O'Shea [2] extended the definition of r-complete partitions, the k-relaxed r-complete partitions.

Definition 1.6 ([2])**.** Let k be a nonnegative integer and r be a positive integer. A partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n is called a k-relaxed r-complete partition (shortly, (k,r)-partition) if no $k+1$ consecutive integers between 1 and rn are absent from the set $\{\sum_{i=1}^{\ell} \alpha_i \lambda_i : \alpha_i \in \{0, 1, \ldots, r\}\}$.

For example, the partition $(1,3)$ is not a complete partition of 4, but it is a $(1,1)$-partition since $1 = 1$, $3 = 3$, and $4 = 1 + 3$.

The concepts of (k,r)-partitions and k-step partitions are introduced independently. However, the following theorem deduces that a partition λ is $(k-1,1)$-partition if and only if it is a k-step partition.

Theorem 1.7 ([2, Theorem 1])**.** *A partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_1 \le k+1$ is a (k,r)-partition if and only if $\lambda_i \le (k+1) + r\sum_{j=1}^{i-1} \lambda_j$ for $i = 2, \ldots, \ell$.*

In Section 2, we enumerate the number of (k,r)-partitions of n in various ways and give a matrix equation of this number. As a special case of these enumerations, we obtain the number of r-complete partitions. Let $p_r(n,k)$ be the number of (k,r)-partitions of n. It is clear that $p_r(n,0) = c_r(n)$, the number of r-complete partitions of n, and $p_1(n,k) = s(n,k+1)$, the number of $(k+1)$-step partitions of n. The following theorem is one of the main results.

Theorem 1.8. *For each nonnegative integer k,*

$$\sum_{n=0}^{\infty} p_r(n,k) q^n (1-q)(1-q^2)\cdots(1-q^{rn+k+1}) = 1.$$

In particular,

$$\sum_{n=0}^{\infty} c_r(n) q^n (1-q)(1-q^2)\cdots(1-q^{rn+1}) = 1.$$

In Section 3, we focus on the double-complete partitions. We write $dc(n)$ as the number of double-complete partitions of n and $dc_1(n)$, $dc_2(n)$, and $dc_3(n)$ as the number of such partitions with additional conditions as follows.

Let $DC_1(n)$ (resp. $DC_2(n)$) be the set of all double-complete partitions $(\lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3}, \ldots, \lambda_t^{m_t})$ of n satisfying $\lambda_1 = 1$, $m_1 \ge 3$, $\lambda_2 = 2$, $m_2 \ge 1$, (resp. $\lambda_1 = 1$, $m_1 \ge 2$, $\lambda_2 = 2$, $m_2 \ge 1$, $\lambda_3 = 3$, $m_3 \ge 1$), and $DC_3(n) = DC_1(n) \cap DC_2(n)$. We denote by $dc_i(n)$ $(i = 1,2,3)$ the cardinality of $DC_i(n)$. We enumerate $dc(n)$ by establishing the identities of $dc_1(n)$, $dc_2(n)$, and $dc_3(n)$.

Theorem 1.9. *$dc(n) = dc_1(n) + dc_2(n) - dc_3(n)$ for $n \ge 5$ and*

$$\sum_{n=5}^{\infty} dc_1(n) q^n (1-q)(1-q^2)\cdots(1-q^{n-1}) = q^5,$$

$$\sum_{n=7}^{\infty} dc_2(n)q^n(1-q)(1-q^2)\cdots(1-q^{n-1}) = q^7,$$

$$\sum_{n=8}^{\infty} dc_3(n)q^n(1-q)(1-q^2)\cdots(1-q^{n-1}) = q^8.$$

2. Results on (k, r)-partitions

First, we introduce previous results about the (k, r)-partitions. We use $\lceil x \rceil$ and $\lfloor x \rfloor$ for the least integer greater than or equal to x and the greatest integer less than or equal to x, respectively. Bruno and O'Shea showed the following proposition.

Proposition 2.1 ([2, Equation (1) and Proposition 1])**.** *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a (k, r)-partition of n. Then λ satisfies the following conditions*:

(a) $\lambda_i \leq (k+1)(1+r)^{i-1}$ *for* $i = 1, \ldots, \ell$.
(b) $\ell \geq \lceil \log_{(1+r)}(\frac{rn}{k+1} + 1) \rceil$.

From this, we can easily prove that the largest part is at most $\lfloor \frac{k+1+rn}{1+r} \rfloor$.

Proposition 2.2. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a (k, r)-partition of n. Then $\lambda_\ell \leq \lfloor \frac{k+1+rn}{1+r} \rfloor$.*

Proof. It follows from $n - \lambda_\ell = \sum_{i=1}^{\ell-1} \lambda_i$ that $\lambda_\ell \leq (k+1) + r\sum_{i=1}^{\ell-1} \lambda_i = (k+1) + r(n - \lambda_\ell) = (k+1) + rn - r\lambda_\ell$. Hence, $\lambda_\ell \leq \lfloor \frac{k+1+rn}{1+r} \rfloor$. □

The relation between r-complete partitions and (k, r)-partitions is summarized as follows.

Proposition 2.3. *Let $c_r(n)$ be the number of r-complete partitions of n and $p_r(n, k)$ be the number of (k, r)-partitions of n. Then $c_r(n) = p_r(n-1, r)$.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be an r-complete partition of n and $\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_\ell)$. Since $\lambda_2 \leq r+1$ by definition, $\overline{\lambda}$ is an (r, r)-partition of $n-1$. Similarly, for an (r, r)-partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of $n-1$, $\lambda^* = (1, \lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is an r-complete partition of n. □

Proposition 2.4. *Let $c_r(n, k)$ be the number of r-complete partitions of n with exactly k ones. Then $c_r(n, k) = p_r(n-k, rk) - p_r(n-k-1, rk+r)$.*

Proof. We prove that $p_r(n-k, rk) = c_r(n, k) + p_r(n-k-1, rk+r)$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be an (rk, r)-partition of $n-k$. We consider two cases. If $\lambda_1 = 1$, then $\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_\ell)$ is an $(rk+r, r)$-partition of $n-k-1$ since $\lambda_2 \leq (rk+1) + r$. If $\lambda_1 \neq 1$, then $\lambda^* = (1^k, \lambda_1, \lambda_2, \ldots, \lambda_\ell)$, the partition with k copies of 1 added to λ, is an r-complete partition of n with exactly k ones since $\lambda_1 \leq rk+1$. □

For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_1 \le k+1$, let $\lambda^{(k,r)}$ be a partition $(\lambda_1, \ldots, \lambda_m)$ such that $m \le \ell$ is the largest integer satisfying $\lambda_i \le (k+1) + r\sum_{j=1}^{i-1} \lambda_j$ for $i = 2, \ldots, m$. If $\lambda_1 > k+1$, then we set $\lambda^{(k,r)} = \emptyset$. We now prove Theorem 1.8.

Proof of Theorem 1.8. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, $\lambda_1 \le k+1$, and $\lambda^{(k,r)} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ with $\sum_{i=1}^{m} \lambda_i = n$. If $m < \ell$ and $\lambda_{m+1} \le rn+k+1$, then it contradicts the fact that m is the largest integer satisfying $\lambda_i \le (k+1)+r\sum_{j=1}^{i-1} \lambda_j$ for $i = 2, \ldots, m$. Therefore, $m < \ell$ implies that $\lambda_{m+1} > rn + k + 1$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\lambda_1 > k+1$, then $\lambda^{(k,r)} = \emptyset$. Hence, we can divide $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ into a (k,r)-partition $\lambda^{(k,r)} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n and a partition $(\lambda_{m+1}, \ldots, \lambda_\ell)$ whose parts are greater than $rn + k + 1$. Therefore, we have

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} \frac{p_r(n,k)q^n}{\prod_{j=rn+k+2}^{\infty}(1-q^j)}.$$

The second identity is straightforward by putting $k = 0$. □

In the rest of this section, we give an alternative method to count the number of (k,r)-partitions by using the following matrix relation. For positive integers r and s with $s \le r$, let $\Gamma_n^{(r,s)}$ be the $n \times n$ matrix whose entries are $\gamma_{i,j}^{(r,s)} = p_r(i-j, rj-s)$. The matrix $\Gamma_n^{(r,s)}$ is lower triangular since $p_r(i,j) = 0$ for $i < 0$. Figure 1 shows $\Gamma_{10}^{(3,2)}$. The entry $\gamma_{4,1}^{(3,2)} = p_3(3,1)$, for instance, is 2 because $(1,1,1)$ and $(1,2)$ are $(1,3)$-partitions but (3) is not a $(1,3)$-partition.

$$\Gamma_{10}^{(3,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 9 & 7 & 5 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 12 & 10 & 7 & 5 & 3 & 2 & 1 & 1 & 0 & 0 \\ 18 & 14 & 11 & 7 & 5 & 3 & 2 & 1 & 1 & 0 \\ 25 & 21 & 15 & 11 & 7 & 5 & 3 & 2 & 1 & 1 \end{pmatrix}$$

FIGURE 1. The matrix $\Gamma_{10}^{(3,2)}$ with $\gamma_{i,j}^{(3,2)} = p_3(i-j, 3j-2)$

Let $M_e^{(r,s)}(n,k)$ (resp. $M_o^{(r,s)}(n,k)$) be the set of partitions of $n-k$ into an even (resp. odd) number of distinct parts, whose sizes are less than or equal to $rk-(s-1)$. We write $\mu_e^{(r,s)}(n,k) = \left|M_e^{(r,s)}(n,k)\right|$ and $\mu_o^{(r,s)}(n,k) = \left|M_o^{(r,s)}(n,k)\right|$. The matrix $\mathcal{M}_n^{(r,s)}$ is the $n \times n$ matrix whose entries are $\mu_{i,j}^{(r,s)} =$

$\mu_e^{(r,s)}(i,j)-\mu_o^{(r,s)}(i,j)$. $\mathcal{M}_n^{(r,s)}$ is also lower triangular; see Figure 2 for example. The matrix $\mathcal{M}_{10}^{(3,2)}$ has the entries $\mu_{i,j}^{(3,2)}=\mu_e^{(3,2)}(i,j)-\mu_o^{(3,2)}(i,j)$ and the entry $\mu_{7,4}^{(3,2)}=0$ since $M_e^{(3,2)}(7,4)=\{(1,2)\}$ and $M_o^{(3,2)}(7,4)=\{(3)\}$.

$$\mathcal{M}_{10}^{(3,2)}=\begin{pmatrix}1&0&0&0&0&0&0&0&0&0\\-1&1&0&0&0&0&0&0&0&0\\-1&-1&1&0&0&0&0&0&0&0\\1&-1&-1&1&0&0&0&0&0&0\\0&0&-1&-1&1&0&0&0&0&0\\0&0&0&-1&-1&1&0&0&0&0\\0&1&0&0&-1&-1&1&0&0&0\\0&1&1&0&0&-1&-1&1&0&0\\0&1&0&1&0&0&-1&-1&1&0\\0&-1&1&0&1&0&0&-1&-1&1\end{pmatrix}$$

FIGURE 2. The matrix $\mathcal{M}_{10}^{(3,2)}$ with $\mu_{i,j}^{(3,2)}=\mu_e^{(3,2)}(i,j)-\mu_o^{(3,2)}(i,j)$

The two matrices $\mathcal{M}_n^{(r,s)}$ and $\Gamma_n^{(r,s)}$ do not seem relevant, but the following theorem gives a connection between them.

Theorem 2.5. *$\mathcal{M}_n^{(r,s)}\cdot\Gamma_n^{(r,s)}=I_n$, the identity matrix.*

Proof. We show that

$$\sum_{h=1}^{n}\left\{\mu_e^{(r,s)}(i,h)-\mu_o^{(r,s)}(i,h)\right\}p_r(h-j,rj-s)=\begin{cases}0 \text{ if } i\neq j,\\1 \text{ if } i=j,\end{cases}$$

where $i,j\in\{1,2,\dots,n\}$.

Let $M^{(r,s)}(n,k)=M_e^{(r,s)}(n,k)\cup M_o^{(r,s)}(n,k)$ and $P_r(n,k)$ be the set of (k,r)-partitions of n. For sets A and B, $A\times B$ is the set of ordered pairs (a,b) when $a\in A$ and $b\in B$. First, for $i\neq j$, we prove

$$\sum_{h=1}^{n}\mu_e^{(r,s)}(i,h)p_r(h-j,rj-s)=\sum_{h=1}^{n}\mu_o^{(r,s)}(i,h)p_r(h-j,rj-s),$$

by constructing an involution on $\bigcup_{h=1}^{n}\left(M^{(r,s)}(i,h)\times P_r(h-j,rj-s)\right)$. Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_\ell)$ and $\tau=(\tau_1,\tau_2,\dots,\tau_m)$ with $(\lambda,\tau)\in M^{(r,s)}(i,h)\times P_r(h-j,rj-s)$. We choose $\phi(\lambda,\tau)=(\bar\lambda,\bar\tau)$ as follows.

If $\lambda_\ell\geq\tau_m$, we set $\bar\lambda=(\lambda_1,\lambda_2,\dots,\lambda_{\ell-1})$ and $\bar\tau=(\tau_1,\tau_2,\dots,\tau_m,\lambda_\ell)$. Hence, $\bar\lambda\in M^{(r,s)}(i,h+\lambda_\ell)$ and $\bar\tau\in P_r(h-j+\lambda_\ell,rj-s)$ since $\lambda_\ell\leq rh-(s-1)=(rj-s+1)+r\sum_{i=1}^{m}\tau_i$. Therefore, $\phi(\lambda,\tau)=(\bar\lambda,\bar\tau)\in M^{(r,s)}(i,h+\lambda_\ell)\times P_r(h-j+\lambda_\ell,rj-s)$.

If $\lambda_\ell<\tau_m$, let $\bar\lambda=(\lambda_1,\lambda_2,\dots,\lambda_\ell,\tau_m)$ and $\bar\tau=(\tau_1,\tau_2,\dots,\tau_{m-1})$. We have $\bar\lambda\in M^{(r,s)}(i,h-\tau_m)$ and $\bar\tau\in P_r(h-j-\tau_m,rj-s)$ since $\tau_m\leq(rj-$

$s+1)+r(\sum_{i=1}^{m-1}\tau_i) = r(h-\tau_m)-(s-1)$. Therefore, $\phi(\lambda,\tau) = (\bar{\lambda},\bar{\tau}) \in M^{(r,s)}(i,h-\tau_m)\times P_r(h-j-\tau_m, rj-s)$.

In both cases, the numbers of parts of λ and $\bar{\lambda}$ differ by 1, so they have opposite parities. Thus, the map ϕ is an involution on $\bigcup_{h=1}^{n}\big(M^{(r,s)}(i,h)\times P_r(h-j,\, rj-s)\big)$ and we have

$$\sum_{h=1}^{n}\mu_e^{(r,s)}(i,h)p_r(h-j,rj-s) = \sum_{h=1}^{n}\mu_o^{(r,s)}(i,h)p_r(h-j,rj-s).$$

Now, it remains to show that

$$\sum_{h=1}^{n}\mu_e^{(r,s)}(i,h)p_r(h-i,ri-s) - \sum_{h=1}^{n}\mu_o^{(r,s)}(i,h)p_r(h-i,ri-s) = 1.$$

For $\lambda \in M^{(r,s)}(i,h)$ and $\tau \in P_r(h-i,ri-s)$, it must be $h=i$ since λ and τ are partitions of $i-h$ and $h-i$, respectively. Therefore, there is a unique partition pair $(\emptyset,\emptyset)\in\bigcup_{h=1}^{n}\Big(M_e^{(r,s)}(i,h)\times P_r(h-i,ri-s)\Big)$ and there is no element in $\bigcup_{h=1}^{n}\Big(M_o^{(r,s)}(i,h)\times P_r(h-i,ri-s)\Big)$, which completes the proof. $\square$

For example, two partition pairs $((1),(1,1,1))$ and $((1),(1,2))$ are elements in the set $M_o^{(3,2)}(5,4)\times P_3(3,1)$. According to the proof of Theorem 2.5, $\phi(\lambda,\tau) = (\emptyset,(1,1,1,1)) \in M_e^{(3,2)}(4,4)\times P_3(4,1)$ when $(\lambda,\tau) = ((1),(1,1,1))$ and $\phi(\lambda,\tau) = ((1,2),(1)) \in M_e^{(3,2)}(7,4)\times P_3(1,1)$ when $(\lambda,\tau) = ((1),(1,2))$.

3. Results on double-complete partitions

We first rewrite Definition 1.3 and Theorem 1.4 by using the notation for a partition $\lambda = (\lambda_1,\ldots,\lambda_\ell)$, where the λ_i are nondecreasing.

Definition 3.1. A double-complete partition of n is a partition $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ such that each integer m with $2\le m\le n-2$ can be represented in at least two different ways as $\sum_{i=1}^{\ell}\alpha_i\lambda_i$ with $\alpha_i\in\{0,1\}$.

Theorem 3.2. *For a positive integer $n\ge 5$, a partition $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ of n is double-complete if and only if λ satisfies one of the following*:

(a) $\lambda_1=\lambda_2=\lambda_3=1$, *there is* $4\le i\le\ell$ *such that* $\lambda_i=2$, *and* $\lambda_i\le -1+\sum_{j=1}^{i-1}\lambda_j$ *for each* $i=4,\ldots,\ell$.

(b) $\lambda_1=\lambda_2=1$, $\lambda_3=2$, *there is* $4\le i\le\ell$ *such that* $\lambda_i=3$, *and* $\lambda_i\le -1+\sum_{j=1}^{i-1}\lambda_j$ *for each* $i=4,\ldots,\ell$.

Proof. ($\Rightarrow$) Suppose that there exists $i\ge 4$ such that $\lambda_i\ge\sum_{j=1}^{i-1}\lambda_j$. For such i, $\sum_{j=1}^{i-1}\lambda_j - 1 = \sum_{j=2}^{i-1}\lambda_j$ cannot be represented in two different ways as a sum of parts of λ, which is a contradiction.

($\Leftarrow$) We prove it by using the induction on ℓ. First, partitions of $n\ge 5$ into 4 parts satisfying the conditions (a) or (b) are $(1,1,1,2)$ and $(1,1,2,3)$, and they are double-complete partitions by definition. Suppose $(\lambda_1, \ldots, \lambda_\ell)$ is a partition of $n \geq 5$ into $\ell \geq 4$ parts satisfying the conditions (a) or (b), and it is double-complete. We claim that $(\lambda_1, \ldots, \lambda_\ell, \lambda_{\ell+1})$ with $\lambda_{\ell+1} \leq n-1$ is a double-complete partition of $n + \lambda_{\ell+1}$. Since $(\lambda_1, \ldots, \lambda_\ell)$ is a double-complete partition of n, each integer m with $2 \leq m \leq n-2$ can be represented in at least two different ways as a sum of its parts. Hence, it remains to show that each integer m' with $n - 1 \leq m' \leq n + \lambda_{\ell+1} - 2$ can be represented in at least two different ways as a sum of parts of $(\lambda_1, \ldots, \lambda_\ell, \lambda_{\ell+1})$.

If $\lambda_{\ell+1} + 2 \leq m' \leq n + \lambda_{\ell+1} - 2$, then $m' - \lambda_{\ell+1}$ can be represented in at least two different ways as a sum of parts of $(\lambda_1, \ldots, \lambda_\ell)$, which follows that m' can be represented in at least two different ways as a sum of parts of $(\lambda_1, \ldots, \lambda_\ell, \lambda_{\ell+1})$.

We now consider for $n-1 \leq m' \leq \lambda_{\ell+1}+1$, that is either $m' = n-1 = \lambda_{\ell+1}$ or $m' = 1+\lambda_{\ell+1}$ since $\lambda_{\ell+1} \leq n-1$. In the former case, $m' = n-1$ can be written as $\sum_{i=2}^{\ell} \lambda_i$ and $\lambda_{\ell+1}$. In the latter case, either $m' = n-1 = \sum_{i=2}^{\ell} \lambda_i = 1+\lambda_{\ell+1}$ or $m' = n = \sum_{i=1}^{\ell} \lambda_i = 1 + \lambda_{\ell+1}$ holds. This completes the proof. □

Now, we provide a proof of Theorem 1.9 by using Theorem 3.2.

Proof of Theorem 1.9. Note that $dc(n) = dc_1(n) + dc_2(n) - dc_3(n)$ by Theorem 3.2 and the inclusion-exclusion principle. The proof of the identities for $dc_1(n)$, $dc_2(n)$, and $dc_3(n)$ are almost the same, so we here only prove the identity for $dc_1(n)$. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition with at least three 1's and one 2 as its parts and $\lambda^* = (\lambda_1, \ldots, \lambda_m)$ such that $m \leq \ell$ is the greatest integer satisfying $\lambda_i \leq -1 + \sum_{j=1}^{i-1} \lambda_j$ for $i = 4, \ldots, m$. It follows from Theorem 3.2 that if $\sum_{i=1}^{m} \lambda_i = n$ and $m < \ell$, then $\lambda_{m+1} \geq n$. Hence, similarly to the proof of Theorem 1.8, we have

$$\sum_{n=5}^{\infty} p(n-5)q^n = q^5 \sum_{n=0}^{\infty} p(n)q^n = q^5 \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=5}^{\infty} \frac{dc_1(n)q^n}{\prod_{j=n}^{\infty}(1-q^j)}.$$ □

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