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COMMUTING STRUCTURE JACOBI OPERATOR FOR SEMI-INVARIANT SUBMANIFOLDS OF CODIMENSION 3 IN COMPLEX SPACE FORMS

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Abstract. Let M be a semi-invariant submanifold with almost contact metric structure (ϕ, ξ, η, g) of codimension 3 in a complex space form $M_{n+1}(c), c \neq 0$. We denote by S and R_ξ be the Ricci tensor of M and the structure Jacobi operator in the direction of the structure vector ξ, respectively. Suppose that the third fundamental form t satisfies $dt(X, Y) = 2\theta g(\phi X, Y)$ for a certain scalar $\theta(\neq 2c)$ and any vector fields X and Y on M. In this paper, we prove that M satisfies $R_\xi S = SR_\xi$ and at the same time $R_\xi\phi = \phi R_\xi$, then M is a Hopf hypersurface of type (A) provided that the scalar curvature s of M holds $s - 2(n-1)c \leq 0$.

1. Introduction

A submanifold M is called a CR $submanifold$ of a Kaehlerian manifold $\tilde{M}$ with complex structure J if there exists a differentiable distribution $\triangle : p \to \triangle_p \subset M_p$ on M such that $\triangle$ is J-invariant and the complementary orthogonal distribution $\triangle^\perp$ is totally real, where M_p denotes the tangent space at each point p in M ([1], [27]). In particular, M is said to be a *semi-invariant submanifold* provided that $\dim\triangle^\perp = 1$. The unit normal in $J\triangle^\perp$ is called the *distinguished normal* to the semi-invariant submanifold ([4], [25]). In this case, M admits an induced almost contact metric structure (ϕ, ξ, η, g). A typical example of a semi-invariant submanifold is real hypersurfaces. And new examples of nontrivial semi-invariant submanifolds in a complex projective space $P_n\mathbb{C}$ are constructed in [16] and [22]. Therefore we may expect to generalize some results which are valid in a real hypersurface to a semi-invariant submanifold.

An n-dimensional complex space form $\tilde{M}_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature $4c$. As is well known, complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbb{C}$, or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0$ or $c < 0$.

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For the real hypersurface of $\tilde{M}_n(c), c \neq 0$, many results are known. One of them, Takagi([23], [24]) classified all the homogeneous real hypersurfaces of $P_n\mathbb{C}$ as six model spaces which are said to be A_1, A_2, B, C, D and E, and Cecil-Ryan ([5]) and Kimura ([17]) proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds when the structure vector field ξ is principal.

On the other hand, real hypersurfaces in $H_n\mathbb{C}$ have been investigated by Berndt [2], Montiel and Romero [18] and so on. Berndt [2] classified all real hypersurfaces with constant principal curvatures in $H_n\mathbb{C}$ and showed that they are realized as the tubes of constant radius over certain submanifolds. Also such kinds of tubes are said to be real hypersurfaces of type A_0, A_1, A_2 or type B.

Let M be a real hypersurface of type A_1 or type A_2 in a complex projective space $P_n\mathbb{C}$ or that of type A_0, A_1 or A_2 in a complex hyperbolic space $H_n\mathbb{C}$. Now, hereafter unless otherwise stated, such hypersurfaces are said to be of *type* (A) for our convenience sake.

Characterization problems for a real hypersurface of type (A) in a complex space form were studied by many authors ([7], [8], [13], [18], [20] etc.).

Two of them, we introduce the following characterization theorems due to Okumura [20] for $c > 0$ and Montiel and Romero [18] for $c < 0$ respectively.

Theorem O. *Let M be a real hypersurface of $P_n\mathbb{C}$, $n \geq 2$. If it satisfies*

$$g((A\phi - \phi A)X, Y) = 0 \tag{1.1}$$

for any vector fields X and Y, then M is locally congruent to a tube of radius r over one of the following Kaehlerian submanifolds :

(A_1) *a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \pi/2$,*
(A_2) *a totally geodesic $P_k\mathbb{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$.*

Theorem MR. *Let M be a real hypersurface of $H_n\mathbb{C}$, $n \geq 2$. If it satisfies* (1.1), *then M is locally congruent to one of the following hypersurface :*

(A_0) *a horosphere in $H_n\mathbb{C}$, i.e., a Montiel tube,*
(A_1) *a geodesic hypersphere, or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*
(A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ $(c \leq k \leq n-2)$.*

Denoting by R the curvature tensor of the submanifold, we define the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ with respect to the structure vector ξ. Then R_ξ is a self adjoint endomorphism on the tangent space of a CR submanifold.

Using several conditions on the structure Jacobi operator R_ξ, characterization problems for real hypersurfaces of type (A) have recently studied (cf. [7], [8], [19]). In the previous paper [7], Cho and one of the present authors gave another characterization of real hypersurface of type (A) in a complex projective space $P_n\mathbb{C}$.

Ki, Nagai and Takagi ([13]) proved the following characterizing homogeneous real hypersurfaces of type (A) and some special classes of Hopf hypersurfaces.

Theorem KNT([13]). *Let M be a real hypersurface in a nonflat complex space form $M_n(c), n \geq 2$. If M satisfies $R_\xi \phi = \phi R_\xi$ and at the same time satisfies $R_\xi S = SR_\xi$, then M is a Hopf hypersurface. Further, M is locally congruent to one of homogeneous real hypersurfaces of type A_0, A_1, A_2, or to a Hopf hypersurface with $g(A\xi, \xi) = 0$, where S denote the Ricci tensor of M.*

On the other hand, semi-invariant submanifolds of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$ have been studied in [10]$\sim$ [12], [14]$\sim$ [16] and so on by using properties of induced almost contact metric structure and those of the third fundamental form of the submanifold. In the preceding work, Takagi and the present authors assert the following:

Theorem KST([16]). *Let M be a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$ with constant holomorphic sectional curvature $4c$. If the structure vector ξ is an eigenvector for the shape operator in the direction of the distinguished normal and the third fundamental form t satisfies $dt = 2\theta\omega$ for a certain scalar $\theta(< 2c)$, where $\omega(X, Y) = g(\phi X, Y)$ for any vectors X and Y on M, then M is a Hopf real hypersurface in a complex projective space $P_n\mathbb{C}$.*

In this paper, we discuss the version with respect to semi-invariant submanifolds of Theorem KNT, that is, we consider a semi-invariant submanifold M of codimension 3 in a complex space form $M_{n+1}(c), c \neq 0$ which satisfies $R_\xi \phi = \phi R_\xi$ and at the same time $R_\xi S = SR_\xi$ such that the third fundamental form t satisfies $dt = 2\theta\omega$ for a certain scalar $\theta(\neq 2c)$. In this case, we prove that M is a real hypersurface is of type (A) in $M_n(c)$ provided that the scalar curvature s of M satisfies $s - 2(n-1)c \leq 0$.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the semi-invariant are supposed to be orientable.

2. Preliminaries

Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold with parallel almost complex structure J and a Riemannian metric tensor G. Let M be a real $(2n-1)$-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$. We denote by g the Riemannian metric tensor on M from that of $\tilde{M}$.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation with respect to the metric tensor G on $\tilde{M}$ and by ∇ the one on M. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX,Y)C + g(KX,Y)D + g(LX,Y)E, \tag{2.1}$$

$$\begin{aligned} \tilde{\nabla}_X C &= -AX + l(X)D + m(X)E, \\ \tilde{\nabla}_X D &= -KX - l(X)C + t(X)E, \\ \tilde{\nabla}_X E &= -LX - m(X)C - t(X)D \end{aligned} \tag{2.2}$$

for any vector fields X and Y tangent to M and any vector field C, D and E normal to M, where A, K, L are called the *second fundamental forms* with respect to the normal vector C, D and E respectively, and l, m and t being the *third fundamental forms.*

As is well-known, a submanifold of a Kaehlerian manifold is said to be a *CR submanifold* ([1], [27]) if it is endowed with a pair of mutually orthogonal and complementary differentiable distribution $(\Delta, \Delta^{\perp})$ such that for any point $p \in M$ we have $J\Delta_p = \Delta_p$, $J\Delta_p^{\perp} \subset \Delta_p^{\perp}M$, where $\Delta_p^{\perp}M$ denote the normal space of M at p. In particular, M is said to be *semi-invariant submanifold* provided that $dim\Delta^{\perp} = 1$([4], [25]). In this case the unit vector field in $J\Delta^{\perp}$ is called a *distinguished normal* to the semi-invariant submanifold and denote by C([4], [25]).

More precisely, we choose an orthonormal basis $e_1, \cdots, e_{2n-2}, \xi$ of M_p in such a way that $e_1, e_2, \cdots, e_{2n-2} \in \Delta$, where M_p denotes the tangent space to M at each point p in M. Then we see that

$$G(J\xi, e_i) = -G(\xi, Je_i) = 0$$

for $i = 1, \cdots, 2n-2$.

From now on we consider M is a real $(2n-1)$-dimensional semi-invariant submanifold of a Kaehlerian manifold $\tilde{M}$ of real dimension $2(n+1)$. Then we can write ([4], [26])

$$JX = \phi X + \eta(X)\mathcal{C}, \quad J\mathcal{C} = -\xi, \quad JD = -E, \quad JE = D, \tag{2.3}$$

where we have put $g(\phi X, Y) = G(JX, Y), \eta(X) = G(JX, \mathcal{C})$ for any vector fields X and Y tangent to M.

By the Hermitian property of J, we see, using (2.3), that the aggregate (ϕ, ξ, η, g) is an *almost contact metric structure* on M, that is, we have

$$\begin{gathered} \phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(X) = g(\xi, X), \\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \end{gathered} \tag{2.4}$$

for any vectors X and Y on M.

In the sequel, we denote the normal components of $\tilde{\nabla}_X C$ by $\nabla^{\perp} C$. The distinguished normal C is said to be *parallel* in the normal bundle if we have $\nabla^{\perp} C = 0$, that is, l and m vanish identically.

From the Kaehler condition $\tilde{\nabla} J = 0$ and take account of the Gauss and Weingarten formulas,we obtain from (2.3)

$$\nabla_X \xi = \phi AX, \tag{2.5}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.6}$$

$$KX = \phi LX - m(X)\xi, \tag{2.7}$$

$$LX = -\phi KX + l(X)\xi \tag{2.8}$$

for any vectors X and Y on M. The last two relationships give

$$l(X) = g(L\xi, X), \quad m(X) = -g(K\xi, X), \tag{2.9}$$

$$m(\xi) = -k, \quad l(\xi) = TrL, \tag{2.10}$$

where, we have put $k = TrK$.

We notice here that there is no loss of generality such that we may assume $T_r L = 0$. Therefore we have by (2.10)

$$l(\xi) = 0. \tag{2.11}$$

Applying (2.8) by ϕ and using (2.7), we find

$$-g(KX, Y) - m(X)\eta(Y) = g(\phi KX, \phi Y) - \eta(X)l(\phi Y).$$

If we take the skew-symmetric part of this with respect to X and Y, then we obtain

$$-m(X)\eta(Y) + m(Y)\eta(X) = \eta(X)l(\phi Y) - \eta(Y)l(\phi X),$$

which together with (2.10) gives

$$l(\phi X) = m(X) + k\eta(X). \tag{2.12}$$

Similarly we have

$$m(\phi X) = -l(X) \tag{2.13}$$

because of (2.10).

Transforming (2.7) by L and using (2.8) and (2.9), we obtain

$$g(KLX, Y) + g(LKX, Y) = -l(X)m(Y) - l(Y)m(X). \tag{2.14}$$

In the rest of this paper we shall suppose that $\tilde{M}$ is a Kaehlerian manifold of constant holomorphic sectional curvature $4c$, which is called a *complex space form* and denote by $M_{n+1}(c)$. Then equations of the Gauss and Codazzi are given by

$$\begin{aligned} R(X, Y)Z = c\{&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY \\ &+ g(KY, Z)KX - g(KX, Z)KY + g(LY, Z)LX - g(LX, Z)LY, \end{aligned} \tag{2.15}$$

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X - l(X)KY + l(Y)KX \\ - m(X)LY + m(Y)LX = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},\end{aligned} \tag{2.16}$$

$$(\nabla_X K)Y - (\nabla_Y K)X + l(X)AY - l(Y)AX - t(X)LY + t(Y)LX = 0, \tag{2.17}$$

$$\begin{aligned}(\nabla_X L)Y - (\nabla_Y L)X + m(X)AY - m(Y)AX \\ + t(X)KY - t(Y)KX = 0,\end{aligned} \tag{2.18}$$

where R is the Riemann-Christoffel curvature tensor on M, and those of the Ricci by

$$\begin{aligned}(\nabla_X l)Y - (\nabla_Y l)X + g(KAX, Y) - g(AKX, Y) \\ + m(X)t(Y) - m(Y)t(X) = 0,\end{aligned} \tag{2.19}$$

$$\begin{aligned}(\nabla_X m)Y - (\nabla_Y m)X + g(LAX, Y) - g(ALX, Y) \\ + t(X)l(Y) - t(Y)l(X) = 0,\end{aligned} \tag{2.20}$$

$$\begin{aligned}(\nabla_X t)Y - (\nabla_Y t)X + g(LKX, Y) - g(KLX, Y) \\ + l(X)m(Y) - l(Y)m(X) = 2cg(\phi X, Y).\end{aligned} \tag{2.21}$$

In what follows, to write our formulas in a convention form, we denote by $\alpha = \eta(A\xi), \beta = \eta(A^2\xi), \gamma = \eta(A^3\xi), TrA = h, TrK = k, Tr({}^tAA) = h_{(2)}$ and for a function f we denote by ∇f the gradient vector field of f.

Now, we put $\nabla_\xi \xi = U$ in the sequel. Then U is orthogonal to ξ because of (2.5). From now on we put

$$A\xi = \alpha\xi + \mu W, \tag{2.22}$$

where W is a unit vector field orthogonal to ξ. Then we have

$$U = \mu\phi W \tag{2.23}$$

because of (2.5). So, W is orthogonal to U. Further, we have

$$\mu^2 = \beta - \alpha^2. \tag{2.24}$$

From (2.22) and (2.23) we have

$$\phi U = -A\xi + \alpha\xi, \tag{2.25}$$

which together with (2.5) and (2.22) yields

$$g(\nabla_X \xi, U) = \mu g(AW, X), \quad \mu g(\nabla_X W, \xi) = g(AU, X) \tag{2.26}$$

because W is orthogonal to ξ.

Differentiating (2.25) covariantly along M and using (2.5) and (2.6), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU + \nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX, \tag{2.27}$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha - 2kL\xi. \tag{2.28}$$

Because of (2.5), (2.26) and (2.27), we verify that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha - 2k(K\xi - k\xi). \tag{2.29}$$

Finally, we introduce the Jacobi operator R_ξ with respect to the structure vector field ξ which is defined by $R_\xi X = R(X, \xi)\xi$ for any vector X.

If we transform (2.8) by L and take account of (2.7) and (2.9), then we get

$$L^2X - K^2X = l(X)L\xi - m(X)K\xi.$$

Because of (2.9), (2.10) and this, it is seen from (2.15) that

$$\begin{aligned} R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX \\ - m(X)K\xi - l(X)L\xi. \end{aligned} \tag{2.30}$$

Suppose that $R_\xi\phi = \phi R_\xi$ holds on M. Then from (2.30) we have

$$\begin{aligned} \alpha(\phi AX - A\phi X) = g(A\xi, X)U + g(U, X)A\xi + 2kLX \\ - 2k\{l(X)\xi + \eta(X)L\xi\}, \end{aligned} \tag{2.31}$$

where we have used (2.5), (2.8) and (2.12).

3. The third fundamental forms of semi-invariant submanifolds

In this section we shall suppose that M is a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ and that the third fundamental form t satisfies

$$dt = 2\theta\omega, \quad \omega(X, Y) = g(\phi X, Y) \tag{3.1}$$

for a certain scalar θ and any vectors X and Y, where d denotes the exterior differential operator. Then we have from (2.14) and (2.21)

$$g(LKX, Y) + l(X)m(Y) = -(\theta - c)g(\phi X, Y), \tag{3.2}$$

which together with (2.9)$\sim$(2.11) implies that

$$KL\xi = kL\xi, \quad LK\xi = 0. \tag{3.3}$$

Differentiating (3.1) covariantly along M and using (2.6) and the first Bianchi identity, we find

$$(X\theta)\omega(Y,Z)+(Y\theta)\omega(Z,X)+(Z\theta)\omega(X,Y)=0,$$

which implies $(n-2)X\theta=0$. Thus $\theta(\geq c)$ is constant if $n>2$.

For the case where $\theta=c$ in (3.1) we have $dt=2c\omega$. In this case, the normal connection of M is said to be *L-flat*([10]).

Lemma 3.1. *Let M be a semi-invariant submanifold with L-flat normal connection in $M_{n+1}(c)$, $c\neq 0$. If $A\xi=\alpha\xi$, then we have $\nabla^{\perp}C=0$ and $K=L=0$.*

Proof. From (3.2) we have

$$T_r({}^tKK)-\|k\xi\|^2+\|L\xi\|^2=2(n-1)(\theta-c)$$

because of (2.7), (2.9) and (2.12), which implies

$$\|K-k\eta\otimes\xi\|^2+\|L\xi\|^2=2(n-1)(\theta-c),$$

where $\|F\|^2=g(F,F)$ for any vector field F on M. Thus, by our hypothesis $\theta=c$, we have $K=k\eta\otimes\xi$.

In the same way, we see from (2.8), (2.10), (2.13) and (3.2) that $L=0$. And hence $m(X)=-k\eta(X)$ and $l=0$ because of (2.9). Therefore, it suffices to show that $k=0$. Using these facts, (2.19) reformed as

$$k\{\eta(X)A\xi-g(A\xi,X)\xi\}=k\{\eta(X)t-t(X)\xi\},$$

which together with $A\xi=\alpha\xi$ gives

$$k\{t-t(\xi)\xi\}=0. \tag{3.4}$$

We also have from (2.18)

$$k\{\eta(X)(AY+t(Y)\xi)-\eta(Y)(AX+t(X)\xi)\}=0,$$

which implies $k(h-\alpha)=0$. Form this and (3.4) we verify that $k=0$. This completes the proof. □

Applying (3.2) by ϕ and taking account of (2.7) and (2.13), we find

$$K^2X+\eta(X)K^2\xi+l(X)L\xi=(\theta-c)\{X-\eta(X)\xi\}, \tag{3.5}$$

which implies $\eta(X)K^2\xi-g(K^2\xi,X)\xi=0$. Thus, it follows that

$$K^2\xi=(\|K\xi\|^2)\xi \tag{3.6}$$

by virtue of (2.9). Thus, (3.5) becomes

$$K^2X+l(X)L\xi+\|K\xi\|^2\eta(X)\xi=(\theta-c)(X-\eta(X)\xi).$$

Putting $X=L\xi$ in (2.8) and taking account of (2.12) and (3.3), we obtain

$$L^2\xi = kK\xi + (\|K\xi\|^2 + k^2)\xi. \tag{3.7}$$

If we put $X = L\xi$ in (3.2) and make use of (2.13) and (3.2), we find

$$(\theta - c - \|K\xi\|^2)L\xi = 0.$$

Similarly, we verify, using (3.2) and (3.7), that

$$(\theta - c - \|L\xi\|^2 - k^2)(\|K\xi\|^2 - k^2) = 0.$$

Let $\|L\xi\| \neq 0$ at every point of M and suppose that this subset does not void. Then we have $\|K\xi\|^2 = \theta - c$ and $\|L\xi\|^2 + k^2 = \theta - c$ on the subset. Using these facts, we can verify that (for detail, see (2.22) and (2.24) of [16])

$$\nabla k = 2AL\xi, \tag{3.8}$$

$$\nabla_X L\xi = t(X)K\xi - AKX - kAX \tag{3.9}$$

on the set. Differentiating (3.8) covariantly and taking the skew-symmetric part obtained, we find

$$(\theta - 2c)\{\eta(X)K\xi - m(X)\xi\} = 0,$$

where we have used (2.12), (2.16), (3.3) and (3.9), which shows that $(\theta - 2c)(m(X) + k\eta(X)) = 0$ and hence $\theta = 2c$ on this subset.
Thus, from the first equation of (2.3) we have the following :

Lemma 3.2. *Let M be a semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$, $c \neq 0$ satisfying (3.1). If $\theta - 2c \neq 0$, then $\nabla^{\perp} C = -k\xi E$ on M.*

In the following we assume that M satisfies (3.1) with $\theta - 2c \neq 0$. Then we have

$$L\xi = 0, \quad K\xi = k\xi \tag{3.10}$$

because of (2.9). It is, using (3.10), clear that (2.7), (2.8) and (3.2) are reduced respectively to

$$\phi LX = KX - k\eta(X)\xi, \tag{3.11}$$

$$L = K\phi, \tag{3.12}$$

$$g(LKX, Y) + (\theta - c)g(\phi X, Y) = 0. \tag{3.13}$$

From the last two equations, we obtain

$$L^2 X = (\theta - c)(X - \eta(X)\xi). \tag{3.14}$$

Further, if we take account of (3.10), then the other structure equations (2.16)∼(2.21) reformed as

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X \\ = k\{\eta(Y)LX - \eta(X)LY\} + c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},\end{aligned} \tag{3.15}$$

$$(\nabla_X K)Y - (\nabla_Y K)X = t(X)LY - t(Y)LX, \tag{3.16}$$

$$(\nabla_X L)Y - (\nabla_Y L)X = k\{\eta(X)AY - \eta(Y)AX\} - t(X)KY + t(Y)KX, \tag{3.17}$$

$$KAX - AKX = k\{\eta(X)t - t(X)\xi\}, \tag{3.18}$$

$$LAX - ALX = (Xk)\xi - \eta(X)\nabla k + k(\phi AX + A\phi X), \tag{3.19}$$

where we have used (2.5).

Putting $X = \xi$ in (3.18) and using (3.10), we find

$$KA\xi = kA\xi + k\{t' - t(\xi)\xi\}, \tag{3.20}$$

where $g(t', X) = t(X)$ for any vector X. From now on we will use the same letter t instead of t'.

Replacing X by ξ in (3.19) and using (2.5), (3.10) and (3.12), we get

$$KU = (\xi k)\xi - \nabla k + kU. \tag{3.21}$$

If we apply (3.20) by ϕ and make use of (2.22) (3.11) and (3.12), then we find

$$KU = k(t\phi - U), \tag{3.22}$$

which together with (3.21) yields

$$\nabla k = (\xi k)\xi + k(-t\phi + 2U). \tag{3.23}$$

If we transform (3.19) by ϕ and take account of (2.22), (3.11) and the last equation, then we obtain

$$\phi ALX - KAX = -k\{(t - t(\xi)\xi)\eta(X) + 2\mu\eta(X)W + 2g(A\xi, X)\xi - AX + \phi A\phi X\},$$

which connected to (3.18) gives

$$\phi AL = -LA\phi. \tag{3.24}$$

Since θ is constant if $n > 2$, differentiating (3.14) covariantly, we find

$$(\nabla_X L^2)Y = (c - \theta)\{\eta(Y)\phi AX + g(\phi AX, Y)\xi\},$$

or, using (3.13) and (3.17), it is verified that (see, [16])

$$\begin{aligned}2(\nabla_X L)LY =&(\theta - c)\{2t(X)\phi Y - \eta(Y)(\phi A + A\phi)X + g((A\phi - \phi A)X, Y)\xi \\ &- \eta(X)(\phi A - A\phi)Y\} - k\{\eta(Y)(AL + LA)X \\ &- g((AL + LA)X, Y)\xi - \eta(X)(LA - AL)Y\},\end{aligned}$$

which together with (3.10) and (3.22) yields

$$\begin{aligned}&(\theta - c)(A\phi - \phi A)X + (k^2 + \theta - c)(u(X)\xi + \eta(X)U) \\ &\quad + k\{(AL + LA)X + k\{-t(\phi X)\xi + \eta(X)\phi \circ t\} = 0,\end{aligned} \tag{3.25}$$

where $u(X) = g(U, X)$ for any vector X.

In the following we consider the case where (2.22) with $\mu = 0$, that is $A\xi = \alpha\xi$. Differentiating this covariantly and using (2.5), we find

$$(\nabla_X A)\xi = -A\phi AX + \alpha\phi AX + (X\alpha)\xi,$$

which together with (3.10) and (3.15) gives

$$-2A\phi AX + \alpha(\phi A + A\phi)X + 2c\phi X = \eta(X)\nabla\alpha - (X\alpha)\xi. \tag{3.26}$$

If we put $X = \xi$ in this and using (2.22) with $\mu = 0$, we find

$$\nabla\alpha = (\xi\alpha)\xi. \tag{3.27}$$

Differentiating the second equation of (3.10) covariantly along M, and using (2.5), we find $\nabla_X m = -(Xk)\xi + k\phi AX$, from which taking the skew-symmetric part and making use of (2.20) with $l = 0$,

$$LAX - ALX - k(\phi AX + A\phi X) = (Xk)\xi - \eta(X)\nabla k.$$

Since $A\xi = \alpha\xi$ was assumed, then we have

$$\nabla k = (\xi k)\xi \tag{3.28}$$

because of (3.10). From the last two equations, it follows that

$$LA - AL = k(\phi A + A\phi). \tag{3.29}$$

If we put $X = \xi$ in (3.18) and remember (2.21) with $\mu = 0$ and (3.10), then we get

$$k\{t(X) - t(\xi)\eta(X)\} = 0. \tag{3.30}$$

Since we have $A\xi = \alpha\xi$, differentiating (3.28) covariantly, and taking the skew-symmetric part obtained, we get

$$(\xi k)(A\phi + \phi A) = 0. \tag{3.31}$$

From this and (3.27) we can write (3.26) as $\alpha(A^2\phi + c\phi) = 0$. By the properties of the almost contact metric structure, it follows that

$$\xi k\{h_{(2)} - \alpha^2 + 2(n-1)c\} = 0,$$

which implies $\xi k = 0$ if $c > 0$.

4. Semi-invariant submanifolds satisfying $R_\xi \phi = \phi R_\xi$

We will continue our arguments under the same hypotheses $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ as those stated in section 3. Further suppose, throughout this paper, that $R_\xi \phi = \phi R_\xi$, which means that the eigenspace of the structure Jacobi operator R_ξ is invariant by the structure operator ϕ. Then (2.31) reformed as

$$\alpha(\phi AX - A\phi X) = g(A\xi, X)U + g(U, X)A\xi + 2kLX \tag{4.1}$$

by virtue of (3.10).

Transforming this by A, and taking the trace obtained, we have $g(A^2\xi, U) = 0$ because of (3.26), which together with (2.22) yields

$$\mu g(AW, U) = 0. \tag{4.2}$$

Applying (4.1) by L and using (2.25), (3.11) and (3.19), we find

$$\begin{aligned}\alpha\{AKX - k\eta(X)A\xi - \phi ALX\} + g(LU, X)A\xi + g(KU, X)U \\ = -2kL^2X,\end{aligned} \tag{4.3}$$

which together with (3.18) and (3.22) yields

$$\begin{aligned}&k\alpha\{t(X)\xi - \eta(X)t + g(A\xi, X)\xi - \eta(X)A\xi\} \\ &\quad + g(LU, X)A\xi - g(A\xi, X)LU - u(X)KU + g(KU, X)U = 0,\end{aligned}$$

where $u(X) = g(U, X)$ for any vector X. If we take the inner product with ξ to this and use (3.10), then we get

$$k\alpha\{t(X) - t(\xi)\eta(X) + g(A\xi, X) - \alpha\eta(X)\} + \alpha g(LU, X) = 0. \tag{4.4}$$

Combining the last two equations and taking account of (2.24), we obtain

$$\mu\{w(X)LU - g(LU, X)W\} + u(X)KU - g(KU, X)U = 0, \tag{4.5}$$

where $w(X) = g(W, X)$ for any vector X.

We notice here that the following fact :

Remark 4.1. $\alpha \neq 0$ on Ω.

In fact, if not, then we have $\alpha = 0$ on this subset. We discuss our arguments on such a place. So (4.1) reformed as

$$\mu\{w(X)U + u(X)W\} + 2kLX = 0 \tag{4.6}$$

because of (2.22) with $\alpha = 0$. Putting $X = U$ or W in this we have respectively

$$LU = -\frac{\mu\beta}{2k}W, \quad LW = -\frac{\mu}{2k}U \tag{4.7}$$

by virtue of (2.24) with $\alpha = 0$. Using this and (3.14), we can write (4.3) as

$$-\frac{\beta^2}{2k}w(X)W + g(KU, X)U = -2k(\theta - c)(X - \eta(X)\xi).$$

Taking the inner product with W to this, we obtain $\beta^2 = 4k^2(\theta - c)$.

On the other hand, combining (4.6) and (4.7) to (3.14) we also have $\beta^2 = 4(n-1)k^2(\theta - c)$, which implies $(n-2)(\theta - c)k = 0$, a contradiction because of our assumption and Lemma 2.1. Thus, $\alpha = 0$ is not impossible on Ω.

Now, putting $X = U$ in (4.4) and remembering Remark 4.1, we find $kt(U) + g(LU, U) = 0$.

By the way, replacing X by U in (4.1) and using (2.22) and (2.25), we find

$$\alpha(\phi AU + \mu AW) = \mu^2 A\xi + 2kLU.$$

If we take the inner product with U and make use of (4.2) and Lemma 3.3, then we obtain $g(LU, U) = 0$ and hence $t(U) = 0$.

By putting $X = U$ in (4.5), we then have

$$KU = \tau U, \tag{4.8}$$

where τ is given by $\tau\mu^2 = g(KU, U)$ by virtue of Lemma 3.3. Applying this by ϕ and using (3.12), we find

$$LU = \tau\mu W. \tag{4.9}$$

It is, using (4.8) and (4.9), seen that

$$\tau^2 = \theta - c. \tag{4.10}$$

because of (3.13).

Remark 4.2. $\Omega = \emptyset$ if $\theta = c$.

Since we have $\theta = c$, then (3.14) gives $L = 0$ and thus $KX = k\eta(X)\xi$ by virtue of (3.11). Hence, (3.17) reformed as

$$k\{\eta(X)AY - \eta(Y)AX + \eta(X)t(Y)\xi - t(X)\eta(Y)\xi\} = 0,$$

which shows $k(t(X) + g(A\xi, X) - \sigma\eta(X)) = 0$, where we have put $\sigma = \alpha + t(\xi)$. Thus, the last two equations imply

$$AX = \eta(X)A\xi + g(A\xi, X)\xi - \alpha\eta(X)\xi.$$

Since U is orthogonal to ξ and W, it is clear that $AU = 0$ and $AW = \mu\xi$.

If we put $X = \mu W$ in (4.1) and remember (2.23) and the fact that $L = 0$, then we obtain $\mu^2 U = 0$ and hence $A\xi = \alpha\xi$. Owing to Lemma 3.1, we conclude that $k = 0$ and thus $\Omega = \emptyset$.

By Remark 4.2, we may only consider the case where $\tau \neq 0$ on Ω. Because of (3.22) and (4.9) we have

$$t(\phi X) = (1 + \frac{\tau}{k})g(U, X). \tag{4.11}$$

Therefore, by (2.4), it is clear that

$$t = t(\xi)\xi - \mu(1 + \frac{\tau}{k})W. \tag{4.12}$$

Using (2.22), we can write (3.20) as

$$\mu KW = k\mu W + k(t - t(\xi)\xi),$$

which together with (4.12) implies that

$$KW = -\tau W \tag{4.13}$$

because of Lemma 3.3.

If we take account of (3.25) and (4.11), then we find

$$\tau^2(A\phi X - \phi AX) + \tau(\tau - k)(u(X)\xi + \eta(X)U) + k(ALX + LAX) = 0. \tag{4.14}$$

Differentiating (4.8) covariantly along Ω, we find

$$(\nabla_X K)U + K\nabla_X U = \tau\nabla_X U,$$

which together with (3.16) and (4.9) yields

$$\begin{aligned} \mu\tau\{t(X)w(Y) - t(Y)w(X)\} + g(K\nabla_X U, Y) - g(K\nabla_Y U, X) \\ = \tau\{g(\nabla_X U, Y) - g(\nabla_Y U, X)\}. \end{aligned} \tag{4.15}$$

By the way, because of (2.22) and (2.24), we can write (2.29) as

$$\nabla_\xi U = 3\phi AU + \alpha\mu W - \mu^2\xi + \phi\nabla\alpha. \tag{4.16}$$

Replacing X by ξ in (4.15) and taking account of the last two relationships, we find

$$\begin{aligned} \mu^2(\tau - k)\xi + \mu\tau(t(\xi) - 2\alpha)W + \mu(k - \tau)AW \\ + 3(LAU - \tau\phi AU) = \tau\phi\nabla\alpha - L\nabla\alpha, \end{aligned} \tag{4.17}$$

where we have used the first equation of (2.26).

In a direct consequence of (3.12) and (4.8), we obtain

$$\mu LW = \tau U \tag{4.18}$$

because of $\mu \neq 0$ on Ω.

In the same way as above, we see from (4.13)

$$\frac{\tau}{\mu}\{t(X)u(Y)-t(Y)u(X)\}+g(K\nabla_X W,Y)-g(K\nabla_Y W,X) \\ =\tau\{g(\nabla_Y W,X)-g(\nabla_X W,Y)\}. \tag{4.19}$$

In the next place, from (2.22) and (2.25) we have $\phi U=-\mu W$. Differentiating this covariantly and using (2.6), we find

$$g(AU,X)\xi-\phi\nabla_X U=(X\mu)W+\mu\nabla_X W.$$

Putting $X=\xi$ in this and making use of (2.29), we get

$$\mu\nabla_\xi W=3AU-\alpha U+\nabla\alpha-(\xi\alpha)\xi-(\xi\mu)W, \tag{4.20}$$

which enables us to obtain

$$W\alpha=\xi\mu. \tag{4.21}$$

From now on we assume that

$$A^2\xi=\rho A\xi+(\beta-\rho\alpha)\xi. \tag{4.22}$$

From this, and (2.22) and (2.24) we see that

$$AW=\mu\xi+(\rho-\alpha)W. \tag{4.23}$$

In the next place, differentiating (4.23) covariantly along Ω, we find

$$(\nabla_X A)W+A\nabla_X W=(X\mu)\xi+\mu\nabla_X\xi+X(\rho-\alpha)W+(\rho-\alpha)\nabla_X W. \tag{4.24}$$

By taking the inner product with W to this and using (2.26) and (4.23), we obtain

$$g((\nabla_X A)W,W)=-2g(AU,X)+X\rho-X\alpha \tag{4.25}$$

because W is a unit orthogonal vector to ξ.

Applying (4.24) by ξ and using (2.26), we also obtain

$$\mu g((\nabla_X A)W,\xi)=(\rho-2\alpha)g(AU,X)+\mu(X\mu), \tag{4.26}$$

which connected to (3.15) gives

$$\mu(\nabla_\xi A)W=(\rho-2\alpha)AU+\mu\nabla\mu-k\mu LW-cU, \tag{4.27}$$

or, using (3.10), (3.15) and (4.26),

$$\mu(\nabla_W A)\xi=(\rho-2\alpha)AU-2cU+\mu\nabla\mu. \tag{4.28}$$

Putting $X=\xi$ in (4.25) and taking account of (4,26), we have

$$W\mu=\xi\rho-\xi\alpha. \tag{4.29}$$

Replacing X by ξ in (4.24) and using (4.27), we find

$$(\rho - 2\alpha)AU - k\mu LW - cU + \mu\nabla\mu + \mu(A\nabla_\xi W - (\rho - \alpha)\nabla_\xi W)$$
$$= \mu(\xi\mu)\xi + \mu^2 U + \mu(\xi\rho - \xi\alpha)W.$$

Substituting (4.20) and (4.21) into this and making use of (4.18), we find

$$\begin{aligned} 3A^2U - 2\rho AU + (\alpha\rho - \beta - c - k\tau)U + A\nabla\alpha + \frac{1}{2}\nabla\beta - \rho\nabla\alpha \\ = 2\mu(W\alpha)\xi + (2\alpha - \rho)(\xi\alpha)\xi + \mu(\xi\rho)W. \end{aligned} \tag{4.30}$$

On the other hand, if we put $X = \mu W$ in (4.1) and take account of (2.23), (2.24) and (4.23), then we find $\alpha AU + (\beta - \rho\alpha + 2k\tau)U = 0$, which shows

$$AU = \lambda U, \tag{4.31}$$

where the function λ is defined, using Remark 4.1, by

$$\alpha\lambda = \rho\alpha - \beta - 2k\tau. \tag{4.32}$$

Differentiating (4.31) covariantly along Ω, we find

$$(\nabla_X A)U + A\nabla_X U = (X\lambda)U + \lambda\nabla_X U.$$

If we take the skew-symmetric part of this, then we get

$$\mu(k\tau - c)(\eta(Y)w(X) - \eta(X)w(Y)) + g(A\nabla_X U, Y) - g(A\nabla_Y U, X)$$
$$= (X\lambda)u(Y) - (Y\lambda)u(X) + \lambda\{g(\nabla_X U, Y) - g(\nabla_Y U, X)\},$$

where we have used (2.22), (2.25), (3.15) and (4.9). Replacing X by U in this and using (4.31), we get

$$A\nabla_U U - \lambda\nabla_U U = (U\lambda)U - \mu^2\nabla\lambda. \tag{4.33}$$

Taking the inner product with W to this and remembering (4.23), we obtain

$$\mu g(\xi, \nabla_U U) + \mu^2(W\lambda) + (\rho - \alpha - \lambda)g(W, \nabla_U U) = 0. \tag{4.34}$$

By the way, from $KU = \tau U$, we have

$$(\nabla_X K)U + K\nabla_X U = \tau\nabla_X U, \tag{4.35}$$

which implies that $g((\nabla_X K)U, U) = 0$. Because of (3.16), (4.9) and the last relationship give $(\nabla_U K)U = 0$, which connected to (4.13) and (4.35) yields $g(W, \nabla_U U) = 0$. Thus, (4.34) reformed as

$$\mu g(\xi, \nabla_U U) + \mu^2(W\lambda) = 0.$$

However, the first term of this vanishes identically because of (2.26) and (4.23), which shows $\mu(W\lambda) = 0$ and hence

$$W\lambda = 0. \tag{4.36}$$

In the same way, we verify, using (2.26) and (4.23), that

$$\xi\lambda = 0. \tag{4.37}$$

Now, differentiating (2.25) covariantly and using (2.5), we find

$$(\nabla_X A)\xi + A\phi AX = (X\alpha)\xi + \alpha\phi AX + (X\mu)W + \mu\nabla_X W.$$

If we put $X = \mu W$ in this and use (4.23), (4.28) and (4.31), then we find

$$\mu^2\nabla_W W - \mu\nabla\mu = (2\rho\lambda - 3\alpha\lambda + \alpha^2 - \alpha\rho - 2c)U - \mu(W\alpha)\xi - \mu(W\mu)W. \tag{4.38}$$

5. Semi-invariant submanifolds satisfying $R_\xi S = SR_\xi$

We will continue our arguments under the same hypotheses $R_\xi\phi = \phi R_\xi$ and $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ as those in section 4. Further, we assume that

$$R_\xi S = SR_\xi \tag{5.1}$$

holds on M.

From (2.15) the Ricci tensor S of type (1,1) of M is given by

$$SX = c\{(2n+1)X - 3\eta(X)\xi\} + hAX - A^2X + kKX - K^2X - L^2X$$

by virtue of (3.10).

By the way, we see, using (3.5) and (3.10), that

$$K^2X = (\theta - c)(X - \eta(X)\xi) + k^2\eta(X)\xi. \tag{5.2}$$

Substituting this and (3.14) into the last equation and using (4.10), we obtain

$$SX = \{(2n+1)c - 2(\theta - c)\}X + (2(\theta - c) - k^2 - 3c)\eta(X)\xi + hAX - A^2X + kKX, \tag{5.3}$$

which connected to (3.10) yields

$$S\xi = 2(n-1)c\xi + hA\xi - A^2\xi. \tag{5.4}$$

Because of (3.10), we can write (2.30) as

$$R_\xi X = cX - (k^2 + c)\eta(X)\xi + \alpha AX - \eta(AX)A\xi + kKX.$$

Combining this to (5.3), the condition (5.1) gives

$$(k^2+c)\{\eta(X)S\xi - g(S\xi,X)\xi\} + \alpha(AS-SA)X + k(KS-SK)X \\ + g(SA\xi,X)A\xi - g(A\xi,X)SA\xi = 0. \tag{5.5}$$

On the other hand, from (5.4) we have

$$g(S\xi,X)\xi - \eta(X)S\xi = h\{g(A\xi,X)\xi - \eta(X)A\xi\} + \eta(X)A^2\xi - g(A^2\xi,X)\xi.$$

Because of (3.18) and (3.20) we also have

$$(A^2K - KA^2)X = k\{t(X)A\xi - g(A\xi,X)t\} + k\{t(AX)\xi - \eta(X)At\},$$

which together with (5.3) implies that

$$(SK-KS)X = kh\{t(X)\xi - \eta(X)t\} + k\{g(A\xi,X)t - t(X)A\xi\} \\ + k\{\eta(X)At - t(AX)\xi\}.$$

However, we see from (3.20) and (5.3)

$$SA\xi = \{(2n+1)c - 2\tau^2 + k^2\}A\xi + k^2(t - t(\xi)\xi) - \alpha(k^2-\tau^2)\xi \\ + hA^2\xi - A^3\xi,$$

which enables us to obtain

$$g(SA\xi,X)A\xi - g(A\xi,X)SA\xi \\ = k^2\{t(X)A\xi - g(A\xi,X)t\} - \{\alpha(k^2-\tau^2) + t(\xi)k^2\}(\eta(X)A\xi - g(A\xi,X)\xi) \\ + h\{g(A^2\xi,X)A\xi - g(A\xi,X)A^2\xi\} + g(A\xi,X)A^3\xi - g(A^3\xi,X)A\xi.$$

Substituting above three equations into (5.5), we find

$$g(A^3\xi,X)A\xi - g(A\xi,X)A^3\xi \\ = \{hA\xi - (k^2+c)\xi\}g(A^2\xi,X) - \{hg(A\xi,X) - (k^2+c)\eta(X)\}A^2\xi \\ + k^2\{\eta(X)At - t(AX)\xi\} + k^2(h-\alpha)(t(X)\xi - \eta(X)t) \\ + \{h(k^2+c) + t(\xi)k^2\}(g(A\xi,X)\xi - \eta(X)A\xi). \tag{5.6}$$

Putting $X = \xi$ in this, we have

$$\gamma A\xi - \alpha A^3\xi = \beta\{hA\xi - (k^2+c)\xi\} - (h\alpha - k^2 - c)A^2\xi \\ + k^2(At - t(AX)\xi) + k^2(h-\alpha)(t(\xi)\xi - t) + \{h(k^2+c) + t(\xi)k^2\}(\alpha\xi - A\xi). \tag{5.7}$$

By the way, we see from (2.22) and (4.12)

$$k(t - t(\xi)\xi) = (k+\tau)(\alpha\xi - A\xi),$$

which tells us that

$$kAt = \{kt(\xi) + \alpha(k+\tau)\}A\xi - (k+\tau)A^2\xi. \tag{5.8}$$

Using these facts, (5.7) reformed as

$$\alpha A^3\xi = (h\alpha + k\tau - c)A^2\xi + \{\gamma - \beta h + h(c - k\tau)\}A\xi + (\beta - h\alpha)(c - k\tau)\xi. \tag{5.9}$$

Substituting this and (5.8) into (5.6), we find

$$\begin{aligned}(k\tau - c)\{g(A^2\xi, X)A\xi - g(A\xi, X)A^2\xi\}\\ = \alpha(k\tau - c)\{g(A^2\xi, X)\xi - \eta(X)A^2\xi\} + y\{g(A\xi, X)\xi - \eta(X)A\xi\}\end{aligned}$$

for some function y, which connected to (2.22) implies that

$$\mu(k\tau - c)\{w(X)A^2\xi - g(A^2\xi, X)W\} = x\mu\{w(X)\xi - \eta(X)W\}$$

for some function x. If we put $X = A\xi$ in this, then we get

$$(k\tau - c)\{\mu^2 A^2\xi - \mu\gamma W\} = x\mu(\mu\xi - \alpha W). \tag{5.10}$$

We notice here that $k\tau - c \neq 0$.

In fact, if not, then we have $k\tau - c = 0$ and hence k is a constant on this subset. Thus (3.23) implies that $t(\phi X) = 2u(X)$ for any vector X, which together with (4.12) gives $k - \tau = 0$ on the set. And consequently we have $\tau^2 - c = 0$, a contradiction because $\theta - 2c \neq 0$ was assumed.
Therefore (5.10) yields

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi \tag{5.11}$$

on Ω, where we have used (2.22), and have put $\mu\rho = g(A^2\xi, W)$. Thus, (5.1) implies (4.22) and hence (4.23). From (4.22) we have

$$\alpha A^3\xi = \alpha(\rho^2 + \beta - \rho\alpha)A\xi + \rho\alpha(\beta - \rho\alpha)\xi.$$

Comparing this with (5.9), we obtain

$$(h - \rho)(\beta - \rho\alpha + k\tau - c) = 0. \tag{5.12}$$

From (4.8), we can write (3.21) as

$$\nabla k = (\xi k)\xi + (k - \tau)U. \tag{5.13}$$

On the other hand, if we put $X = \mu W$ in (4.14) and take account of (2.23) and (4.23), then we get

$$(\theta - c)\{AU - (\rho - \alpha)U\} + k\tau\{AU + (\rho - \alpha)U\} = 0,$$

which connected to (4.31) yields

$$\lambda(k + \tau) + (\rho - \alpha)(k - \tau) = 0. \tag{5.14}$$

In the next place, we will prove the following lemma :

Lemma 5.1. *If M satisfies* (4.1), (4.23) *and* $dt = 2\theta\omega$ *for a scalar* $\theta(\neq 2c)$, *then we have* $k - \tau \neq 0$ *on* Ω.

Proof. If not, then we have $k - \tau = 0$ on an open subset of Ω. We discuss our argument on such a place. Then we have $\lambda = 0$ because of (5.14) and Remark 4.2. So (4.31) and (4.32) turn out respectively to

$$AU = 0, \tag{5.15}$$

$$\beta - \rho\alpha + 2\tau^2 = 0. \tag{5.16}$$

We also have from (4.11) $t = t(\xi)\xi - 2\phi U$, which shows $t(Y) = t(\xi)\eta(Y) - 2g(\phi U, Y)$ for any vector Y. Differentiating this covariantly and using (2.5), (2.6) and (5.15), we find

$$(\nabla_X t)Y = X(t(\xi))\eta(Y) + t(\xi)g(\phi AX, Y) - 2g(\phi\nabla_X U, Y),$$

from which, taking the skew-symmetric part with respect to X and Y and using (3.1),

$$\begin{aligned}2\theta g(\phi X, Y) = X(t(\xi))\eta(Y) - Y(t(\xi))\eta(X) + t(\xi)\{g(\phi AX, Y) - g(\phi AY, X)\}\\ + 2\{g(\phi\nabla_Y U, X) - g(\phi\nabla_X U, Y)\}.\end{aligned}$$

On the other hand, we verify from (2.27) that

$$\begin{aligned}&g(\phi\nabla_X U, Y) - g(\phi\nabla_Y U, X) + (X\alpha)\eta(Y) - (Y\alpha)\eta(X)\\ &\quad = -2cg(\phi X, Y) - 2g(A\phi AX, Y) + \alpha\{g(\phi AX, Y) - g(\phi AY, X)\}.\end{aligned}$$

Combining the last two equations, it follows that

$$\begin{aligned}&2(\theta - 2c)g(\phi X, Y) + t(\xi)\{g(\phi AX, Y) - g(\phi AY, X)\}\\ &\quad = X(t(\xi))\eta(Y) - Y(t(\xi))\eta(X) + 2\{2g(A\phi AX, Y) + \alpha(g(\phi AX, Y)\\ &\quad - g(\phi AY, X)) + (X\alpha)\eta(Y) - (Y\alpha)\eta(X)\}.\end{aligned}$$

Putting $Y = \xi$ in this and remembering (5.15), we find

$$X(t(\xi)) + 2(X\alpha) = \{\xi(t(\xi)) + 2\xi\alpha\}\eta(X) + (t(\xi) + 2\alpha)u(X). \tag{5.17}$$

Substituting this into the last equation, we obtain

$$\begin{aligned}2(\theta - 2c)g(\phi X, Y) = (t(\xi) + 2\alpha)\{u(X)\eta(Y) - u(Y)\eta(X)\\ + g(\phi AX, Y) - g(\phi AY, X)\} + 4g(A\phi AX, Y).\end{aligned}$$

If we put $X = \mu W$ in this and take account of (2.23), (4.23) and (5.15), then we get

$$2(\theta - 2c) = (t(\xi) + 2\alpha)(\rho - \alpha). \tag{5.18}$$

In the next step, differentiating (4.13) covariantly, we find

$$(\nabla_X K)W + K\nabla_X W + \tau\nabla_X W = 0,$$

from which, taking the skew-symmetric part and using (3.16) and (4.9),

$$\begin{aligned}\frac{\tau}{\mu}\{t(Y)u(X) - t(X)u(Y)\} + g(K\nabla_X W, Y) - g(K\nabla_Y W, X)\\ = \tau\{(\nabla_Y W)X - (\nabla_X W)Y\}.\end{aligned} \tag{5.19}$$

If we put $X = \xi$ in this and make use of (2.26), (4.13), (4.20) and (5.15), then we find

$$K\nabla\alpha + \tau\nabla\alpha = 2\tau(\xi\alpha)\xi + \tau(2\alpha + t(\xi))U. \tag{5.20}$$

Replacing X by W in (5.19) and making use of (4.38), we have

$$\mu(K\nabla\mu + \tau\nabla\mu) = 2\tau(\mu^2 - \alpha^2 + \rho\alpha + 2c)U + 2\mu\tau(W\alpha)\xi.$$

If we take the inner product with U to this and take account of (4.8), then we obtain $\mu(U\mu) = (\mu^2 - \alpha^2 + \rho\alpha + 2c)\mu^2$, which together with (2.24) and (5.16) gives

$$\mu(U\mu) = 2(\mu^2 + \tau^2 + c)\mu^2. \tag{5.21}$$

On the other hand, differentiating (5.15) covariantly with respect to ξ, we find $(\nabla_\xi A)U + A\nabla_\xi U = 0$, which together with (4.16), (5.11) and (5.15) implies that

$$(\nabla_\xi A)U + (\alpha\rho - \beta)A\xi - \alpha(\beta - \rho\alpha)\xi + A\phi\nabla\alpha = 0.$$

Applying by ϕ, we have

$$\phi(\nabla_\xi A)U + (\alpha\rho - \beta)U + \phi A\phi\nabla\alpha = 0. \tag{5.22}$$

Since we see from (3.15)

$$(\nabla_U A)\xi - (\nabla_\xi A)U = \mu(\tau^2 + c)W$$

by virtue of (2.25), (3.10) and (4.9), it follows that

$$\phi(\nabla_U A)\xi = \phi(\nabla_\xi A)U + (\tau^2 + c)U. \tag{5.23}$$

We also have from (2.27)

$$\nabla_X U + g(A^2\xi, X)\xi = \phi(\nabla_X A)\xi + \phi A\phi AX + \alpha AX,$$

which connected to (5.15) gives $\nabla_U U = \phi(\nabla_U A)\xi$. Thus, (5.23) reformed as

$$\nabla_U U = \phi(\nabla_\xi A)U + (\tau^2 + c)U.$$

Combining this to (5.22) and using (5.16), it follows that

$$\nabla_U U = (c - \tau^2)U - \phi A \phi \nabla \alpha. \tag{5.24}$$

If we apply by A and take account of (4.33) with $\lambda = 0$ and (5.15), then we have $A\phi A\phi\nabla\alpha = 0$.

Now, taking the inner product with U to (4.30) and making use of (2.22) $\sim$ (2.25) and (4.23), we obtain

$$\mu(U\mu) = (c - \tau^2)\mu^2 + (\rho - \alpha)U\alpha. \tag{5.25}$$

However, applying (5.20) by U and using (4.8), we find $2U\alpha = (t(\xi) + 2\alpha)\mu^2$, which connected to (5.18) gives $(\rho - \alpha)U\alpha = (\theta - 2c)\mu^2$. Substituting (5.21) and this into (5.25), we find $2\mu^2 + 3c + 3\tau^2 = \theta$, which together with (4.10) gives $\mu^2 + \tau^2 + c = 0$ and consequently μ is a constant. Thus, we see, using (2.24) and (5.16), that

$$\alpha(\rho - \alpha) = \tau^2 - c. \tag{5.26}$$

Therefore, $\alpha(\rho - \alpha) = const$. Differentiation gives

$$(\rho - \alpha)\nabla\alpha + \alpha(\nabla\rho - \nabla\alpha) = 0,$$

which connected to (4.29) implies that $(\rho - \alpha)\xi\alpha = 0$, where we have used $\mu = const$. Accordingly we have $\xi\alpha = 0$ by virtue of (5.26) and the fact that $\theta - 2c \neq 0$.

Using (4.10) and (5.26), we can write (5.18) as

$$2(\theta - 2c)\alpha = (\theta - 2c)(t(\xi) + 2\alpha).$$

Thus, it follows that $t(\xi) = 0$ provided that $\theta - 2c \neq 0$. Hence, (5.17) turns out to be $\nabla\alpha = \alpha U$, which implies $du = 0$. Therefore, it is clear that $\nabla_U U = 0$ because of $\mu = const$, which connected to (5.24) yields $(c - \tau^2)U = \alpha\phi A\phi U$. So we have $c - \tau^2 = \alpha(\rho - \alpha)$, where we have used (2.23), (2.25) and (4.23). From this and (5.26) it follows that $\theta - 2c = 0$, a contradiction. Hence, Lemma 5.1 is proved. □

Lemma 5.2. *Under the same hypotheses as those in Lemma 5.1, we have on* Ω

$$\nabla k = (k - \tau)U. \tag{5.27}$$

Proof. Differentiating (5.13) covariantly along Ω, we find

$$Y(Xk) = Y(\xi k)\eta(X) + (\xi k)g(\phi AX, Y) + (Yk)u(X) + (k - \tau)\nabla_Y u(X),$$

from which, taking the skew-symmetric part, it follows that

$$\begin{aligned}\eta(X)Y(\xi k)-&\eta(Y)X(\xi k)+(\xi k)\{\eta(Y)u(X)-\eta(X)u(Y)\\&+g(\phi AY,X)-g(\phi AX,Y)\}=(k-\tau)du(X,Y),\end{aligned}\tag{5.28}$$

where d denotes the operator of the exterior derivative.

Now, we assumed that $\beta-\rho\alpha+k\tau-c\neq 0$ on Ω. Then we have $\rho=h$ because of (5.12), So (5.14) reformed as $\lambda(k+\tau)+(h-\alpha)(k-\tau)=0$.

Differentiation this with respect to ξ gives

$$(h-\alpha+\lambda)\xi k+(k-\tau)(\xi h-\xi\alpha)=0,\tag{5.29}$$

where we have used (4.37).

On the other hand, we take an orthonormal frame filed $\{e_0=\xi, e_1=W, e_2,\cdots,e_{n-1}, e_n=\phi e_1=\frac{1}{\mu}U, e_{n+1}=\phi e_2,\cdots,e_{2n-2}=\phi e_{n-1}\}$ of M. Taking the trace of (2.27), we obtain

$$\sum_{i=0}^{2n-2} g(\phi\nabla_{e_i}U,e_i)=\xi\alpha-\xi h.$$

Putting $X=\phi e_i$ and $Y=e_i$ in (5.28) and summing up for $i=1,2,\cdots,n-1$, we have

$$(k-\tau)\sum_{i=0}^{2n-2} du(\phi e_i,e_i)=\xi k(\alpha-h),$$

where we have used (2.22), (2.25), (4.23) and (4.31). Combining the last two relationships, we get

$$(h-\alpha)\xi k=(k-\tau)(\xi h-\xi\alpha).$$

From this and (5.29) we see that $(2h-2\alpha+\lambda)\xi k=0$.

If $\xi k\neq 0$ on Ω, then we have $\lambda=2(\alpha-h)$, which together with (5.14) implies that $(h-\alpha)(k+3\tau)=0$ on this subset. We discuss our arguments on such a place. So we have $h-\alpha=0$ from the last equation and hence $\lambda=0$. Consequently we have $\mu^2+2k\tau=0$ by virtue of (2.24) and (4.32) with $\rho=h$. Differentiation with respect to ξ gives $\mu(\xi\mu)+\tau(\xi k)=0$.

However, if we take the inner product with U to (4.28) and remember (2.24), (4.31) and the fact that $h-\alpha=0$ and $\lambda=0$, then we have $\mu\nabla\mu=(\mu^2+k\tau+c)U$ and consequently $\xi\mu=0$. Hence we have $\tau(\xi k)=0$, a contradiction. Thus, we have (5.27) provided that $h=\rho$.

Accordingly, we may only consider that the case where

$$\beta-\rho\alpha=c-k\tau\tag{5.30}$$

because of (5.12). Combining this to (4.32), we obtain

$$\alpha\lambda+k\tau+c=0.\tag{5.31}$$

Because of (5.30) and (5.31) we can write (5.14) as

$$(k-\tau)(\beta-\alpha^2)+(k-\tau)(k\tau-c)=(k+\tau)(k\tau+c),$$

which together with (2.24) yields $(k-\tau)\mu^2=k(\tau^2+\tau+2c)$. Differentiation gives

$$2\mu(k-\tau)\nabla\mu=\{2(\tau^2+c)-\mu^2\}\nabla k,$$

which connected to (5.13) and Lemma 5.1 gives

$$W\mu=0. \tag{5.32}$$

Since we have already showed that $g(W,\nabla_U U)=0$, it is seen that $du(W,U)=0$ because of (5.32). So if we put $X=U$ and $Y=W$ in (5.28) and make use of (4.23) and (4.31), then we obtain $\xi k(\lambda+\rho-\alpha)=0$ and hence $\lambda+\rho-\alpha=0$ if $\xi k\neq 0$, which together with (5.14) gives $\lambda=0$. From this and (5.31) we see that $k\tau+c=0$, a contradiction. This completes the proof of Lemma 5.2. □

Owing to Lemma 5.1 and Lemma 5.2, we verify from (5.28) that $du=0$. Hence we have $du(\xi,X)=0$ for any vector X on Ω, which together with (2.5), (2.26), (4.16) and (4.31) implies that

$$3\lambda\phi U+A\xi-\beta\xi+\phi\nabla\alpha+\mu AW=0,$$

or, using (2.22), (2.23) and (5.14)

$$\nabla\alpha=(\xi\alpha)\xi+(\rho-3\lambda)U. \tag{5.33}$$

We are now going to prove that $\xi\alpha=0$.

Differentiation (5.14) with respect to ξ gives $\xi\rho-\xi\alpha=0$ with the aid of (4.37), Lemma 5.1 and Lemma 5.2.

Using (4.31), (5.33) and this fact, we can write (4.30) as

$$\frac{1}{2}\nabla\beta+(2\rho\lambda+\alpha\rho-\beta-c-k\tau-\rho^2)U=\{2\mu(W\alpha)+\alpha(\xi\alpha)\}\xi. \tag{5.34}$$

Since we have $W\mu=0$ because of (4.29), if we take the inner product W to the last equation and take account of (2.24), then we obtain $\alpha(W\alpha)=0$ and hence $W\alpha=0$ by virtue of Remark 4.1.

Differentiating (4.32) with respect to ξ and making use of (4.37), Lemma 5.1 and the fact that $\xi\rho-\xi\alpha=0$, we find $\xi\beta=(\rho+\alpha-\lambda)\xi\alpha$.

On the other hand, if we differentiate (2.24) with respect to ξ and remember $W\alpha=0$ and (4.21), then we have $\xi\beta=2\alpha(\xi\alpha)$. From this and the last relationship we get $(\lambda+\alpha-\rho)\xi\alpha=0$.

Now, if $\xi\alpha\neq 0$ on Ω, the we have $\lambda=\rho-\alpha$ on this subset. We discuss our arguments on this subset. Then (5.14) yields $\lambda k=0$ and hence $\lambda=0$ and $\rho-\alpha=0$. So (5.33) and (5.34) are reduced respectively to

$$\nabla\alpha = (\xi\alpha)\xi + \alpha U, \qquad \frac{1}{2}\nabla\beta = \alpha(\xi\alpha)\xi + (\beta + k\tau + c)U.$$

We also have from (4.32) $\beta = \alpha^2 - 2k\tau$, which together with (5.27) yields $\nabla\beta = 2\alpha\nabla\alpha - 2\tau(k-\tau)U$. Combining above equations, it follows that $\tau^2 = c$, that is, $\theta - 2c = 0$, a contradiction. Thus, (5.33) reformed as

$$\nabla\alpha = (\rho - 3\lambda)U. \tag{5.35}$$

6. Main theorem

First of all, we will prove the following lemma.

Lemma 6.1. *Let M be a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ satisfying $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$. Suppose that M satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $R_\xi S = SR_\xi$. Then the distinguished normal is parallel in the normal bundle, where S denotes the Ricci tensor of M.*

Proof. We already know that $du = 0$. So we have from (4.35)

$$g(K\nabla_X U, Y) - g(K\nabla_Y U, X) + \mu\tau\{t(X)w(Y) - t(Y)w(X)\} = 0,$$

where we have used (3.16) and (4.9). Putting $X = \xi$ in this and using (2.25), (2.26), (4.16) and (4.31), we find

$$K(3\lambda\mu W + \alpha A\xi - \beta\xi + \phi\nabla\alpha) + k\mu AW + \mu\tau t(\xi)W = 0,$$

which connected to (2.22), (3.10), (3.12), (4.13), (4.23) and (5.35) gives

$$\tau t(\xi) + (\rho - \alpha)(k + \tau) = 0, \tag{6.1}$$

or, using (5.14)

$$\tau(k-\tau)t(\xi) = \lambda(k+\tau)^2. \tag{6.2}$$

On the other hand, differentiating (4.12) covariantly along Ω, and taking account of (2.5), (2.6), (4.31) and (5.27), we get

$$\begin{aligned}(\nabla_X t)Y = X(t(\xi))\eta(Y) + t(\xi)g(\phi AX, Y) + \frac{\tau}{k^2}(k-\tau)\mu u(X)w(Y) \\ - (1 + \frac{\tau}{k})\{\lambda u(X) - g(\phi\nabla_X U, Y) + t(\nabla_X Y),\end{aligned}$$

from which taking the skew-symmetric part and using (2.25) and (3.1),

$$2\theta g(\phi X,Y)+\frac{\tau}{k^2}(k-\tau)\mu(u(Y)w(X)-u(X)w(Y)) \tag{6.3}$$
$$=X(t(\xi))\eta(Y)-Y(t(\xi))\eta(X)+t(\xi)\{g(\phi AX,Y)-g(\phi AY,X)\}$$
$$-(1+\frac{\tau}{k})\{\lambda(u(X)\eta(Y)-u(Y)\eta(X))-g(\phi\nabla_X U,Y)+g(\phi\nabla_Y U,X)\}.$$

By the way, we have from (2.27) and (3.15)

$$g(\phi\nabla_X U,Y)-g(\phi\nabla_Y U,X)+(\rho+\lambda-3\alpha)(u(X)\eta(Y)-u(Y)\eta(X))$$
$$=-2cg(\phi X,Y)-2g(A\phi AX,Y)+\alpha(g(\phi AX,Y)-g(\phi AY,X)),$$

where we have used (3.10), (4.31) and (5.35).

Combining the last two equations, we obtain

$$2\theta g(\phi X,Y)+\frac{\tau}{k^2}(k-\tau)\mu\{u(Y)w(X)-u(X)w(Y)\}-t(\xi)\{g(\phi AX,Y)-g(\phi AY,X)\}$$
$$=X(t(\xi))\eta(Y)-Y(t(\xi))\eta(X)+(1+\frac{\tau}{k})\{2cg(\phi X,Y)+(\rho-3\lambda)(u(X)\eta(Y)$$
$$-u(Y)\eta(X))-2g(A\phi AX,Y)+\alpha(g(\phi AX,Y)-g(\phi AY,X))\}.$$

Putting $Y=\xi$ in this and making use of (2.5) and (4.31), we find

$$X(t(\xi))=\xi(t(\xi))\eta(X)+\{t(\xi)+(1+\frac{\tau}{k})(\lambda+\alpha-\rho)\}u(X), \tag{6.4}$$

which together with (6.1) yields

$$X(t(\xi))=\xi(t(\xi))\eta(X)+(1+\frac{\tau}{k})(\lambda+t(\xi))u(X).$$

Substituting this into the last equation and using (5.14), we find

$$2\theta g(\phi X,Y)+\frac{\tau}{k^2}\mu(k-\tau)\{w(X)u(Y)-w(Y)u(X)\} \tag{6.5}$$
$$=(1+\frac{\tau}{k})\{(\rho-2\lambda+t(\xi))(u(X)\eta(Y)-u(Y)\eta(X))$$
$$+2cg(\phi X,Y)+2g(A\phi AX,Y)+(\rho+t(\xi))(g(\phi AX,Y)-g(\phi AY,X))\}.$$

Differentiating (6.1) covariantly and remembering (5.27), we find

$$\tau X(t(\xi))=(\alpha-\rho)(k-\tau)u(X)+(k+\tau)(X\alpha-X\rho),$$

which connected to (5.14) yields

$$\tau X(t(\xi))=(k+\tau)(X\alpha-X\rho+\lambda u(X)). \tag{6.6}$$

By the way, we see already that $\xi\rho-\xi\alpha=0$. Thus, from the last equation, it follows that $\xi(t(\xi))=0$ and hence (6.4) can be written as

$$X(t(\xi))=\{t(\xi)+(1+\frac{\tau}{k})(\lambda-\rho+\alpha)\}u(X),$$

which together with (6.1) gives

$$\tau X(t(\xi)) = \{(k + 2\tau + \frac{\tau^2}{k})(\alpha - \rho) + \tau\lambda(1 + \frac{\tau}{k})\}u(X).$$

Combining this to (6.6), we get

$$(k + \tau)(\nabla\alpha - \nabla\rho + \lambda U) = (1 + \frac{\tau}{k})\{(k + \tau)(\alpha - \rho) + \tau\lambda\}U,$$

which together with (5.14) gives

$$k(\nabla\alpha - \nabla\rho) = 2\tau(\lambda + \alpha - \rho)U, \tag{6.7}$$

where we have used $k + \tau \neq 0$.

If we differentiate (6.2) and take account of Lemma 5.1 and itself, we find

$$\lambda(k + \tau)^2 U + \tau(k - \tau)\nabla t(\xi) = (k + \tau)^2\nabla\lambda + 2\lambda(k^2 - \tau^2)U,$$

which together with (6.6), and Lemma 5.1 and Lemma 5.2 implies that $(k + \tau)\nabla\lambda = (k - \tau)(\nabla\alpha - \nabla\rho) + 2\tau\lambda U$, or using (5.14) and (6.7),

$$(k + \tau)\nabla\lambda = 6\tau\lambda U. \tag{6.8}$$

Now, if we put $X = U$ and $Y = W$ in (6.5) and using (2.23), (4.23) and (4.31), then we find

$$2\theta + \frac{\tau}{k^2}(k - \tau)\mu^2 = (1 + \frac{\tau}{k})\{2c - 2\lambda(\rho - \alpha) + (t(\xi) + \rho)(\lambda + \rho - \alpha)\}.$$

By the way, it is seen, using (4.32) and (5.14), that $(k - \tau)^2\mu^2 + 2k(\alpha\lambda + \tau k - \tau^2) = 0$. Thus, the last equation can be written as

$$\begin{aligned}\theta k(k - \tau) - \tau\alpha\lambda(k - \tau) - \tau^2(k - \tau)^2 \\ = c(k^2 - \tau^2) + \lambda^2(k + \tau)^2 - \tau\lambda(k + \tau)(t(\xi) + \rho).\end{aligned}$$

If we multiply $k - \tau$ to this and take account of (4.10), (5.14) and (6.2), then we obtain

$$\lambda^2(k + \tau)^2 + 2\tau\alpha\lambda(k - \tau) + (k - \tau)^2(\tau^2 - c) = 0. \tag{6.9}$$

Differentiating this covariantly and using (5.27) and (6.8), we find

$$(k - \tau)\nabla(\alpha\lambda) = \lambda\{\alpha(k - \tau) - 4\lambda(k + \tau)\}U.$$

From this and (5.14) and (5.35), we have

$$\alpha(k - \tau)\nabla\lambda + 6\tau\lambda^2 U = 0,$$

which together with (6.8) yields $\lambda\{\alpha(k - \tau) + \lambda(k + \tau)\} = 0$. Thus, it follows that $\alpha(k - \tau) + \lambda(k + \tau) = 0$ by virtue of (6.9), which connected to (5.14) gives $\rho = 2\alpha$. Further, we have from the last relationship $(k + \tau)\nabla\lambda + (k - \tau)\nabla\alpha = 0$, which together with (5.35) and (6.8) gives $6\tau\lambda + (k - \tau)(2\alpha - 3\lambda) = 0$. Thus,

it follows that $(8\tau - 5k)\lambda = 0$, and hence $5k = 8\tau$ because of (6.9).
So, we see, using (5.27), that k is a constant on Ω and hence $U = 0$, a contradiction. This completes the proof. □

According to Lemma 6.1 we prove the following :

Lemma 6.2. *Under the same hypotheses as those in Lemma 6.1, we have* $K = L = 0$, *provided that the scalar curvature* s *of* M *satisfies*

$$s - 2(n-1)c \leq 0.$$

Remark 6.3. This lemma proved in [16] for the case where $\theta - 2c < 0$ and $c > 0$. But, we need the condition $s - 2(n-1)c \leq 0$ for the case where $c < 0$, where s is the scalar curvature of M. So we introduce the outline of the proof.

The sketch of Proof. By Lemma 2.2 and Lemma 6.1, we have $k = 0$ and hence $m = 0$ on M because of (3.10). Thus, (3.15)∼(3.20) turn out to be

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{6.10}$$

$$(\nabla_X K)Y - (\nabla_Y K)X = t(X)LY - t(Y)LX, \tag{6.11}$$

$$(\nabla_X L)Y - (\nabla_Y L)X = 0, \tag{6.12}$$

$$KA - AK = 0, \quad LA - AL = 0, \tag{6.13}$$

Since we have $K\xi = 0$ because of (3.10), differentiating $K\xi = 0$ covariantly along M and using (2.5) and (3.12), we find

$$(\nabla_X K)\xi = -LAX. \tag{6.14}$$

Since $k = 0$, (5.2) reformed as

$$K^2 X = \tau'(X - \eta(X)\xi)., \tag{6.15}$$

where $\tau' = \theta - c$.

Differentiating (6.15) covariantly along M and using (2.5), we find

$$(\nabla_X K)KY + K(\nabla_X K)Y = -\tau'\{\eta(Y)\phi AX + g(\phi AX, Y)\xi\}.$$

Using the quite same method as those used to (3.14), we can derive from the last equation the following :

$$\begin{aligned} 2(\nabla_X K)KY = \tau'\{&-2t(X)\phi Y + \eta(X)(\phi A - A\phi)Y \\ &+ g((\phi A - A\phi)X, Y)\xi) + \eta(Y)(\phi A + A\phi)X\}, \end{aligned} \tag{6.16}$$

where we have used (3.13) and (6.11), which together with (2.5), (3.11) and (6.11) implies that $\tau' U = 0$ and hence $A\xi = \alpha\xi$.
Therefore, if we take account of Lemma 5.3 and (3.26), then we obtain

$$\tau'(A\phi - \phi A) = 0. \tag{6.17}$$

In the following, we assume that $\tau' \neq 0$ on M. Then, from this and (6.10) we can verify the following (cf. [6]) :

$$A^2 = \alpha A + c(I - \eta \otimes \xi), \tag{6.18}$$

$$(\nabla_X A)Y = -c\{\eta(Y)\phi X + g(\phi X, Y)\xi\}. \tag{6.19}$$

Using (6.17), we can write (6.16) as

$$K(\nabla_X K)Y = \tau'\{-t(X)\phi Y + \eta(X)\phi AY + g(\phi AX, Y)\xi\}.$$

If we transform this by K and make use of (3.12), (6.11), (6.14) and (6.15), then we have

$$(\nabla_X K)Y = t(X)LY - \eta(X)ALX - \eta(Y)LAX - g(ALX, Y)\xi. \tag{6.20}$$

Differentiating (3.12) covariantly along M and using (2.6) and the last equation, we find

$$(\nabla_X L)Y = -t(X)KY + \eta(X)AKY + \eta(Y)AKX + g(AKX, Y)\xi. \tag{6.21}$$

If we take the trace of L in this and remember (3.20) and the fact that $TrK = TrL = 0$ and $A\xi = \alpha\xi$, we verify that

$$Tr(AK) = 0, \tag{6.22}$$

which connected to (6.18) gives

$$Tr(A^2K) = 0. \tag{6.23}$$

Differentiating (6.20) covariantly along M and using (6.22), (6.23) and the previously obtained formulas and the Ricci indentity for K, we have (for detail, see (4.19) of [16]).

$$(h + 3\alpha)AL = 2\{(n+1)\theta - 2(n+2)c\}L,$$

which connected to (3.14) yields

$$(h + 3\alpha)(AX - \alpha\eta(X)\xi) = 2\{(n+1)\theta - 2(n+2)c\}(X - \eta(X)\xi).$$

Taking the trace of this, we have

$$(h + 3\alpha)(h - \alpha) = 4(n-1)\{(n+1)\theta - 2(n+2)c\}, \tag{6.24}$$

where we put

$$\delta = 4(n-1)\{(n+1)\theta - 2(n+2)c\}, \tag{6.25}$$

In the same way as above, we also obtain (for detail, see (4.21) of [16])

$$(4\theta - 12c - h_{(2)} - 3\alpha^2)AK = \{4c\alpha - (\theta - 2c)(h-\alpha)\}K,$$

which connected to (6.15) yields

$$(4\theta - 12c - h_{(2)} - 3\alpha^2)(h-\alpha) = 2(n-1)\{4c\alpha - (\theta - 2c)(h-\alpha)\}. \tag{6.26}$$

Since we have $h_{(2)} = \alpha h + 2(n-1)c$ from (6.18), combining (6.24) to (6.26), we obtain

$$(\theta - 3c)(h-\alpha) = 2(n-1)\alpha(\theta - 2c). \tag{6.27}$$

On the other hand, from (5.3) we verify that the scalar curvature s of M is given by

$$s = 4(n^2-1)c - 4(n-1)\tau' + h^2 - h_{(2)},$$

which connected to (6.18) gives

$$s = 2(n-1)(2n+1)c - 4(n-1)\tau' + h(h-\alpha). \tag{6.29}$$

By the way, it is seen, using (4.10), that $\theta - 3c \neq 0$ for $c < 0$. But we also have $\theta - 3c \neq 0$ for $c > 0$ if $s - 2(n-1)c \leq 0$.

In fact, if not, then we have $\theta = 3c$ on this open subset of M. We discuss our arguments on such a place. So we have $\alpha = 0$ because of (6.28). Hence (6.18) and (6.25) reformed respectively as $h_{(2)} = 2(n-1)c$, $h^2 = 4(n-1)^2c$. Using these facts and (4.10), we can write (6.29) as $s - 2(n-1)c = 4(n-1)(2n-3)c$, a contradiction because $s - 2(n-1)c \leq 0$.
Thus, we can write (6.27) as

$$h - \alpha = \frac{2(n-1)}{\theta - 3c}(\theta - 2c)\alpha.$$

Substituting this into (6.24), we obtain

$$4(n-1)(\theta - 2c)\{(n+1)\theta - 2(n+2)c\}\alpha * 2 = \delta(\theta - 3c)^2.$$

which together (6.25) gives

$$\delta\{(\theta - 3c)^2 - (\theta - 2c)\alpha^2\} = 0. \tag{6.30}$$

We notice here that $\delta \neq 0$ if $c < 0$. We also see that $\delta \neq 0$ for $c > 0$. In fact. if not, them we have $\delta = 0$. Then we have by (6.25)

$$\theta - c = \frac{n+3}{n+1}c.$$

Using this fact and (6.24), we can write (6.29) as

$$s - 2(n-1)c = \frac{4(n-1)}{n+1}(n^2-3)c + \epsilon^2,$$

where $\epsilon^2 = 0$ or $12\alpha^2$, a contradiction because $c > 0$ and $s - 2(n-1)c \leq 0$ was assumed. Therefore (6.30) turns out to be

$$(\theta - 3c)^2 = (\theta - 2c)\alpha^2 = 0. \tag{6.31}$$

Thus, if we combine (6.27) to (6.31). then we obtain $\alpha(h-\alpha) = 2(n-1)(\theta - 3c)$, which together with (6.24) yields

$$h(h-\alpha) = 2(n-1)(2n-1)\tau' - 4n(n-1)c.$$

Using this, we can write (6.29) as

$$s - 2(n-1)c = 2(n-1)(2n-3)\tau'.$$

Therefore we have $\tau' = 0$ if $s - 2(n-1)c \leq 0$. This completes the proof of Lemma 6.2. □

Let $N_0(p) = \{v \in T_p^{\perp}(M) : A_v = 0\}$ and $H_0(p)$ be the maximal J-invariant subspace of $N_0(p)$. As a consequence of Lemma 6.2, we have $K = L = 0$, the orthogonal complement of $H_0(p)$ is invariant under parallel translation with respect to the normal connection because of $\nabla^{\perp} C = 0$. Thus, by the reduction theorem in [9], [21] and by Lemma 3.2 and Lemma 3.3, we conclude that

Theorem 6.4. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ with constant holomorphic sectional curvature $4c$ such that the third fundamental form t satisfies $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$, where $\omega(X,Y) = g(\phi X, Y)$ for any vector fields X and Y on M. If M satisfies $R_\xi\phi = \phi R_\xi$ and at the same time $R_\xi S = SR_\xi$, then M is a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, provided that $s - 2(n-1)c \leq 0$, where s denotes the scalar curvature of M.*

Since we have $\nabla^{\perp} C = 0$, we can write (2.16) and (4.1) as

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},$$
$$\alpha(\phi AX - A\phi X) - g(A\xi, X)U - g(U, X)A\xi = 0$$

respectively. Making use of (2.5), (2.6) and the above equations, it is prove in [16] that $g(U,U) = 0$, that is, M is a Hopf real hypersuface. Hence, we conclude that $\alpha(A\phi - \phi A) = 0$ and hence $A\xi = 0$ or $A\phi = \phi A$. Here, we note that the case $\alpha = 0$ correspond to the case of tube of radius $\pi/4$ in $P_n\mathbb{C}$([5],[6]). But, in the case $H_n\mathbb{C}$ it is known that α never vanishes for Hopf hypersurfaces (cf. [3]) Thus, owing to Theorem 6.4, Theorem O and Theorem MR, we have

Main Theorem. *Let M be a real $(2n-1)$-dimensional $(n>2)$ semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ with constant holomorphic sectional curvature $4c$ such that the Ricci tensor S satisfies $R_\xi S = SR_\xi$ and the third fundamental form t satisfies $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ where, S denotes the Ricci tensor of M. Then $R_\xi \phi = \phi R_\xi$ holds on M if and only if $A\xi = 0$ or M is locally congruent to one of the following hypersurfaces :*

- (I) *in case that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$ if $s - 2(n-1)c \leq 0$,*
  - (A_1) *a geodesic hypersphere of radius r, where $0 < r < \pi/2$ and $r \neq \pi/4$,*
  - (A_2) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ for some $k \in \{1, ..., n-2\}$, where $0 < r < \pi/2$ and $r \neq \pi/4$;*
- (II) *in case that $M_n(c) = H_n\mathbb{C}$ if $s - 2(n-1)c \leq 0$,*
  - (A_0) *a horosphere,*
  - (A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,*
  - (A_2) *a tube over a totally geodesic $H_k\mathbb{C}$ for some $k \in \{1, ..., n-2\}$,*

where, s denotes the scalar curvature of M.

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