East Asian Math. J.
Vol. 38 (2022), No. 5, pp. 643–647
http://dx.doi.org/10.7858/eamj.2022.039

# THE FOCK-DIRICHLET SPACE AND THE FOCK-NEVANLINNA SPACE

Hong Rae Cho* and Soohyun Park

Abstract. Let $F^2$ denote the space of entire functions $f$ on $\mathbb{C}$ that are square integrable with respect to the Gaussian measure $dG(z) = \frac{1}{\pi}e^{-|z|^2}dA(z)$, where $dA(z) = dxdy$ is the ordinary area measure. The Fock-Dirichlet space $F^2_{\mathcal{D}}$ consists of all entire functions $f$ with $f' \in F^2$. We estimate Taylor coefficients of functions in the Fock-Dirichlet space. The Fock-Nevanlinna space $F^2_{\mathcal{N}}$ consists of entire functions that possesses just a bit more integrability than square integrability. In this note we prove that $F^2_{\mathcal{D}} = F^2_{\mathcal{N}}$.

## 1. The Fock-Dirichlet space

We consider the Gaussian probability measure

$$dG(z) = \frac{1}{\pi}e^{-|z|^2}dA(z),$$

where $dA(z) = dxdy$ is the ordinary area measure on $\mathbb{C}$. We define

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}\, dG(z).$$

Let $F^2$ denote the space of entire functions $f$ such that the norm

$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2\, dG(z)$$

is finite. The Fock space $F^2$ is the Hilbert space with the reproducing kernel $K(z, w) = e^{z\bar{w}}$ (see [5]).

The Dirichlet energy is a measure of how variable a function is. The Dirichlet energy of $f$ in $F^2$ is defined by

$$Q(f, f) = \int_{\mathbb{C}} |f'(z)|^2\, dG(z).$$

Received June 7, 2022; Accepted September 30, 2022.
2010 *Mathematics Subject Classification.* 30H20.
*Key words and phrases.* Fock-Dirichlet space, Fock-Nevanlinna space.
This work was financially supported by a 2-Year Research Grant of Pusan National University.
*Corresponding author.


(pISSN 1226-6973, eISSN 2287-2833)

It is a quadratic functional on $F^2$.

**Definition 1.** The Fock-Dirichlet space $F^2_{\mathcal{D}}$ consists of all entire functions $f$ such that the Dirichlet energy $Q(f,f) = \|f'\|^2$ is finite.

*Remark* 1. In fact, the Fock-Dirichlet space is a Fock-Sobolev space of order 1 (see [1], [2], [3]).

Let $f$ be an entire function on $\mathbb{C}$. Then we have the following power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Thus

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 n! \quad \text{and} \quad \|f'\|^2 = \sum_{n=1}^{\infty} n|a_n|^2 n!.$$

Therefore

$$\|f\|^2 \leq |f(0)|^2 + \|f'\|^2$$

and so $F^2_{\mathcal{D}} \subset F^2$. However, the converse is not true. Every element $f$ in $F^2$ is automatically infinitely differentiable, and $f'$ is automatically entire again. Nevertheless, given $f \in F^2$, there is no reason that $f'$ must be again square-integrable with respect to $dG$. In fact, even though the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n = (n!(n+1)^2)^{-1/2}$ belongs to $F^2$, $f'$ does not belong to $F^2$. Thus having $f'$ be in $F^2$ is a nontrivial regularity condition on $f$.

Since $F^2_{\mathcal{D}} \subset F^2$, we can define the Fock-Dirichlet norm $\|f\|_{\mathcal{D}}$ by

$$\|f\|^2_{\mathcal{D}} = \|f\|^2 + Q(f,f).$$

Another possibility is to let

$$|||f|||^2_{\mathcal{D}} = |f(0)|^2 + Q(f,f).$$

**Lemma 1.1.**

$$f \in F^2_{\mathcal{D}} \quad \textit{if and only if} \quad zf(z) \in F^2.$$

*Proof.* Let $f, g \in F^2_{\mathcal{D}}$. We define

$$Q(f,g) = \langle f, g\rangle_{\mathcal{D}} = \int_{\mathbb{C}} f'(z)\overline{g'(z)}\, dG(z).$$

By the integration by parts, we know that

$$\int_{\mathbb{C}} f'(z)\overline{g(z)}\, dG(z) = \int_{\mathbb{C}} f(z)\overline{zg(z)}\, dG(z). \tag{1}$$

By (1), we have

$$\langle f, g\rangle + Q(f,g) = \int_{\mathbb{C}} |z|^2 f(z)\overline{g(z)}\, dG(z) = \langle zf, zg\rangle.$$

If we take $f = g$, then we have

$$\|zf\|^2 = \|f\|^2 + Q(f,f) = \|f\|_{\mathcal{D}}^2.$$

Thus we get the result. □

**Lemma 1.2.**

$$\langle z^m, z^n \rangle = \begin{cases} n! & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

**Lemma 1.3.**

$$\langle z^m, z^n \rangle_{\mathcal{D}} = \begin{cases} (n+1)! & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

*Proof.*

$$\langle z^m, z^n \rangle_{\mathcal{D}} = \langle z^m, z^n \rangle + mn \langle z^{m-1}, z^{n-1} \rangle.$$

□

**Proposition 1.4.** *The reproducing kernel for $F_{\mathcal{D}}^2$ is given by*

$$K_{\mathcal{D}}(z,w) = \frac{1}{z\bar{w}}(e^{z\bar{w}} - 1).$$

*Proof.* Since $\{z^n/\|z^n\|_{\mathcal{D}}\}$ is an orthonormal basis for $F_{\mathcal{D}}^2$, we have

$$\begin{aligned} K_{\mathcal{D}}(z,w) &= \sum_{n=0}^{\infty} \frac{z^n}{\|z^n\|_{\mathcal{D}}} \frac{\bar{w}^n}{\|w^n\|_{\mathcal{D}}} \\ &= \sum_{n=0}^{\infty} \frac{(z\bar{w})^n}{(n+1)!} \\ &= \frac{1}{z\bar{w}}(e^{z\bar{w}} - 1). \end{aligned}$$

□

Note that

$$\begin{aligned} |f(z)|^2 &= |\langle f, K_{\mathcal{D}}(z,\cdot) \rangle_{\mathcal{D}}| \\ &\leq \|f\|_{\mathcal{D}}^2 \|K_{\mathcal{D}}(z,\cdot)\|_{\mathcal{D}}^2. \end{aligned}$$

Here

$$\|K_{\mathcal{D}}(z,\cdot)\|_{\mathcal{D}}^2 = K_{\mathcal{D}}(z,z) = \frac{1}{|z|^2}(e^{|z|^2} - 1).$$

Thus elements $f$ of $F_{\mathcal{D}}^2$ satisfy the following pointwise bounds

$$|f(z)|^2 \leq \frac{e^{|z|^2} - 1}{|z|^2} \|f\|_{\mathcal{D}}^2, \quad z \in \mathbb{C}. \tag{2}$$

## 2. Taylor coefficients of functions in the Fock-Dirichlet space

**Theorem 2.1.** *Let $f(z) = \sum a_n z^n \in F^2_{\mathcal{D}}$ be the Taylor series at the origin on $\mathbb{C}$. Then we have*

$$|a_n| \le \left(\frac{e}{n+1}\right)^{\frac{n+1}{2}} \|f\|_{\mathcal{D}}, \quad n \ge 0.$$

*Proof.* Let $r > 0$. By Cauchy's estimates and (2), Taylor coefficients $a_n$ of $f$ satisfy

$$\begin{aligned} |a_n| &\le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(re^{i\theta})|}{r^n} d\theta \\ &\le \frac{e^{\frac{r^2}{2}}}{r^{n+1}} \|f\|_{\mathcal{D}}. \end{aligned}$$

We consider the function

$$h(r) = \frac{e^{\frac{r^2}{2}}}{r^{n+1}}.$$

Then

$$\begin{aligned} h'(r) &= (r^{-(1+n)} e^{\frac{r^2}{2}})' \\ &= -(1+n) r^{-(1+n+1)} e^{\frac{r^2}{2}} + r r^{-(1+n)} e^{\frac{r^2}{2}} \\ &= r^{-(1+n)} e^{\frac{r^2}{2}} \left\{ -(n+1) r^{-1} + r \right\}. \end{aligned}$$

Thus it has its minimum value at $r = \sqrt{n+1}$, which implies that

$$|a_n| \le \left(\frac{e}{n+1}\right)^{\frac{n+1}{2}} \|f\|_{\mathcal{D}}, \quad n \ge 0.$$

□

## 3. The Fock-Nevanlinna space

Let

$$\log^+ x = \begin{cases} 0 & \text{if } \ 0 \le x \le 1, \\ \log x & \text{if } \ x > 1. \end{cases}$$

We define the Fock-Nevanlinna space $F^2_{\mathcal{N}}$ by

$$F^2_{\mathcal{N}} = \left\{ f \in F^2 : \int_{\mathbb{C}} |f(z)|^2 \log^+ |f(z)| \, dG(z) < \infty \right\}.$$

**Lemma 3.1** (Young's inequality)**.** *Let $s, t \ge 0$. Then*

(3) $$st \le s \log s - s + e^t.$$

**Theorem 3.2.**

$$F^2_{\mathcal{D}} = F^2_{\mathcal{N}}.$$

*Proof.* Let $f \in F^2_{\mathcal{D}}$. By using the inequality such that

$$|f(z)| \le e^{\frac{1}{2}|z|^2}\|f\|,$$

we have

$$\int_{\mathbb{C}} |f(z)|^2 \log^+ |f(z)|\, dG(z) - \|f\|^2 \log^+ \|f\| \le \frac{1}{2}\|zf\|^2 = \frac{1}{2}\|f\|^2_{\mathcal{D}}.$$

Hence $F^2_{\mathcal{D}} \subset F^2_{\mathcal{N}}$.

Let $f \in F^2_{\mathcal{N}}$. For $c > 1$ if we choose $s = c|f(z)|^2, t = \frac{1}{c}|z|^2$, by Young's inequality (3.1), we have

$$\begin{aligned}\|f\|^2_{\mathcal{D}} &= \int_{\mathbb{C}} |z|^2|f(z)|^2\, dG(z)\\ &\le 2c\int_{\mathbb{C}} |f(z)|^2 \log^+ |f(z)|\, dG(z) + (c\log c - c)\|f\|^2 + \int_{\mathbb{C}} e^{-(1-1/c)|z|^2}\,\frac{dA}{\pi}.\end{aligned}$$

This means that if a function possesses just a bit more integrability than square integrability, then the function has finite expected energy. Thus $F^2_{\mathcal{N}} \subset F^2_{\mathcal{D}}$. We get the result. □

The concepts of the Dirichlet space and the Nevanlinna space have been studied a lot in the unit disk [4]. In this study, the concepts of the Fock-Dirichlet space and the Fock-Nevanlinna space in the entire complex plane were introduced and simple results were obtained. Based on the presented ideas, more advanced research will be conducted in future research.

## References


[1] H. R. Cho, B. R. Choe and H. Koo, *Fock-Sobolev spaces of fractional order*, Potential Anal. **43** (2015), 199-240.

[2] H. R. Cho and S. Park, *Fractional Fock-Sobolev spaces*, Nagoya Math. J., **237** (2020), 79-97.

[3] H. R. Cho and K. Zhu, *Fock-Sobolev spaces and their Carleson measures*, J. Funct. Anal. **263** (2012), no. 8, 2483-2506.

[4] K. Zhu, *Operator theory in function spaces*. Monographs and Textbooks in Pure and Applied Mathematics, **139**. Marcel Dekker, Inc., New York, 1990. xii+258 pp.

[5] K. Zhu, *Analysis on Fock spaces*, Graduate Texts in Mathematics, **263**, Springer, New York, 2012, x+344 pp.



Hong Rae Cho
Department of Mathematics, Pusan National University, 2, Busandaehak-ro 63beon-gil, Geumjeong-gu, Busan, 46241, Rep. of KOREA
*Email address*: chohr@pusan.ac.kr

Soohyun Park
Department of Mathematics, Pusan National University, 2, Busandaehak-ro 63beon-gil, Geumjeong-gu, Busan, 46241, Rep. of KOREA
*Email address*: shpark7@pusan.ac.kr