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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR SINGULAR GENERALIZED LAPLACIAN PROBLEMS WITH A PARAMETER

Chan-Gyun Kim

Abstract. In this paper, we consider singular φ-Laplacian problems with nonlocal boundary conditions. Using a fixed point index theorem on a suitable cone, the existence results for one or two positive solutions are established under the assumption that the nonlinearity may not satisfy the L^1-Carathéodory condition.

1. Introduction

In this paper, we study the existence and multiplicity of positive solutions to the following boundary value problem

$$\begin{cases} (q(t)\varphi(u'(t)))' + \lambda h(t) f(u(t)) = 0, \ t \in (0,1), \\ u(0) = \int_0^1 u(r) d\alpha_1(r), u(1) = \int_0^1 u(r) d\alpha_2(r), \end{cases} \tag{1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $q \in C([0,1],(0,\infty))$, $\lambda \in [0,\infty) := \mathbb{R}_+$ is a parameter, $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $f(s) > 0$ for $s > 0$, $h \in C((0,1), \mathbb{R}_+)$, and the integrator functions α_i $(i = 1, 2)$ are nondecreasing on $[0,1]$.

All integrals in (1) are meant in the sense of Riemann–Stieltjes. Throughout this paper, we assume the following hypotheses, unless otherwise stated.

(H_1) There exist increasing homeomorphisms $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(x)\psi_1(y) \le \varphi(yx) \le \varphi(x)\psi_2(y) \text{ for all } \ x, y \in \mathbb{R}_+.$$

(H_2) For $i = 1, 2$, $\hat{\alpha}_i := \alpha_i(1) - \alpha_i(0) \in [0, 1)$.

Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. Then we denote by $\mathcal{H}_\xi$ the set

$$\left\{ g \in C((0,1),(0,\infty)) : \int_0^1 \xi^{-1}\left(\left| \int_s^{\frac{1}{2}} g(\tau) d\tau \right| \right) ds < \infty \right\}.$$

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It is well known that

$$\varphi^{-1}(x)\psi_2^{-1}(y) \le \varphi^{-1}(xy) \le \varphi^{-1}(x)\psi_1^{-1}(y) \text{ for all } x, y \in \mathbb{R}_+ \tag{2}$$

and $L^1(0,1) \cap C(0,1) \subseteq \mathcal{H}_{\psi_1} \subseteq \mathcal{H}_{\varphi} \subseteq \mathcal{H}_{\psi_2}$ (see, e.g., [9, Remark 1]).

The nonlocal boundary value problems play an important role in physics and applied mathematics (see, e.g., [2, 7, 8]), and the existence of positive solutions for nonlocal boundary value problems have been extensively studied. For example, Liu [17] showed, under several assumptions on the nonlinearity, the existence of one or two positive solutions to four-point boundary value problems which is a special case of problem (1). Webb and Infante [19] studied the existence and multiplicity of positive solutions to semilinear elliptic problems with several nonlocal boundary conditions involving a Stieltjes integral. Ko and Lee [16] studied the existence, nonexistence and multiplicity of positive solutions to semilinear elliptic systems subject to integral boundary conditions with positive parameter. Recently, under several assumptions on the nonlinearity, Son and Wang [18] showed the existence and multiplicity of positive solutions to p-Laplacian systems with nonlinear boundary conditions. For other interesting results on problems with nonlocal boundary conditions, we refer the reader to [4, 5, 10, 11, 13, 14] and the references therein.

When $\varphi(s) = |s|^{p-2}s$ for some $p \in (1,\infty)$, $q \equiv 1$, $\hat{\alpha}_1 = \hat{\alpha}_2 = 0$ and $h \in \mathcal{H}_{\varphi} \setminus \{0\}$, Agarwal, Lü and O'Regan [1] showed the existence and multiplicity of positive solutions to problem (1) under several assumptions on $f_0 := \lim_{s\to 0} \frac{f(s)}{\varphi(s)}$ and $f_\infty := \lim_{s\to\infty} \frac{f(s)}{\varphi(s)}$. Recently, Kim [12] extended the results of [1] to singular generalized Laplacian problem (1) with the assumptions that q may not be 1, $\hat{\alpha}_1 = \hat{\alpha}_2 = 0$ and $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$.

Motivated by the previous results mentioned above, we study the existence of one or two positive solutions to the problem (1). The rest of this paper is organized as follows. In Section 2, we give preliminary results which are essential for proving the main result (Theorem 3.3) in this paper. In Section 3, the main result is proved.

2. Preliminaries

For convenience, we use some notations used in [10] (or [15]) as follows. The usual maximum norm in a Banach space $C[0,1]$ is denoted by $\|u\|_\infty := \max_{t\in[0,1]} |u(t)|$ for $u \in C[0,1]$. For $h \in \mathcal{H}_\varphi \setminus \{0\}$, let $\alpha_h := \inf\{x \in (0,1) : h(x) > 0\}$, $\beta_h := \sup\{x \in (0,1) : h(x) > 0\}$, $\bar{\alpha}_h := \sup\{x \in (0,1) : h(y) > 0 \text{ for all } y \in (\alpha_h, x)\}$, $\bar{\beta}_h := \inf\{x \in (0,1) : h(y) > 0 \text{ for all } y \in (x, \beta_h)\}$, $\gamma_h^1 := \frac{1}{4}(3\alpha_h + \bar{\alpha}_h)$ and $\gamma_h^2 := \frac{1}{4}(\bar{\beta}_h + 3\beta_h)$. From $h \in C((0,1), \mathbb{R}_+) \setminus \{0\}$, it follows that

$$h(t) > 0 \text{ for } t \in (\alpha_h, \bar{\alpha}_h) \cup (\bar{\beta}_h, \beta_h), \text{ and } 0 \le \alpha_h < \gamma_h^1 < \gamma_h^2 < \beta_h \le 1. \tag{3}$$

Let $\rho_h := \rho_1 \min\{\gamma_h^1, 1 - \gamma_h^2\} \in (0,1)$, where

$$q_0 := \min_{t\in[0,1]} q(t) > 0 \text{ and } \rho_1 := \psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{q_0}\right)\right]^{-1} \in (0,1].$$

Then $\mathcal{K} := \{u \in C([0,1],\mathbb{R}_+) : u(t) \geq \rho_h\|u\|_\infty \text{ for } t \in [\gamma_h^1, \gamma_h^2]\}$ is a cone in $C[0,1]$. For $r > 0$, let $\mathcal{K}_r := \{u \in \mathcal{K} : \|u\|_\infty < r\}$, $\partial\mathcal{K}_r := \{u \in \mathcal{K} : \|u\|_\infty = r\}$ and $\overline{\mathcal{K}}_r := \mathcal{K}_r \cup \partial\mathcal{K}_r$.

For $g \in \mathcal{H}_\varphi$, consider the following problem

$$\begin{cases} (q(t)\varphi(u'(t)))' + g(t) = 0, \ t \in (0,1), \\ u(0) = \int_0^1 u(r)d\alpha_1(r), u(1) = \int_0^1 u(r)d\alpha_2(r). \end{cases} \tag{4}$$

Define a function $T : \mathcal{H}_\varphi \to C[0,1]$ by $T(0) = 0$ and, for $g \in \mathcal{H}_\varphi \setminus \{0\}$,

$$T(g)(t) = \begin{cases} A_1 \int_0^1 \int_0^r I_g(s,\sigma)dsd\alpha_1(r) + \int_0^t I_g(s,\sigma)ds, & \text{if } 0 \leq t \leq \sigma, \\ A_2 \int_0^1 \int_r^1 I_g(\sigma,s)dsd\alpha_2(r) + \int_t^1 I_g(\sigma,s)ds, & \text{if } \sigma \leq t \leq 1, \end{cases}$$

where $A_i := (1-\hat{\alpha}_i)^{-1} \in [1,\infty)$ for $i \in \{1,2\}$, $I_g(x,y) := \varphi^{-1}\left(\frac{1}{q(s)}\int_x^y g(\tau)d\tau\right)$ for $x, y \in (0,1)$ and $\sigma = \sigma(g)$ is a constant satisfying

$$\begin{aligned} & A_1 \int_0^1 \int_0^r I_g(s,\sigma)dsd\alpha_1(r) + \int_0^\sigma I_g(s,\sigma)ds \\ = \ & A_2 \int_0^1 \int_r^1 I_g(\sigma,s)dsd\alpha_2(r) + \int_\sigma^1 I_g(\sigma,s)ds. \end{aligned} \tag{5}$$

Then T is well defined, and although $\sigma = \sigma(g)$ is not necessarily unique, $T(g)$ is independent of the choice of σ satisfying (5) (see [10, Lemma 1 and Remark 2]).

Lemma 2.1. ([10, Lemma 2]) *Assume that* $(H_1), (H_2)$ *and* $g \in \mathcal{H}_\varphi$ *hold. Then* $T(g)$ *is a unique solution to problem* (4), *and the following properties are satisfied:*

- *(i)* $T(g)(t) \geq \min\{T(g)(0), T(g)(1)\} \geq 0$ *for* $t \in [0,1]$;
- *(ii) for any* $g \not\equiv 0$, $\max\{T(g)(0), T(g)(1)\} < \|T(g)\|_\infty$;
- *(iii)* σ *is a constant satisfying* (5) *if and only if* $T(g)(\sigma) = \|T(g)\|_\infty$;
- *(iv)* $T(g)(t) \geq \rho_1 \min\{t, 1-t\}\|T(g)\|_\infty$ *for* $t \in [0,1]$ *and* $T(g) \in \mathcal{K}$.

Define a function $F : \mathbb{R}_+ \times \mathcal{K} \to C(0,1)$ by $F(\lambda,u)(t) := \lambda h(t)f(u(t))$ for $(\lambda,u) \in \mathbb{R}_+ \times \mathcal{K}$ and $t \in (0,1)$. Clearly, $F(\lambda,u) \in \mathcal{H}_\varphi$ for any $(\lambda,u) \in \mathbb{R}_+ \times \mathcal{K}$, since $h \in \mathcal{H}_\varphi$. Let us define an operator $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ by $H(\lambda,u) := T(F(\lambda,u))$ for $(\lambda,u) \in \mathbb{R}_+ \times \mathcal{K}$. By Lemma 2.1 *(iv)*, $H(\mathbb{R}_+ \times \mathcal{K}) \subseteq \mathcal{K}$, and consequently H is well defined. Moreover, u is a solution to the problem (1) if and only if $H(\lambda,u) = u$ for some $(\lambda,u) \in \mathbb{R}_+ \times \mathcal{K}$.

Lemma 2.2. ([13, Lemma 4] or [14, Lemma 4]) *Assume that* $(H_1), (H_2)$ *and* $h \in \mathcal{H}_\varphi \setminus \{0\}$ *hold. Then the operator* $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ *is completely continuous.*

Finally, we recall a well-known theorem of the fixed point index theory.

Theorem 2.3. ([3, 6]) *Assume that, for some $m > 0$, $\mathcal{H} : \overline{\mathcal{K}}_m \to \mathcal{K}$ is completely continuous. Then the following assertions are true.*

(i) $i(\mathcal{H}, \mathcal{K}_m, \mathcal{K}) = 1$ if $\|\mathcal{H}(u)\|_\infty < \|u\|_\infty$ for $u \in \partial\mathcal{K}_m$;

(ii) $i(\mathcal{H}, \mathcal{K}_m, \mathcal{K}) = 0$ if $\|\mathcal{H}(u)\|_\infty > \|u\|_\infty$ for $u \in \partial\mathcal{K}_m$.

3. Main results

Let $\mathcal{C}_1 := \psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right) \min\left\{\int_{\gamma_h^1}^{\gamma_h} \psi_2^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right) ds, \int_{\gamma_h}^{\gamma_h^2} \psi_2^{-1}\left(\int_{\gamma_h}^{s} h(\tau)d\tau\right) ds\right\}$ and $\mathcal{C}_2 := \psi_1^{-1}\left(\frac{1}{q_0}\right) \max\left\{A_1 \int_0^{\gamma_h} \psi_1^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right) ds, A_2 \int_{\gamma_h}^{1} \psi_1^{-1}\left(\int_{\gamma_h}^{s} h(\tau)d\tau\right) ds\right\}$. Here, $\gamma_h := \frac{\gamma_h^1 + \gamma_h^2}{2}$ and $A_i := (1 - \hat{\alpha}_i)^{-1} \geq 1$ for $i = 1, 2$. Clearly, by (3),

$$\mathcal{C}_1 > 0 \text{ and } \mathcal{C}_2 > 0.$$

Define continuous functions $f_*, f^* : \mathbb{R}_+ \to \mathbb{R}_+$ by, for $r \in \mathbb{R}_+$,

$$f_*(r) := \min\{f(y) : \rho_h r \leq y \leq r\} \text{ and } f^*(r) := \max\{f(y) : 0 \leq y \leq r\}.$$

Define $S_1, S_2 : (0,\infty) \to (0,\infty)$ by

$$S_1(r) := \frac{1}{f_*(r)}\varphi\left(\frac{r}{C_1}\right) \text{ and } S_2(r) := \frac{1}{f^*(r)}\varphi\left(\frac{r}{C_2}\right) \text{ for } r \in (0,\infty).$$

By (2) and (H_2), $\psi_2^{-1}(y) \leq \psi_1^{-1}(y)$ for all $y \in \mathbb{R}_+$ and $A_i = (1 - \hat{\alpha}_i)^{-1} \geq 1$ for $i = 1, 2$. Consequently, $0 < \mathcal{C}_1 < \mathcal{C}_2$ and

$$0 < S_2(r) < S_1(r) \text{ for all } r \in (0,\infty). \tag{6}$$

Remark 1. For any $L \in C(\mathbb{R}_+, \mathbb{R}_+)$, let $L_c := \lim_{r\to c} \frac{L(r)}{\varphi(r)}$ for $c \in \{0, \infty\}$. Then it is well known that $(f_*)_c = (f^*)_c = 0$ if $f_c = 0$, and $(f_*)_c = (f^*)_c = \infty$ if $f_c = \infty$ (see, e.g., [12, Remark 2]). For $i \in \{1, 2\}$, it follows from (2) that

$$\lim_{r\to 0^+} S_i(r) = 0 \text{ if } f_0 = \infty, \text{ and } \lim_{r\to\infty} S_i(r) = 0 \text{ if } f_\infty = \infty; \tag{7}$$

$$\lim_{r\to 0^+} S_i(r) = \infty \text{ if } f_0 = 0, \text{ and } \lim_{r\to\infty} S_i(r) = \infty \text{ if } f_\infty = 0. \tag{8}$$

Lemma 3.1. *Assume that $(H_1), (H_2)$ and $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$ hold. Let $r \in (0,\infty)$ be fixed. Then, for any $\lambda \in (0, S_2(r))$, $\|H(\lambda, v)\|_\infty < \|v\|_\infty$ for all $v \in \partial\mathcal{K}_r$ and $i(H(\lambda, \cdot), \mathcal{K}_r, \mathcal{K}) = 1$.*

Proof. Let $\lambda \in (0, S_2(r))$ and $v \in \partial\mathcal{K}_r$ be fixed. Then

$$0 \leq \lambda f(v(t)) \leq \lambda f^*(r) = \frac{\lambda}{S_2(r)}\varphi\left(\frac{r}{\mathcal{C}_2}\right) < \varphi\left(\frac{r}{\mathcal{C}_2}\right) \text{ for } t \in [0, 1]. \tag{9}$$

Let σ be an element of $(0, 1)$ satisfying $H(\lambda, v)(\sigma) = \|H(\lambda, v)\|_\infty$. We have two cases: either (i) $\sigma \in (0, \gamma_h)$ or (ii) $\sigma \in [\gamma_h, 1)$. We only consider the case (i) since the case (ii) can be proved similarly. First, we show that

$$\|H(\lambda, u)\|_\infty \leq A_1 \int_0^\sigma I_{F(\lambda,u)}(s, \sigma)ds. \tag{10}$$

Since $I_{F(\lambda,u)}(s,x) \geq 0$ for $x \geq s$ and $I_{F(\lambda,u)}(s,x) \leq 0$ for $x \leq s$,

$$\begin{aligned}&\int_0^1 \int_\sigma^r I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) \\ = \quad & -\int_0^\sigma \int_r^\sigma I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \int_\sigma^1 \int_\sigma^r I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) \leq 0.\end{aligned}$$

Consequently,

$$\begin{aligned}H(\lambda,u)(\sigma) &= A_1 \int_0^1 \int_0^r I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \int_0^\sigma I_{F(\lambda,u)}(s,\sigma)ds \\ &= A_1 \left[\int_0^1 \int_0^r I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \left(1 - \int_0^1 d\alpha_1(r)\right) \int_0^\sigma I_{F(\lambda,u)}(s,\sigma)ds\right] \\ &= A_1 \left[\int_0^1 \int_\sigma^r I_{F(\lambda,u)}(s,\sigma)dsd\alpha_1(r) + \int_0^\sigma I_{F(\lambda,u)}(s,\sigma)ds\right] \\ &\leq A_1 \int_0^\sigma I_{F(\lambda,u)}(s,\sigma)ds.\end{aligned}$$

From (2),(9),(10) and the definition of $\mathcal{C}_2$, it follows that

$$\begin{aligned}\|H(\lambda,v)\|_\infty &\leq A_1 \int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)} \int_s^\sigma \lambda h(\tau)f(v(\tau))d\tau\right) ds \\ &< A_1 \int_0^{\gamma_h} \varphi^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau \frac{1}{q_0}\varphi\left(\frac{r}{\mathcal{C}_2}\right)\right) ds \\ &\leq A_1 \int_0^{\gamma_h} \psi_1^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right) ds\varphi^{-1}\left(\frac{1}{q_0}\varphi\left(\frac{r}{\mathcal{C}_2}\right)\right) \\ &\leq A_1 \int_0^{\gamma_h} \psi_1^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right) ds\psi_1^{-1}\left(\frac{1}{q_0}\right)\frac{r}{\mathcal{C}_2} \leq r = \|v\|_\infty.\end{aligned}$$

By Theorem 2.3, for any $\lambda \in (0, S_2(r))$, $i(H(\lambda,\cdot),\mathcal{K}_r,\mathcal{K}) = 1$. □

Lemma 3.2. *Assume that* $(H_1),(H_2)$ *and* $h \in \mathcal{H}_{\psi_2} \setminus \{0\}$ *hold. Let* $r \in (0,\infty)$ *be fixed. Then, for any* $\lambda \in (S_1(r),\infty)$, $\|H(\lambda,v)\|_\infty > \|v\|$ *for all* $v \in \partial\mathcal{K}_r$ *and* $i(H(\lambda,\cdot),\mathcal{K}_r,\mathcal{K}) = 0$.

Proof. Let $\lambda \in (S_1(r),\infty)$ and $v \in \partial\mathcal{K}_r$ be fixed. Then $\rho_h r \leq v(t) \leq r$ for $t \in [\gamma_h^1,\gamma_h^2]$ and

$$\lambda f(v(t)) \geq \lambda f_*(r) = \frac{\lambda}{S_1(r)}\varphi\left(\frac{r}{\mathcal{C}_1}\right) > \varphi\left(\frac{r}{\mathcal{C}_1}\right) \text{ for } t \in [\gamma_h^1,\gamma_h^2]. \qquad (11)$$

Let σ be an element of $(0,1)$ satisfying $H(\lambda,v)(\sigma) = \|H(\lambda,v)\|_\infty$. Then we have two cases: either (i) $\sigma \in [\gamma_h,1)$ or (ii) $\sigma \in (0,\gamma_h)$. We only consider the case (i) since the case (ii) can be proved similarly. By Lemma 2.1 (i), $H(\lambda,v)(0) \geq 0$, and it follows from (2), (11) and the definition of $\mathcal{C}_1$ that

$$\begin{aligned}\|H(\lambda,v)\|_\infty &= H(\lambda,v)(0)+\int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma \lambda h(\tau)f(v(\tau))d\tau\right)ds\\ &> \int_{\gamma_h^1}^{\gamma_h}\varphi^{-1}\left(\int_s^{\gamma_h}h(\tau)d\tau\frac{1}{\|q\|_\infty}\varphi\left(\frac{r}{\mathcal{C}_1}\right)\right)ds\\ &\geq \int_{\gamma_h^1}^{\gamma_h}\psi_2^{-1}\left(\int_s^{\gamma_h}h(\tau)d\tau\right)ds\varphi^{-1}\left(\frac{1}{\|q\|_\infty}\varphi\left(\frac{r}{\mathcal{C}_1}\right)\right)\\ &\geq \int_{\gamma_h^1}^{\gamma_h}\psi_2^{-1}\left(\int_s^{\gamma_h}h(\tau)d\tau\right)ds\psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right)\frac{r}{\mathcal{C}_1}\geq r=\|v\|_\infty.\end{aligned}$$

By Theorem 2.3, for any $\lambda\in(S_1(r),\infty)$, $i(H(\lambda,\cdot),\mathcal{K}_r,\mathcal{K})=0$. □

Now we give the main result for the existence and multiplicity of positive solutions to the problem (1).

Theorem 3.3. *Assume that* $(H_1),(H_2)$ *and* $h\in\mathcal{H}_{\psi_1}\setminus\{0\}$ *hold.*

- (i) *Assume that there exist* r_1 *and* r_2 *such that* $0<r_1<r_2$ *(resp.,* $0<r_2<r_1$*) and* $S_1(r_1)<S_2(r_2)$*. Then the problem* (1) *has a positive solution* $u=u(\lambda)$ *satisfying* $r_1<\|u\|_\infty<r_2$ *(resp.,* $r_2<\|u\|_\infty<r_1$*) for any* $\lambda\in(S_1(r_1),S_2(r_2))$*.*
- (ii) *Assume that there exist* r_1,r_2 *and* R_1 *(resp.,* R_2*) such that* $0<r_1<r_2<R_1$ *(resp.,* $0<r_2<r_1<R_2$*) and* $S_*<S_2(r_2)$ *(resp.,* $S_1(r_1)<S^*$*). Then the problem* (1) *has two positive solutions* $u_1=u_1(\lambda)$ *and* $u_2=u_2(\lambda)$ *satisfying* $r_1<\|u_1\|_\infty<r_2<\|u_2\|_\infty<R_1$ *for any* $\lambda\in(S_*,S_2(r_2))$ *(resp.,* $r_2<\|u_1\|_\infty<r_1<\|u_2\|_\infty<R_2$ *for any* $\lambda\in(S_1(r_1),S^*)$*).*

Here, $S_*:=\max\{S_1(r_1),S_1(R_1)\}$ *and* $S^*:=\min\{S_2(r_2),S_2(R_2)\}$*.*

Proof. Since the proofs are similar, we only give the proof of Theorem 3.3 (i) with $0<r_1<r_2$. Let $\lambda\in(S_1(r_1),S_2(r_2))$ be fixed. By Lemma 3.1 and Lemma 3.2, $i(H(\lambda,\cdot),\mathcal{K}_{r_1},\mathcal{K})=0$, $i(H(\lambda,\cdot),\mathcal{K}_{r_2},\mathcal{K})=1$ and $H(\lambda,v)\neq v$ for all $v\in\partial\mathcal{K}_{r_1}$. Then, by the additivity property, $i(H(\lambda,\cdot),\mathcal{K}_{r_2}\setminus\overline{\mathcal{K}}_{r_1},\mathcal{K})=1$. Thus there exists $u\in\mathcal{K}_{r_2}\setminus\overline{\mathcal{K}}_{r_1}$ such that $H(\lambda,u)=u$, and the problem (1) has a positive solution $u=u(\lambda)$ satisfying $r_1<\|u\|_\infty<r_2$. □

Corollary 3.4. *Assume that* $(H_1),(H_2)$ *and* $h\in\mathcal{H}_{\psi_1}\setminus\{0\}$ *hold.*

- (i) *If* $f_0=\infty$ *and* $f_\infty=0$*, then the problem* (1) *has a positive solution* $u(\lambda)$ *for any* $\lambda\in(0,\infty)$ *satisfying* $\|u_\lambda\|_\infty\to 0$ *as* $\lambda\to 0$ *and* $\|u_\lambda\|_\infty\to\infty$ *as* $\lambda\to\infty$*.*
- (ii) *If* $f_0=0$ *and* $f_\infty=\infty$ *, then the problem* (1) *has a positive solution* $u(\lambda)$ *for any* $\lambda\in(0,\infty)$ *satisfying* $\|u_\lambda\|_\infty\to\infty$ *as* $\lambda\to 0$ *and* $\|u_\lambda\|_\infty\to 0$ *as* $\lambda\to\infty$*.*

Proof. We only give the proof of (i) since the proof of (ii) is similar. Since $f_0 = \infty$ and $f_\infty = 0$, it follows from (7) and (8) that

$$S_i(r) \to 0 \text{ as } r \to 0 \text{ and } S_i(r) \to \infty \text{ as } r \to \infty \text{ for } i = 1, 2. \tag{12}$$

Let $\lambda \in (0, \infty)$ be fixed. By (6) and (12), there exist $r_1(\lambda)$ and $r_2(\lambda)$ such that $0 < r_1(\lambda) < r_2(\lambda)$ and $S_1(r_1(\lambda)) < \lambda < S_2(r_2(\lambda))$. By Theorem 3.3 (i), there exists a positive solution u_λ to the problem (1) satisfying $r_1(\lambda) < \|u_\lambda\|_\infty < r_2(\lambda)$. Since $S_i(r) \to 0$ as $r \to 0$ for $i = 1, 2$, we may choose $r_1(\lambda)$ and $r_2(\lambda)$ so that $0 < r_1(\lambda) < r_2(\lambda)$ and $r_2(\lambda) \to 0$ as $\lambda \to 0$. Thus, there exists positive solutions u_λ to the problem (1) for all small $\lambda > 0$ satisfying $\|u_\lambda\|_\infty \to 0$ as $\lambda \to 0$. Similarly, since $S_i(r) \to \infty$ as $r \to \infty$ for $i = 1, 2$, there exists positive solutions u_λ to the problem (1) for all large $\lambda > 0$ satisfying $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$. □

Corollary 3.5. *Assume that* $(H_1), (H_2)$ *and* $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$ *hold.*

- (i) *If* $f_0 = f_\infty = \infty$, *then there exist positive constants* λ^* *and* $\overline{\lambda}$ *such that the problem* (1) *has two positive solutions* $u_1(\lambda)$ *and* $u_2(\lambda)$ *for any* $\lambda \in (0, \lambda^*)$, *it has a positive solution* $u(\lambda^*)$ *for* $\lambda = \lambda^*$, *and it has no positive solutions for* $\lambda \in (\overline{\lambda}, \infty)$.
- (ii) *If* $f_0 = f_\infty = 0$, *then there exist positive constants* λ_* *and* $\underline{\lambda}$ *such that the problem* (1) *has two positive solutions* $u_1(\lambda)$ *and* $u_2(\lambda)$ *for any* $\lambda \in (\lambda_*, \infty)$, *it has a positive solution* $u(\lambda_*)$ *for* $\lambda = \lambda_*$, *and it has no positive solutions for* $\lambda \in (0, \underline{\lambda})$.

Proof. (i) Since $f_0 = f_\infty = \infty$, it follows from (7) that, for $i = 1, 2$, $\lim_{r\to 0} S_i(r) = \lim_{r\to\infty} S_i(r) = 0$. Let $\lambda^* = \max\{S_2(r) : r \in \mathbb{R}_+\} \in (0, \infty)$ and $r^* \in (0, \infty)$ satisfying $S_2(r^*) = \lambda^*$. For any $\lambda \in (0, \lambda^*)$, there exist $r_1(\lambda), r_2(\lambda)$ and $R_1(\lambda)$ such that $0 < r_1(\lambda) < r_2(\lambda) < r^* < R_1(\lambda)$ and $S_* = S_1(r_1(\lambda)) = S_1(R_1(\lambda)) < \lambda < S_2(r_2(\lambda))$. Then, by Theorem 3.4 (ii), there exist two positive solutions $u_1(\lambda)$ and $u_2(\lambda)$ for any $\lambda \in (0, \lambda^*)$.

For each $n \in \mathbb{N}$, let $\lambda_n := \lambda^* - \frac{1}{n}$. Then we may choose $r_1(n)$ and $r_2(n)$ such that $S_1(r_1(n)) < \lambda_n < S_2(r_2(n))$ and $0 < \delta < r_1(n) < r_2(n) < r^*$ for all n. For each n, by Theorem 3.3 (i), there exists $u_n \in \mathcal{K}$ such that $H(\lambda_n, u_n) = u_n$ and $\delta < \|u_n\|_\infty < r^*$. Since $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ is compact and $\{(\lambda_n, u_n)\}$ is bounded in $\mathbb{R}_+ \times \mathcal{K}$, there exist a subsequence $\{(\lambda_{n_k}, u_{n_k})\}$ of $\{(\lambda_n, u_n)\}$ and $u^* \in \mathcal{K}$ such that $H(\lambda_{n_k}, u_{n_k}) = u_{n_k} \to u^*$ in $\mathcal{K}$ as $n_k \to \infty$. Since $\lambda_{n_k} \to \lambda^*$ as $n_k \to \infty$ and H is continuous, $H(\lambda^*, u^*) = u^*$ and $\|u^*\|_\infty \geq \delta > 0$. Thus the problem (1) has a positive solution u^* for $\lambda = \lambda^*$.

Let $\lambda > 0$ be a constant such that there exists a positive solution u_λ to the problem (1), and let σ be a constant satisfying $u_\lambda(\sigma) = \|u_\lambda\|_\infty$. Since $f_0 = f_\infty = \infty$, there exists $C_1 > 0$ such that $f(s) > C_1\varphi(s)$ for $s \in \mathbb{R}_+$. We only consider the case $\sigma \geq \gamma_h$, since the case $\sigma < \gamma_h$ can be proved similarly. Since $u_\lambda(t) \geq u_\lambda(\gamma_h^1)$ for $t \in [\gamma_h^1, \sigma]$, $f(u_\lambda(t)) > C_1\varphi(u_\lambda(\gamma_h^1))$ for $t \in [\gamma_h^1, \gamma]$.

Then

$$
\begin{aligned}
u_\lambda(\gamma_h^1) &\geq \int_0^{\gamma_h^1} \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma \lambda h(\tau) f(u_\lambda(\tau))d\tau\right) ds \\
&\geq \int_0^{\gamma_h^1} \varphi^{-1}\left(\int_{\gamma_h^1}^{\gamma_h} h(\tau)d\tau \|q\|_\infty^{-1}\lambda C_1 \varphi(u_\lambda(\gamma_h^1))\right) ds \\
&\geq \gamma_0 \varphi^{-1}(h_* \|q\|_\infty^{-1}\lambda C_1\varphi(u_\lambda(\gamma_h^1))) \geq \gamma_0 \psi_2^{-1}(h_*\|q\|_\infty^{-1}\lambda C_1)u_\lambda(\gamma_h^1).
\end{aligned}
$$

Here $\gamma_0 = \min\{\gamma_h^1, 1-\gamma_h^2\} > 0$ and $h_* = \min\{\int_{\gamma_h^1}^{\gamma_h} h(\tau)d\tau, \int_{\gamma_h}^{\gamma_h^2} h(\tau)d\tau\} > 0$. Consequently, $\lambda \leq \|q\|_\infty (h_* C_1)^{-1}\psi_2(\gamma_0^{-1}) =: \overline{\lambda}$, and the problem (1) has no positive solutions for $\lambda \in (\overline{\lambda}, \infty)$.

(ii) Since $f_0 = f_\infty = 0$, it follows from (8) that, for $i = 1, 2$, $\lim_{r\to 0} S_i(r) = \lim_{r\to\infty} S_i(r) = \infty$. Then there exists $r_* \in (0,\infty)$ satisfying $S_1(r_*) = \min\{S_1(r) : r \in \mathbb{R}_+\} \in (0,\infty)$. Let $\lambda_* = S_1(r_*)$. For any $\lambda \in (\lambda_*, \infty)$, there exist $r_1(\lambda), r_2(\lambda)$ and $S_2(\lambda)$ such that $0 < r_2(\lambda) < r_1(\lambda) < r_* < S_2(\lambda)$ and $S_1(r_1(\lambda)) < \lambda < S_2(r_2(\lambda)) = S_2(M_2(\lambda)) = S^*$. Then, by Theorem 3.4 (ii), there exist two positive solutions $u_1(\lambda)$ and $u_2(\lambda)$ such that $0 < \|u_1(\lambda)\|_\infty < r_* < \|u_2(\lambda)\|_\infty$. By the argument similar to those in the proof of Corollary 3.5 (i), one can show that the problem (1) has a positive solution $u(\lambda_*)$ for $\lambda = \lambda_*$.

Let $\lambda > 0$ be a constant such that there exists a positive solution u_λ to the problem (1), and let σ be a constant satisfying $u_\lambda(\sigma) = \|u_\lambda\|_\infty$. Since $f_0 = f_\infty = 0$, there exists $C_2 > 0$ such that $f(s) \leq C_2\varphi(s)$ for $s \in \mathbb{R}_+$, and $f(u_\lambda(t)) \leq C_2\varphi(u_\lambda(t)) \leq C_2\varphi(u_\lambda(\sigma))$ for all $t \in [0,1]$. We only consider the case $\sigma \leq \gamma_h$, since the case $\sigma > \gamma_h$ can be proved similarly. By (10),

$$
\begin{aligned}
u_\lambda(\sigma) &\leq A_1 \int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma \lambda h(\tau) f(u_\lambda(\tau))d\tau\right) ds \\
&\leq A_1 \int_0^{\gamma_h} \varphi^{-1}\left(\int_s^\gamma h(\tau)d\tau q_0^{-1}\lambda C_2\varphi(u_\lambda(\sigma))\right) ds \\
&\leq A_* h_{**}\varphi^{-1}(q_0^{-1}\lambda C_2\varphi(u_\lambda(\sigma))) \leq A_* h_{**}\psi_1^{-1}(q_0^{-1}\lambda C_2)u_\lambda(\sigma).
\end{aligned}
$$

Here $h_{**} = \max\left\{\int_0^{\gamma_h} \psi_1^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right) ds, \int_{\gamma_h}^1 \psi_1^{-1}\left(\int_{\gamma_h}^s h(\tau)d\tau\right) ds\right\} > 0$ and $A_* = \max\{A_1, A_2\}$. Consequently, $\lambda \geq q_0 C_2^{-1}\psi_1(A_*^{-1}h_{**}^{-1}) =: \underline{\lambda}$, and the problem (1) has no positive solutions for $\lambda \in (0, \underline{\lambda})$.

□

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Chan-Gyun Kim
Department of Mathematics Education
Chinju National University of Education
Jinju 52673, Korea
Email address: cgkim75@cue.ac.kr