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SINGULAR AND DUAL SINGULAR FUNCTIONS FOR PARTIAL DIFFERENTIAL EQUATION WITH AN INPUT FUNCTION IN $H^1(\Omega)$

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Abstract. In [6, 7] they introduced a new finite element method for accurate numerical solutions of Poisson equations with corner singularities. They consider the Poisson equations with homogeneous boundary conditions, compute the finite element solutions using standard FEM and use the extraction formula to compute the stress intensity factor(s), then they posed new PDE with a regular solution by imposing the nonhomogeneous boundary condition using the computed stress intensity factor(s), which converges with optimal speed. From the solution they could get an accurate solution just by adding the singular part.

They considered a partial differential equation with the input function $f \in L^2(\Omega)$. In this paper we consider a PDE with the input function $f \in H^1(\Omega)$ and find the corresponding singular and dual singular functions. We also induce the corresponding extraction formula which are the basic element for the approach.

1. Introduction

First we let Ω be an open, bounded polygonal domain in $\mathbb{R}^2$ and let Γ_D and Γ_N be a partition of the boundary of Ω such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$, and consider the following Poisson equation with mixed boundary conditions:

$$\left\{ \begin{array}{rcll} -\Delta u &=& f & \text{in } \Omega, \\ u &=& 0 & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \nu} &=& 0 & \text{on } \Gamma_N, \end{array} \right. \tag{1}$$

where $f \in H^1(\Omega)$ and Δ stands for the Laplacian operator. Here, ν denote the outward unit vector normal to the boundary.

We note that the classification of the singular functions and dual singular functions for the various boundary conditions are essential steps in the approach of [6, 7]. We also note that the type of singular function depends on the angle

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of the corner and the boundary conditions near the corner. In many previous papers they consider the case the input function f is in $L^2(\Omega)$(see [2, 3, 5, 8, 9]). The popose of this paper is that we classify the singular functions in the case the input function f is in $H^1(\Omega)$. For simplicity, we consider two cases; a corner with the inner angle w : $\frac{\pi}{2} < \omega \leq 2\pi$ and Dirichlet boundary condition(D/D) and a corner with the inner angle w : $\frac{\pi}{4} < \omega \leq 2\pi$ and mixed boundary condition(D/N).

In this appreach we use a smooth cut-off function η which equals one identically in a neighborhood of the corner and the support of η is small enough so that the function ηs vanishes identically on $\partial\Omega\backslash\Gamma_0$, where Γ_0 is the union of two adjacient boundary lines at the corner.

Here we give an example of a cut-off function. Let (r, θ) be the polar coordinate with the origin at the corner and set

$$B(r_1; r_2) = \{(r, \theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega$$

and

$$B(r_1) = B(0; r_1),$$

and define a smooth enough cut-off function of r as follows:

$$\eta_\rho(r) = \begin{cases} 1 & \text{in} \quad B(\frac{1}{2}\rho), \\ \frac{1}{16}\{8 - 15p(r) + 10p(r)^3 - 3p(r)^5\} & \text{in} \quad \overline{B}(\frac{1}{2}\rho; \rho), \\ 0 & \text{in} \quad \Omega\backslash\overline{B}(\rho), \end{cases} \tag{2}$$

with $p(r) = 4r/\rho - 3$. Here, ρ is a parameter which will be determined so that the singular part $\eta_\rho s_j$ has the same boundary condition as the solution u of the model problem, where s_j is one of the singular functions which will be given in the following sections. Note $\eta(r) = \eta_\rho(r)$ is C^2 and satisfies the following;

$$\eta(\rho) = \eta'(\rho) = \eta''(\rho) = 0 \text{ and } \eta(\rho/2) = 1,\ \eta'(\rho/2) = \eta''(\rho/2) = 0.$$

We will use two cut-off functions $\eta := \eta(r) = \eta_\rho(r)$ and $\eta_2 := \eta_2(r) = \eta_{2\rho}(r)$ to derive the extraction formulas. Note $\eta_2 = 1$ on the support of η.

2. A corner with Dirichlet boundary condition

When the input function f is in $L^2(\Omega)$ and $\Gamma_N = \emptyset$, the singularity is quite simple; we have a singular function s_1 and its dual singular function s_{-1}

$$s = s(r, \theta) = r^{\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \quad s_- = s_-(r, \theta) = r^{-\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \tag{3}$$

for the model problem (1) and the unique solution $u \in H^1_D(\Omega)$ has the representation (see [1, 3, 4]):

$$u = w + \lambda\eta s. \tag{4}$$

From now on we assume that f is in $H^1(\Omega)$ and assume the inner angle w : $\frac{\pi}{2} < \omega \leq 2\pi$.

Case 1) If we have a corner with inner angle $\frac{\pi}{2} < \omega < \pi$ with the boundary condition **D/D**, we have a singular function s_1 and its dual singular function s_{-1}:

$$s_1 = s_1(r,\theta) = r^{\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \quad s_{-1} = s_{-1}(r,\theta) = r^{-\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \tag{5}$$

and the unique solution $u \in H_D^1(\Omega)$ has the representation:

$$u = w + \lambda_1 \eta s_1. \tag{6}$$

Case 2) If we have a corner with inner angle $\pi < \omega \leq \frac{3\pi}{2}$ with the boundary condition **D/D**, we have singular functions s_1 and s_2:

$$s_1 = s_1(r,\theta) = r^{\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \quad s_2 = s_2(r,\theta) = r^{\frac{2\pi}{\omega}} \sin\frac{2\pi\theta}{\omega}, \tag{7}$$

and their dual singular functions s_{-1} and s_{-2}:

$$s_{-1} = s_{-1}(r,\theta) = r^{-\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \quad s_{-2} = s_{-2}(r,\theta) = r^{-\frac{2\pi}{\omega}} \sin\frac{2\pi\theta}{\omega}, \tag{8}$$

and the unique solution $u \in H_D^1(\Omega)$ has the representation:

$$u = w + \lambda_1 \eta s_1 + \lambda_2 \eta s_2. \tag{9}$$

Case 3) If we have a corner with inner angle $\frac{3\pi}{2} < \omega < 2\pi$ with the boundary condition **D/D**, we have singular functions s_1, s_2, and s_3:

$$s_1 = r^{\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \quad s_2 = r^{\frac{2\pi}{\omega}} \sin\frac{2\pi\theta}{\omega}, \quad s_3 = r^{\frac{3\pi}{\omega}} \sin\frac{3\pi\theta}{\omega}, \tag{10}$$

and their dual singular functions s_{-1}, s_{-2} and s_{-3}:

$$s_{-1} = r^{-\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \quad s_{-2} = r^{-\frac{2\pi}{\omega}} \sin\frac{2\pi\theta}{\omega}, \quad s_{-3} = r^{-\frac{3\pi}{\omega}} \sin\frac{3\pi\theta}{\omega}, \tag{11}$$

and the unique solution $u \in H_D^1(\Omega)$ has the representation:

$$u = w + \lambda_1 \eta s_1 + \lambda_2 \eta s_2 + \lambda_3 \eta s_3. \tag{12}$$

Case 4) If we have a corner with inner angle $\omega = 2\pi$ with the boundary condition **D/D**, we have singular functions s_1 and s_3:

$$s_1 = r^{\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \quad s_3 = r^{\frac{3\pi}{\omega}} \sin\frac{3\pi\theta}{\omega}, \tag{13}$$

and their dual singular functions s_{-1} and s_{-3}:

$$s_{-1} = r^{-\frac{\pi}{\omega}} \sin\frac{\pi\theta}{\omega}, \quad s_{-3} = r^{-\frac{3\pi}{\omega}} \sin\frac{3\pi\theta}{\omega}, \tag{14}$$

and the unique solution $u \in H_D^1(\Omega)$ has the representation:

$$u = w + \lambda_1 \eta s_1 + \lambda_3 \eta s_3. \tag{15}$$

3. Extraction formula for the stress intensity: D/D case

In this section we induce the extraction formula for the D/D case. We note that the singular function and dual singular functions in all 4 cases have consistent forms as follows:

$$s_i = r^{\frac{i\pi}{\omega}} \sin \frac{i\pi\theta}{\omega}, \quad s_{-j} = r^{-\frac{j\pi}{\omega}} \sin \frac{j\pi\theta}{\omega}, \tag{16}$$

with i and j are in $L = \{1, 2, 3\}$. Note $\Delta s_i = \Delta s_{-j} = 0$. Before we get the extraction formula, we prove a simple lemma.

Lemma 3.1. *If s_i and s_{-j} are the singular function and dual singular functions for the D/D corners, then we have that*

$$(\Delta(\eta s_i), s_{-j}) = -j\pi\delta_{ij}. \tag{17}$$

Proof. Note that

$$\Delta(\eta s_i) = s_i \Delta\eta + 2\,\nabla\eta \cdot \nabla s_i = s_i(\eta'/r + \eta'') + 2\eta' \cdot \frac{i\pi}{\omega} \cdot r^{-1+\frac{i\pi}{\omega}} \sin \frac{i\pi\theta}{\omega}$$

and that the integrand of $(\Delta(\eta s_i), s_{-j})$ is non-zero on only $B(\rho/2; \rho)$ and has the form:

$$r \cdot \Delta(\eta s_i) \cdot s_{-j} = r^{\frac{(i-j)\pi}{\omega}} (\eta' + r\eta'') \sin \frac{i\pi\theta}{\omega} \sin \frac{j\pi\theta}{\omega} + 2\eta' \cdot \frac{i\pi}{\omega} r^{\frac{(i-j)\pi}{\omega}} \sin \frac{i\pi\theta}{\omega} \sin \frac{j\pi\theta}{\omega}$$

So, we have the lemma by the fact

$$\int_0^\omega \sin \frac{i\pi\theta}{\omega} \sin \frac{j\pi\theta}{\omega} d\theta = \frac{\omega}{2}\delta_{ij}$$

and the properties of η. □

Theorem 3.2. *The stress intensity factors λ_j, for the D/D corners, can be expressed in terms of u and f by the following extraction formula:*

$$\lambda_j = \frac{1}{j\pi} \int_\Omega f\eta s_{-j} dx + \frac{1}{j\pi} \int_\Omega u\Delta(\eta s_{-j}) dx, \quad j \in L. \tag{18}$$

Proof. Note that the solution u of (1) has the form;

$$u = w + \sum_{i \in L} \lambda_i \eta s_i$$

and gives

$$-\Delta u = -\Delta w - \sum_{i \in L} \lambda_i \Delta(\eta s_i) = f.$$

By multiplying $\eta_2 s_{-j}$, we have

$$\begin{aligned} (f, \eta_2 s_{-j}) &= -(\Delta w, \eta_2 s_{-j}) - \textstyle\sum_{i \in L} \lambda_i (\Delta(\eta s_i), \eta_2 s_{-j}) \\ &= -(w, \Delta(\eta_2 s_{-j})) - \textstyle\sum_{i \in L} \lambda_i (\Delta(\eta s_i), s_{-j}) \\ &= -(w, \Delta(\eta_2 s_{-j})) - \lambda_j(-j\pi). \end{aligned} \tag{19}$$

and the result follows by noting the fact that $(\eta s_i, \Delta(\eta_2 s_{-j})) = 0$ and by changing η_2 to η. $\square$

4. A corner with mixed boundary condition

In this section we consider the case we have mixed boundary condition **(D/N)**, assuming that the boundary condition change from Dirichlet to Neumann at that corner.

From now on we assume that f is in $H^1(\Omega)$ and assume the inner angle w satisfies $\frac{\pi}{4} < \omega \le 2\pi$.

Case 1) If we have a corner with inner angle $\frac{\pi}{4} < \omega < \frac{\pi}{2}$ or $\frac{\pi}{2} < \omega \le \frac{3\pi}{4}$ with the boundary condition **D/N**, we have a singular function s_1 and its dual singular function s_{-1}:

$$s_1 = s_1(r,\theta) = r^{\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega}, \quad s_{-1} = s_{-1}(r,\theta) = r^{-\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega}, \tag{20}$$

and the unique solution $u \in H_D^1(\Omega)$ has the representation:

$$u = w + \lambda_1 \eta s_1. \tag{21}$$

Note: We have no singular function if the inner angle $\omega = \frac{\pi}{2}$.

Case 2) If we have a corner with inner angle $\frac{3\pi}{4} < \omega \le \frac{5\pi}{4}$ with the boundary condition **D/N**, we have singular functions s_1 and s_3:

$$s_1 = r^{\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega}, \quad s_3 = r^{\frac{3\pi}{2\omega}} \sin\frac{3\pi\theta}{2\omega}, \tag{22}$$

and their dual singular functions s_{-1} and s_{-3}:

$$s_{-1} = r^{-\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega}, \quad s_{-3} = r^{-\frac{3\pi}{2\omega}} \sin\frac{3\pi\theta}{2\omega}, \tag{23}$$

and the unique solution $u \in H_D^1(\Omega)$ has the representation:

$$u = w + \lambda_1 \eta s_1 + \lambda_3 \eta s_3. \tag{24}$$

Case 3) If we have a corner with inner angle $\frac{5\pi}{4} < \omega < \frac{3\pi}{2}$ or $\frac{3\pi}{2} < \omega \le \frac{7\pi}{4}$ with the boundary condition **D/N**, we have singular functions s_1, s_3, and s_5:

$$s_1 = r^{\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega}, \quad s_3 = r^{\frac{3\pi}{2\omega}} \sin\frac{3\pi\theta}{2\omega}, \quad s_5 = r^{\frac{5\pi}{2\omega}} \sin\frac{5\pi\theta}{2\omega}, \tag{25}$$

and their dual singular functions s_{-1}, s_{-3}, and s_{-5}:

$$s_{-1} = r^{-\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega}, \quad s_{-3} = r^{-\frac{3\pi}{2\omega}} \sin\frac{3\pi\theta}{2\omega}, \quad s_{-5} = r^{-\frac{5\pi}{2\omega}} \sin\frac{5\pi\theta}{2\omega}, \tag{26}$$

and the unique solution $u \in H_D^1(\Omega)$ has the representation:

$$u = w + \lambda_1 \eta s_1 + \lambda_3 \eta s_3 + \lambda_5 \eta s_5. \tag{27}$$

Case 4) If we have a corner with inner angle $\omega = \frac{3\pi}{2}$ with the boundary condition **D/N**, we have singular functions s_1 and s_5:

$$s_1 = r^{\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega}, \quad s_5 = r^{\frac{5\pi}{2\omega}} \sin\frac{5\pi\theta}{2\omega}, \tag{28}$$

and their dual singular functions s_{-1} and s_{-5}:

$$s_{-1} = r^{-\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega}, \quad s_{-5} = r^{-\frac{5\pi}{2\omega}} \sin\frac{5\pi\theta}{2\omega}, \tag{29}$$

and the unique solution $u \in H^1_D(\Omega)$ has the representation:

$$u = w + \lambda_1\eta s_1 + \lambda_5\eta s_5. \tag{30}$$

Case 5) If we have a corner with inner angle $\frac{7\pi}{4} < \omega \le 2\pi$ with the boundary condition **D/N**, we have singular functions s_1, s_3, s_5, and s_7:

$$\begin{aligned} s_1 &= r^{\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega}, \quad s_3 = r^{\frac{3\pi}{2\omega}} \sin\frac{3\pi\theta}{2\omega}, \\ s_5 &= r^{\frac{5\pi}{2\omega}} \sin\frac{5\pi\theta}{2\omega}, \quad s_7 = r^{\frac{7\pi}{2\omega}} \sin\frac{7\pi\theta}{2\omega}, \end{aligned} \tag{31}$$

and their dual singular functions s_{-1}, s_{-3}, s_{-5}, and s_{-7}:

$$\begin{aligned} s_{-1} &= r^{-\frac{\pi}{2\omega}} \sin\frac{\pi\theta}{2\omega}, \quad s_{-3} = r^{-\frac{3\pi}{2\omega}} \sin\frac{3\pi\theta}{2\omega}, \\ s_{-5} &= r^{-\frac{5\pi}{2\omega}} \sin\frac{5\pi\theta}{2\omega}, \quad s_{-7} = r^{-\frac{7\pi}{2\omega}} \sin\frac{7\pi\theta}{2\omega}, \end{aligned} \tag{32}$$

and the unique solution $u \in H^1_D(\Omega)$ has the representation:

$$u = w + \lambda_1\eta s_1 + \lambda_3\eta s_3 + \lambda_5\eta s_5 + \lambda_7\eta s_7. \tag{33}$$

Here we note that $w \in H^3(\Omega) \cap H^1_D(\Omega)$ for all case with one exception that we have $w \in H^{3-\epsilon}(\Omega) \cap H^1_D(\Omega)$ for all $\epsilon > 0$ if we have a corner with inner angle $\omega = 2\pi$ as in **Case 5)**.

5. Extraction formula for the stress intensity factors: D/N case

In this section we induce the extraction formula for the D/N case. We note that the singular function and dual singular functions in all 5 cases have consistent forms as follows:

$$s_i = r^{\frac{i\pi}{2\omega}} \sin\frac{i\pi\theta}{2\omega}, \quad s_{-j} = r^{-\frac{j\pi}{2\omega}} \sin\frac{j\pi\theta}{2\omega}, \tag{34}$$

with i and j are in $M = \{1, 3, 5, 7\}$.

To get the extraction formula for the case D/N, we need a similar lemma.

Lemma 5.1. *If s_i and s_{-j} are the singular function and dual singular functions for the D/N corners, then we have that*

$$(\Delta(\eta s_i), s_{-j}) = -\frac{i\pi}{2}\delta_{ij}. \tag{35}$$

Proof. The proof is similar to that of Lemma 3.1. □

Theorem 5.2. *The stress intensity factors λ_j can be expressed in terms of u and f by the following extraction formula:*

$$\lambda_j = \frac{2}{j\pi}\int_\Omega f\eta s_{-j}dx + \frac{2}{j\pi}\int_\Omega u\Delta(\eta s_{-j})dx, \quad j \in M. \tag{36}$$

Proof. The proof is similar to that of Theorem 3.2. □

6. Suggested algorithms

Motivated by the theorems in section 3, we pose the following algorithm for D/D case ([6, 7]):

A1.: Find a solution $u^{(0)}$ of (1) and

A2.: compute the stress intensity factors $\lambda_j^{(0)}, j \in L$, from (18).

A3.: For $i = 1, 2, \cdots, N$,

A3-1.: Solve, for $w^{(i)}$,

$$\begin{cases} -\Delta w^{(i)} &= f & \text{in } \Omega, \\ w^{(i)} &= -\sum_{j\in L}\lambda_j^{(i-1)}s_j & \text{on } \Gamma_D, \end{cases} \tag{37}$$

A3-2.: Let $u^{(i)} = w^{(i)} + \sum_{j\in L}\lambda_j^{(i-1)}s_j$.

A3-3.: Compute $\lambda_j^{(i)}$ by

$$\lambda_j^{(i)} = \frac{1}{j\pi}\int_\Omega f\eta s_{-j}dx + \frac{1}{j\pi}\int_\Omega u^{(i)}\Delta(\eta s_{-j})dx, \quad j \in L. \tag{38}$$

Motivated by the theorems in section 5, we pose the following algorithm for D/N case:

A1.: Find a solution $u^{(0)}$ of (1) and

A2.: compute the stress intensity factors $\lambda_j^{(0)}, j \in M$, from (36).

A3.: For $i = 1, 2, \cdots, N$,

A3-1.: Solve, for $w^{(i)}$,

$$\begin{cases} -\Delta w^{(i)} &= f & \text{in } \Omega, \\ w^{(i)} &= -\sum_{j\in M}\lambda_j^{(i-1)}s_j & \text{on } \Gamma_D, \\ \frac{\partial w^{(i)}}{\partial \nu} &= -\sum_{j\in M}\lambda_j^{(i-1)}\frac{\partial s_j}{\partial \nu} & \text{on } \Gamma_N. \end{cases} \tag{39}$$

A3-2.: Let $u^{(i)} = w^{(i)} + \sum_{j\in M}\lambda_j^{(i-1)}s_j$.

A3-3.: Compute $\lambda_j^{(i)}$ by

$$\lambda_j^{(i)} = \frac{2}{j\pi}\int_\Omega f\eta s_{-j}dx + \frac{2}{j\pi}\int_\Omega u^{(i)}\Delta(\eta s_{-j})dx, \quad j \in M. \tag{40}$$

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