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BROUWER DEGREE FOR MEAN FIELD EQUATION ON GRAPH

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Abstract. Let u be a function on a connected finite graph $G = (V, E)$. We consider the mean field equation

$$-\Delta u = \rho\left(\frac{he^u}{\int_V he^u d\mu} - \frac{1}{|V|}\right), \tag{1}$$

where Δ is μ-Laplacian on the graph, $\rho \in \mathbb{R}\backslash\{0\}$, $h : V \to \mathbb{R}^+$ is a function satisfying $\min_{x\in V} h(x) > 0$. Following Sun and Wang [15], we use the method of Brouwer degree to prove the existence of solutions to the mean field equation (1). Firstly, we prove the compactness result and conclude that every solution to the equation (1) is uniformly bounded. Then the Brouwer degree can be well defined. Secondly, we calculate the Brouwer degree for the equation (1), say

$$d_{\rho,h} = \begin{cases} -1, & \rho > 0, \\ 1, & \rho < 0. \end{cases}$$

Consequently, the equation (1) has at least one solution due to the Brouwer degree $d_{\rho,h} \neq 0$.

1. Introduction

In a series of works [7–9], Grigor'yan, Lin and Yang solved several discrete differential equations on graphs, say the Yamabe equation, the Kazdan-Warner equation and the Schrödinger equation, by finding critical points for various functionals. Since then, by the variational method, Huang, Lin and Yau [10] solved the mean field equations on graphs, and Zhu [16] solved the mean field equations of the equilibrium turbulence on graphs. Recently, Lin and Yang [12] studied a heat flow for the mean field equation on a finite graph. Their results implied that the solution of heat flow converges to the solution of the mean field equation. Earlier results of the mean field equations on a closed Riemann surface are referred to [5, 6, 14].

In [11], Li defined the Leray-Schauder degree for the mean fields equation on a closed Riemann surface. Chen and Lin [3] gave the specific formula of the

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Leray-Schauder degree. Recently, by the Brouwer degree, Sun and Wang [15] extended the results of [3, 11] to the Kazdan-Warner equations on a connected finite graph. In this paper, we only care about the existence of solutions to mean field equations on finite graphs by method of Brouwer degree [15]. To state our results, we recall some definitions on graphs. Let $G = (V, E)$ be a graph, where V denotes the vertex set and E denotes the edge set. Throughout this paper, we always assume that G satisfies the following conditions (a)-(d).

(a) (Finite) There exist only finite vertexs $x \in V$.
(b) (Connected) For any $x, y \in V$, there exist finite edges connecting x and y.
(c) (Symmetric) Let $w : V \times V \to \mathbb{R}$ be a positive symmetric weight, i.e., $w_{xy} > 0$ and $w_{xy} = w_{yx}$ for any $x, y \in V$.
(d) (Positive finite measure) $\mu : V \to \mathbb{R}^+$ defines a positive finite measure on graph G.

The space of real functions on V is denoted by $V^{\mathbb{R}}$, which is a finite dimensional linear space due to finiteness of G. For any function $u \in V^{\mathbb{R}}$, the μ-Laplacian of u at any vertex x is defined by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x)),$$

where $y \sim x$ means $xy \in E$. For any function $h \in V^{\mathbb{R}}$, the integral of h on V is denoted by

$$\int_V h d\mu = \sum_{x \in V} \mu(x) h(x),$$

and an integral average of h is denoted by

$$\bar{h} = \frac{1}{|V|} \int_V h d\mu = \frac{1}{|V|} \sum_{x \in V} \mu(x) h(x),$$

where $|V| = \sum_{x \in V} \mu(x)$ stands for the volume of V.

According to Liu and Yang [13], $L^p(V)$ on graphs is defined by

$$L^p(V) = \{u \in V^{\mathbb{R}} : \|u\|_{L^p(V)} < +\infty\}, \quad 1 \le p \le \infty,$$

where the norm of $u \in L^p(V)$ is defined by

$$\|u\|_{L^p(V)} = \begin{cases} (\int_V |u|^p d\mu)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max_{x \in V} |u(x)|, & p = \infty. \end{cases}$$

We consider the following mean field equation

$$-\Delta u = \rho \left(\frac{he^u}{\int_V he^u d\mu} - \frac{1}{|V|} \right), \tag{2}$$

where Δ is μ-Laplacian, $\rho \in \mathbb{R}\backslash\{0\}$, $h \in V^{\mathbb{R}^+}$ and $\min_{x\in V} h(x) > 0$. In order to solve the equation (2), let $v = u - \log \int_V he^u d\mu$. Then we can get

$$-\Delta v = \rho h e^v - \frac{\rho}{|V|}. \tag{3}$$

To state the Brouwer degree related to (3), according to Sun and Wang [15], we introduce a map $F_{\rho,h} \in C(V^{\mathbb{R}}, V^{\mathbb{R}})$ denoted by

$$F_{\rho,h} : V^{\mathbb{R}} \to V^{\mathbb{R}}, \quad v \mapsto -\Delta v - \rho h e^v + \frac{\rho}{|V|}.$$

Under the norm $\|\cdot\|_{L^\infty(V)}$, the ball in $V^{\mathbb{R}}$ with center at the origin and radius R is denoted by B_R. If $v_0 \notin F_{\rho,h}(\partial B_R)$ is a regular value, then the Brouwer degree is defined by

$$\deg(F_{\rho,h}, B_R, v_0) = \sum_{v\in B_R, F_{\rho,h}(v)=v_0} \operatorname{sgn}\det(DF_{\rho,h}(v)).$$

According to Chang [2], the constraint that v_0 is a regular value can be relaxed to any value, so $F_{\rho,h}$ can define the Brouwer degree as long as it satisfies $v_0 \notin F_{\rho,h}(\partial B_R)$. To calculate the Brouwer degree for the equation (3), by Chang [2], we introduce two lemmas as follow.

Lemma 1.1 (Homotopic invariance [2])**.** *If $\phi : \bar{B}_R \times [0,1] \to V^{\mathbb{R}}$ is continuous and $v_0 \notin \phi(\partial B_R \times [0,1])$, then*

$$\deg(\phi(\cdot,t), B_R, v_0) = \text{constant}.$$

Lemma 1.2 (Kronecker existence [2])**.** *If $v_0 \notin F_{\rho,h}(\partial B_R)$ and $\deg(F_{\rho,h}, B_R, v_0) \neq 0$, then $F^{-1}_{\rho,h}(v_0) \neq \varnothing$*

Let $v_0 = 0$, according to Lemma 1.2, the equation (3) has at least one solution in B_R as long as $0 \notin F_{\rho,h}(\partial B_R)$ and $\deg(F_{\rho,h}, B_R, 0) \neq 0$. In order to $0 \notin F_{\rho,h}(\partial B_R)$, i.e., the Brouwer degree is well defined, the compactness result of the mean field equation (3) is needed.

Theorem 1.3. *Let $G = (V,E)$ be a graph satisfying conditions* (a)-(d)*. If $\rho \in \mathbb{R}\backslash\{0\}$, $h \in V^{\mathbb{R}^+}$ and $\min_{x\in V} h(x) > 0$, then there exists a constant C only depending on h, ρ and G such that every solution v to* (3) *satisfies*

$$\max_{x\in V} |v(x)| \le C.$$

By Theorem 1.3 we conclude that there is no solution on the boundary ∂B_R for R large. Therefore, the Brouwer degree $\deg(F_{\rho,h}, B_R, 0)$ is well defined as long as R is larger than $C(\rho, h, G)$. Applying the homotopic invariance, we have that $\deg(F_{\rho,h}, B_R, 0)$ is independent of R. Then the Brouwer degree for the equation (3) is defined by

$$d_{\rho,h} := \lim_{\mathbb{R}\to+\infty} \deg(F_{\rho,h}, B_R, 0). \tag{4}$$

Since u and v have the same compactness result, we can prove the existence of solutions to the equation (2) by calculating the Brouwer degree $d_{\rho,h}$ for the equation (3). As for the Brouwer degree $d_{\rho,h}$, we have following results.

Theorem 1.4. *Let $G = (V, E)$ be a graph satisfying conditions* (a)-(d). *If $\rho \in \mathbb{R}\backslash\{0\}$, $h \in V^{\mathbb{R}^+}$ and $\min_{x\in V} h(x) > 0$, then*

$$d_{\rho,h} = \begin{cases} -1, & \rho > 0, \\ 1, & \rho < 0. \end{cases}$$

Hence, Theorem 1.4 and the Kronecker existence show that the mean field equation (2) has at least one solution if $\rho \in \mathbb{R}\backslash\{0\}$ and $\min_{x\in V} h(x) > 0$.

Following the lines of [15], we prove Theorem 1.3 by blow-up analysis, which is due to Brezis and Merle [1]. After establishing the compactness result of the mean field equation, we calculate the Brouwer degree $d_{\rho,h}$ for the mean field equation (3). Compared with [15], where the Kazdan-Warner equation was studied, our results are nontrivial extensions.

The remaining parts of this paper are organized as follow: In Section 2, we give some basic inequalities on finite graphs. In Section 3, we prove that every solution to equation (3) is uniformly bounded and Theorem 1.3 is proved. Then the Brouwer degree $d_{\rho,h}$ for the equation (3) can be well defined. In Section 4, we calculate the Brouwer degree $d_{\rho,h}$ for the equation (3) and prove Theorem 1.4. Throughout this paper, we do not distinguish sequence and its subsequence, we use C to denote absolute constants without distinguishing them.

2. Preliminary analysis

Any two norms on $V^{\mathbb{R}}$ are equivalent since G is a finite graph and $V^{\mathbb{R}}$ is a finite dimensional linear space. Denote

$$V_0^{\mathbb{R}} = \left\{ u \in V^{\mathbb{R}} : \int_V u d\mu = 0 \right\}.$$

Then we can prove that $\max_V |\Delta u|$, $\max_V u - \min_V u$ are norms of u on $V_0^{\mathbb{R}}$. Next, according to Sun and Wang [15], we will prove the following elliptic estimate, which they did not give specific proof.

Lemma 2.1 (Elliptic estimate [15])**.** *There exists a positive constant C such that for all $u \in V^{\mathbb{R}}$*

$$\max_V u - \min_V u \leq C \max_V |\Delta u|. \tag{5}$$

Proof. Firstly, we can prove the elliptic estimate is true for all $u \in V_0^{\mathbb{R}}$. Suppose not. Then for any $k \in \mathbb{N}$, there exists $u_k \in V_0^{\mathbb{R}}$ such that

$$\max_V u_k - \min_V u_k > k \max_V |\Delta u_k|, \quad \int_V u_k d\mu = 0.$$

Let $\tilde{u}_k = u_k/(\max_V u_k - \min_V u_k)$. Then we have

$$\max_V \tilde{u}_k - \min_V \tilde{u}_k = 1, \quad \max_V |\Delta \tilde{u}_k| < \frac{1}{k}, \quad \int_V \tilde{u}_k d\mu = 0.$$

Since

$$|\tilde{u}_k| = |\tilde{u}_k - \frac{1}{|V|}\int_V \tilde{u}_k d\mu| \le \max_V \tilde{u}_k - \min_V \tilde{u}_k = 1,$$

then $\tilde{u}_k$ is bounded in $V_0^{\mathbb{R}}$, and there is a subsequence of $\tilde{u}_k$ and $\tilde{u}_0 \in V_0^{\mathbb{R}}$ such that $\tilde{u}_k \to \tilde{u}_0$ in $V_0^{\mathbb{R}}$ as $k \to \infty$. Thereby, taking $k \to \infty$, we have

$$\max_V \tilde{u}_0 - \min_V \tilde{u}_0 = 1, \quad \max_V |\Delta \tilde{u}_0| = 0, \quad \int_V \tilde{u}_0 d\mu = 0.$$

Then the second and the third equalities imply $\tilde{u}_0 \equiv 0$, which contradicts the first equality.

Secondly, if $u \in V^{\mathbb{R}}$ we can let $u' = u - \bar{u}$, then $u' \in V_0^{\mathbb{R}}$ and repeat the above process. This ends the proof of the lemma. □

Let $u^+ = \max\{u, 0\}$ and $u^- = (-u)^+$. Sun and Wang [15] have proved Kato's inequality.

Lemma 2.2 (Kato's inequality [15])**.**

$$\Delta u^+ \ge \chi_{\{u>0\}} \Delta u. \tag{6}$$

3. Blow-up analysis

We first consider the blow-up behavior of the mean field equation (3).

Lemma 3.1. *Let $G = (V, E)$ be a graph satisfying conditions* (a)-(d). *Let $v_n \in V^{\mathbb{R}}$ be a sequence of solutions to*

$$-\Delta v_n = \rho_n h_n e^{v_n} - \frac{\rho_n}{|V|}, \tag{7}$$

where $h_n \in V^{\mathbb{R}}$ and $\rho_n \in \mathbb{R}$ satisfy

$$\lim_{n\to\infty} h_n = h, \quad \lim_{n\to\infty} \rho_n = \rho.$$

Then after passing to a subsequence, only one of the following alternatives holds.

(I) *v_n is bounded.*

(II) *v_n uniformly diverges to $-\infty$.*

(III) *There exists x_0 such that $v_n(x_0)$ diverges to $+\infty$, furthermore, v_n is bounded from below in V and above in $\{x \in V : \rho h(x) > 0\}$.*

Proof. Firstly, assume v_n is bounded from above. Then (7) implies that Δv_n is bounded. According to Lemma 2.1 and using the elliptic estimate (5) we have

$$\max_V v_n - \min_V v_n \le C \max_V |\Delta v_n| \le C. \tag{8}$$

Next, we discuss the convergence behavior of $\min_V v_n$ in two cases. If $\min_V v_n$ is bounded in below, we obtain the first alternative. If $\liminf_{n\to\infty} \min_V v_n =$

$-\infty$, then (8) implies that up to a subsequence v_n uniformly diverges to $-\infty$, so the second alternative holds.

Secondly, assume v_n is not bounded from above, i.e., $\limsup_{n\to\infty} v_n = +\infty$. Since G is a finite graph, without loss of generality, we may assume that there exists x_0 and up to a subsequence of v_n such that

$$v_n(x_0) = \max_V v_n \to +\infty, \qquad n \to \infty.$$

Then applying Kato's inequality (6) in Lemma 2.2, we have

$$\begin{aligned}
-\Delta v_n^- &= -\Delta(-v_n)^+ \\
&\le -\chi_{\{-v_n>0\}}\Delta(-v_n) \\
&= \chi_{\{v_n<0\}}\big(\frac{\rho_n}{|V|} - \rho_n h_n e^{v_n}\big) \\
&\le \frac{\rho_n^+}{|V|} + \rho_n^+ h_n^- + \rho_n^- h_n^+.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\Delta v_n^-\|_{L^1(V)} &= \int_V |\Delta v_n^-| d\mu \\
&= \int_{\{\Delta v_n^- \ge 0\}} \Delta v_n^- d\mu - \int_{\{\Delta v_n^- < 0\}} \Delta v_n^- d\mu \\
&= -2\int_{\{\Delta v_n^- < 0\}} \Delta v_n^- d\mu \\
&\le 2\int_{\{\Delta v_n^- < 0\}} \frac{\rho_n^+}{|V|} + \rho_n^+ h_n^- + \rho_n^- h_n^+ d\mu \\
&\le C,
\end{aligned}$$

which implies $\max_V |\Delta v_n^-| \le C$. According to Lemma 2.1 there exists a subsequence such that

$$\max_V v_n^- = \max_V v_n^- - \min_V v_n^- \le C.$$

Therefore, v_n is bounded from below in V. Then for any $x_1 \in V$ we have

$$\begin{aligned}
\rho_n h_n(x_1) e^{v_n(x_1)} - \frac{\rho_n}{|V|} &= -\Delta v_n(x_1) \\
&= \frac{1}{\mu(x_1)} \sum_{y\sim x_1} w_{x_1 y}(v_n(x_1) - v_n(y)) \\
&\le C v_n(x_1) + C,
\end{aligned}$$

which implies

$$\rho_n h_n(x_1) \le (C v_n(x_1) + C + \frac{\rho_n}{|V|}) e^{-v_n(x_1)}.$$

Let $n \to \infty$. Then $\rho h(x_1) \le 0$ if and only if $\limsup_{n\to\infty} v_n(x_1) = +\infty$, which implies that v_n is bounded in $\{x \in V : \rho h(x) > 0\}$. □

Next, we prove the compactness result of the mean field equation (3).

Lemma 3.2. *Let $G = (V, E)$ be a graph satisfying conditions* (a)-(d). *Suppose that there exists a positive constant A depending only on h and ρ such that*

(i) $\max_V(|h| + |\rho|) \le A$.
(ii) *If $\rho h(x) > 0$ for some $x \in V$, then $\rho h(x) \ge A^{-1}$.*
(iii) *If $\rho > 0$, then $\rho \ge A^{-1}$.*
(iv) *If $\rho < 0$, then $\rho \le -A^{-1}$ and $\min_V h \ge A^{-1}$.*

Then there exists a positive constant C depending only on A and G such that every solution to (3) *satisfies*

$$\max_{x\in V} |v(x)| \le C.$$

Proof. We give proof by contradiction. Let v_n be a sequence of solution to the equation (7). Suppose v_n blows up as n converge to ∞ satisfying

$$\lim_{n\to\infty} \|v_n\|_{L^\infty(V)} = \infty.$$

Meanwhile, h_n and ρ_n satisfy the conditions (i)-(iv) and

$$\lim_{n\to\infty} h_n = h, \quad \lim_{n\to\infty} \rho_n = \rho.$$

If v_n uniformly diverges to $-\infty$, then we consider

$$-\Delta(v_n - \min_V v_n) = \rho_n h_n e^{v_n} - \frac{\rho_n}{|V|},$$

which yields that $v_n - \min_V v_n$ is bounded according to Lemma 2.1. So $v_n - \min_V v_n$ diverges to a solution φ of the equation

$$-\Delta\varphi = -\frac{\rho}{|V|}, \ \min_V \varphi = 0.$$

But this implies $\rho = 0$ and $\varphi = 0$. We may assume $\rho_n = 0$ by conditions (iii) and (iv). Then in equation (7) we can obtain $v_n \equiv C$, which contradicts that v_n diverges to $-\infty$.

If $\max_V v_n$ diverges to $+\infty$, applying the conclusion (III) of Lemma 3.1, we may assume v_n is bounded from below in V and above in $\Omega = \{x \in V : \rho h(x) > 0\}$. When n is large enough, by condition (ii) we have

$$\begin{aligned}\Omega &\subset \{x \in V : \rho_n h_n(x) > 0\}\\ &\subset \{x \in V : \rho_n h_n(x) \ge A^{-1}\}\\ &\subset \{x \in V : \rho h(x) \ge A^{-1}\}\\ &\subset \Omega.\end{aligned}$$

Then we have

$$\begin{aligned}\rho_n = \int_V \frac{\rho_n}{|V|} d\mu &= \int_V \rho_n h_n e^{v_n} d\mu\\ &= \int_\Omega \rho_n h_n e^{v_n} d\mu + \int_{V\setminus\Omega} \rho_n h_n e^{v_n} d\mu\end{aligned}$$

$$\leq C - \int_V (\rho_n h_n)^- e^{v_n} d\mu,$$

which implies that $\int_V (\rho_n h_n)^- e^{v_n} d\mu \leq C$. Therefore,

$$\begin{aligned}\|\Delta v_n\|_{L^1(V)} &= \int_V |\Delta v_n| d\mu \\ &\leq \int_V |\rho_n h_n| e^{v_n} d\mu + \int_V \frac{|\rho_n|}{|V|} d\mu \\ &= \int_\Omega (\rho_n h_n)^+ e^{v_n} d\mu + \int_V (\rho_n h_n)^- e^{v_n} d\mu + |\rho_n| \\ &\leq C.\end{aligned}$$

By Lemma 2.1, we have

$$\max_V v_n \leq \min_V v_n + C,$$

which implies that $\min_V v_n$ diverges to $+\infty$ and then v_n must diverge to $+\infty$. Hence, $\Omega = \varnothing$. So for any $x \in V$, we have $\rho h(x) \leq 0$.

We may assume $\rho_n h_n(x) \leq 0$ by condition (ii), otherwise, if $\rho_n h_n(x) > 0$, then we have $\rho_n h_n(x) \geq A^{-1}$, which contradicts with $\rho h(x) \leq 0$ as $n \to \infty$. Thus, we have

$$\rho_n = \int_V \rho_n h_n e^{v_n} d\mu \leq 0.$$

According to the previous analysis, we obtain $\rho_n < 0$. Then by condition (iv), we have $\min_V h_n \geq A^{-1}$, thus

$$1 = \int_V h_n e^{v_n} d\mu \geq C A^{-1} e^{\min_V v_n}.$$

Consequently, we have $\min_V v_n \leq C$, which contradicts that $\min_V v_n$ diverges to $+\infty$. This ends the proof of Lemma 3.2. □

Remark 3.3. Actually, the conclusion (III) of Lemma 3.1 is reinforced when $\min_{x\in V} h(x) > 0$. If $\rho > 0$, we can get v_n is bounded in V. If $\rho < 0$, we can only get v_n is bounded from below in V. But this does not affect the proof of Lemma 3.2 because ρ and h in Lemmas 3.1, 3.2 are more general.

Remark 3.4. The conditions (i)-(iv) are necessary in Lemma 3.2. For every positive number ε, taking $\rho = \pm\varepsilon^{\frac{1}{2}}$, $h = \varepsilon^{\frac{1}{2}}$ in the equation (3), we have

$$-\Delta(-\ln \varepsilon^{\frac{1}{2}} |V|) = \pm\varepsilon^{\frac{1}{2}} (\varepsilon^{\frac{1}{2}} e^{-\ln \varepsilon^{\frac{1}{2}} |V|} - \frac{1}{|V|}).$$

When $\rho = \varepsilon^{\frac{1}{2}}$, the condition (i) is necessary since $\lim_{\varepsilon\to+\infty} -\ln \varepsilon^{\frac{1}{2}} |V| = -\infty$, and the condition (ii) and (iii) are necessary since $\lim_{\varepsilon\to 0} -\ln \varepsilon^{\frac{1}{2}} |V| = +\infty$. When $\rho = -\varepsilon^{\frac{1}{2}}$, the first part of condition (iv) is necessary since $\lim_{\varepsilon\to 0} -\ln \varepsilon^{\frac{1}{2}} |V| = +\infty$.

If we let $\rho = -1$ and $h = \varepsilon$, then we have

$$-\Delta(-\ln\varepsilon|V|) = -\varepsilon e^{-\ln\varepsilon|V|} + \frac{1}{|V|}.$$

The second part of condition (iv) is necessary since $\lim_{\varepsilon\to 0} -\ln\varepsilon|V| = +\infty$.

Now we can give the proof of Theorem 1.3 by Lemma 3.2. In fact, it is easy to prove that the conditions (i)-(iv) of Lemma 3.2 hold under the conditions $\rho \in \mathbb{R}\backslash\{0\}$, $h \in V^{\mathbb{R}^+}$ and $\min_{x\in V} h(x) > 0$, i.e., we can find a positive constant A depending only on h and ρ such that

$$\max_{x\in V} |v(x)| \le C(A, G).$$

4. Brouwer degree

In this section, we will prove Theorem 1.4, precisely we will calculate the Brouwer degree $d_{\rho,h}$, which is defined by (4). According to Sun and Wang [15], the Brouwer degree $d_{\rho,h}$ is well defined by the compactness result for the equation (3).

Step 1. If $\rho > 0$, $h \in V^{\mathbb{R}^+}$ and $\min_{x\in V} h(x) > 0$, we have $d_{\rho,h} = -1$.

Proof. Let v_t satisfy

$$(9) \qquad -\Delta v_t = [t + (1-t)\rho][t + (1-t)h]e^{v_t} - \frac{t\varepsilon + (1-t)\rho}{|V|}, \quad t \in [0,1],$$

where $\varepsilon > 0$ is sufficiently small. Applying Lemma 3.2, we have v_t is uniformly bounded with respect to t. Letting $t = 0$ in (9), we have

$$(10) \qquad -\Delta v_0 = \rho h e^{v_0} - \frac{\rho}{|V|}.$$

Since v_0 is uniformly bounded, the Brouwer degree $d_{\rho,h}$ for the equation (10) is well defined. Letting $t = 1$ in (9), we have

$$(11) \qquad -\Delta v_1 = e^{v_1} - \frac{\varepsilon}{|V|}.$$

Similarly, the Brouwer degree for the equation (11) is well defined. According to Lemma 1.1, the equation (10) and the equation (11) have the same Brouwer degree. Thus, we can calculate the Brouwer degree of the equation (11). In fact, the equation (11) only has a unique solution $v_1 = \ln\varepsilon/|V|$ when ε is sufficiently small. As for the specific proof, we refer readers to Sun and Wang [15].

Now we rewrite the operator $F_{\varepsilon,1}$ as follows:

$$F_{\varepsilon,1}(v_1) = -\Delta \begin{pmatrix} v_1(x_1) \\ \vdots \\ v_1(x_n) \end{pmatrix} + \frac{1}{|V|} \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon \end{pmatrix} - \begin{pmatrix} e^{v_1(x_1)} \\ \vdots \\ e^{v_1(x_n)} \end{pmatrix},$$

where $x_i \in V$, $i = 1, \ldots, n$. According to Chung [4], $L := -\Delta = (l_{i,j})_{n\times n}$ is a symmetric nonnegative matrix and 0 is the eigenvalue of L with multiplicity one. Hence, for sufficiently small ε we have

$$\det(DF_{\varepsilon,1}) = \det(-\Delta - \frac{\varepsilon E}{|V|}) < 0.$$

By the homotopic invariance, we have

$$d_{\rho,h} = \lim_{\varepsilon\to 0} d_{\varepsilon,1} = \operatorname{sgn}\det(DF_{\varepsilon,1}) = -1. \qquad \square$$

Step 2. If $\rho < 0$, $h \in V^{\mathbb{R}^+}$ and $\min_{x\in V} h(x) > 0$, we have $d_{\rho,h} = 1$.

Proof. Let v_t satisfy

$$-\Delta v_t = [(1-t)\rho - t][(1-t)h + t]e^{v_t} - \frac{(1-t)\rho - t}{|V|}, \quad t \in [0,1].$$

As the same analysis as Step 1, we can calculate the Brouwer degree of

$$-\Delta v_1 = -e^{v_1} + \frac{1}{|V|}.$$

And we can claim that $v_1 = -\ln|V|$ is the unique solution. In fact, v_1 cannot be anything but constant function. For otherwise, let $v_1(x_0) = \max_V v_1$ and $v_1(x_1) = \min_V v_1$. Then we have

$$-e^{\max_V v_1} + \frac{1}{|V|} = -\Delta v_1(x_0) = \frac{1}{\mu(x_0)}\sum_{y\sim x_0} w_{x_0 y}(v_1(x_0) - v_1(y)) > 0,$$

$$-e^{\min_V v_1} + \frac{1}{|V|} = -\Delta v_1(x_1) = \frac{1}{\mu(x_1)}\sum_{y\sim x_1} w_{x_1 y}(v_1(x_1) - v_1(y)) < 0,$$

which is a contradiction. Hence, we have

$$d_{\rho,h} = d_{-1,1} = \operatorname{sgn}\det(DF_{-1,1}) = \operatorname{sgn}\det(-\Delta + \frac{E}{|V|}) = 1. \qquad \square$$

We finish the proof of Theorem 1.4. Finally, under the conditions $\rho \in \mathbb{R}\backslash\{0\}$ and $\min_{x\in V} h(x) > 0$, the Kronecker existence shows that the mean field equation (2) has at least one solution due to $d_{\rho,h} \neq 0$.

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